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QUASI-PRIME AND d-PRIME IDEALS IN COMMUTATIVE DIFFERENTIAL RINGS

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The paper contains some properties of d-prime ideals and of quasi-prime ideals of commutative differential rings. It is shown, among others, that any quasi-prime ideal is d-prime and that in a noetherian case these two notions coincide. Moreover, an example of a non-noetherian d-ring is given where there are d-prime d-ideals which are not quasi-prime.

1. Preliminaries. Throughout this paper all rings are commutative with identity. For any ring R and for any ideal A of R, $r(A)$ will denote the radical of A. The term *d-ring* will refer to a ring R together with a specified derivation $d: R \to R$.

Let R be a d-ring. An ideal A in R is called a *d-ideal* if $d(A) \subset A$. For an arbitrary subset T of R denote by $[T]$ the smallest d-ideal containing T. A d-ideal P in R is called *quasi-prime* if there is a multiplicative subset S of R such that P is maximal among d-ideals disjoint from S. Some of the properties of the quasi-prime ideals are given in [2]-[4]. A proper d-ideal P of R is called *d-prime* if for d-ideals A and B of R the relation $AB \subset P$ implies either $A \subset P$ or $B \subset P$ (see [1], [6], [7]).

2. Quasi-prime ideals and d-prime d-ideals.

PROPOSITION 2.1. *Every quasi-prime ideal is a d-prime d-ideal.*

Proof. Let P be a quasi-prime ideal in a d-ring R and let S be a multiplicative subset of R such that P is maximal among d-ideals in R disjoint from S. We assume that there are two d-ideals A and B in R such that $AB \subset P$, $A \not\subset P$, and $B \not\subset P$. Choose $a \in A \setminus P$, $b \in B \setminus P$ and consider d-ideals $A_1 = P+[a]$ and $B_1 = P+[b]$. Since $A_1 \supsetneqq P$ and $B_1 \supsetneqq P$, we have $A_1 \cap S \neq \emptyset$ and $B_1 \cap S \neq \emptyset$. Let $s \in A_1 \cap S$ and $t \in B_1 \cap S$. Then

$$st \in A_1 B_1 \subset P+[a][b] \subset P+AB \subset P+P = P, \text{ i.e., } st \in P \cap S.$$

This contradicts the fact that P is disjoint from S.

THEOREM 2.1. *Let R be a noetherian d-ring and let P be a d-ideal in R. Then the following properties are equivalent*:

(1) *P is a quasi-prime ideal*;

(2) *P is a d-prime d-ideal.*

Proof. (1) $\Rightarrow$ (2) follows from Proposition 2.1.

(2) $\Rightarrow$ (1). We prove first that if P is a d-prime d-ideal in a noetherian d-ring R, then P is a primary ideal. Let $xy \in P$. There is a natural number k such that $(P : x^k)$ is a d-ideal (see [5] or [6]). Since $xy \in P$, we have $x^k y \in P$, i.e., $y \in (P : x^k)$, and so $[y] \subset (P : x^k)$. Since $(P : [y])$ is a d-ideal and $x^k \in (P : [y])$, we obtain $[x^k] \subset (P : [y])$, i.e., $[x^k][y] \subset P$. Thus $[x^k] \subset P$ or $[y] \subset P$ and, therefore, $x^k \in P$ or $y \in P$, i.e., P is primary.

Now, let $S = R \setminus r(P)$. Clearly, S is a multiplicative subset of R and P is disjoint from S. Let P_1 be a d-ideal of R maximal among d-ideals containing P and disjoint from S. Then P_1 is a quasi-prime ideal in R. Since $P \subset P_1 \subset r(P)$, we have $r(P_1) = r(P)$. Let n be a natural number such that $r(P)^n \subset P$. Then $r(P_1)^n \subset r(P)^n \subset P$ and, consequently, $P_1^n \subset P$. Since P is a d-prime d-ideal, we get $P_1 \subset P$, and thus $P = P_1$, i.e., P is a quasi-prime ideal.

In what follows we give an example of a non-noetherian d-ring R having d-prime d-ideals which are not quasi-prime. To do this we need some properties of generalized Newton symbols.

3. Generalized Newton symbols.

If $n_1, \ldots, n_k$ are non-negative integers, then we set

$$(n_1, \ldots, n_k) = \frac{(n_1 + \ldots + n_k)!}{n_1! \ldots n_k!}.$$

Since

$$(n_1, \ldots, n_k) = (n_1, n_2 + \ldots + n_k)(n_2, n_3 + \ldots + n_k) \ldots (n_{k-1}, n_k),$$

$(n_1, \ldots, n_k)$ is always a natural number.

We have the following

LEMMA 3.1. *If R is a d-ring and $x_1, \ldots, x_k \in R$, then*

$$d^n(x_1 \ldots x_k) = \sum_{i_1 + \ldots + i_k = n} (i_1, \ldots, i_k) d^{i_1}(x_1) \ldots d^{i_k}(x_k)$$

for any natural number n.

We assume now that p is a prime natural number and that N is the set of all natural numbers. We define two mappings

$$\sigma_p \colon N \to N \cup \{0\} \quad \text{and} \quad \tau_p \colon N \to N \cup \{0\}$$

by the formulas

$$\tau_p(n) = k \text{ if } n = p^k m, \text{ where } p \nmid m, \qquad \tau_p(n) = \sigma_p(n!).$$

Clearly, if $n_1, n_2 \in N$, then $\sigma_p(n_1 n_2) = \sigma_p(n_1) + \sigma_p(n_2)$. The mapping τ_p has the following properties:

(a) $\tau_p(p^m) = (p-1)^{-1}(p^m - 1)$ for any $m \in N$.

(b) If $l \neq m$ and $0 \leqslant a < p$, $0 \leqslant b < p$, then

$$\tau_p(ap^l + bp^m) = a\tau_p(p^l) + b\tau_p(p^m).$$

Using (a) and (b) we get

LEMMA 3.2. *Let $n \in N$. If*

$$n = \sum_{i=0}^{k} a_i p^i, \qquad \textit{where } 0 \leqslant a_i < p \textit{ for } i = 0, 1, \ldots,$$

then

$$\tau_p(n) = (p-1)^{-1}\big(n - s_p(n)\big), \qquad \textit{where } s_p(n) = a_0 + a_1 + \ldots + a_k.$$

As an immediate consequence of Lemma 3.2 we get the following two corollaries:

COROLLARY 3.1. *Let $a_0, a_1, \ldots, a_k$ be integers such that $0 \leqslant a_i < p$ for $i = 0, 1, \ldots, k$ and let*

$$A = (\underbrace{p^k, \ldots, p^k}_{a_k}, \underbrace{p^{k-1}, \ldots, p^{k-1}}_{a_{k-1}}, \ldots, \underbrace{p, \ldots, p}_{a_1}, \underbrace{1, \ldots, 1}_{a_0}).$$

Then p is not a divisor of A.

COROLLARY 3.2. *Let $s_1, \ldots, s_k$ be natural numbers such that $s_1 < \ldots < s_k$ and let $a_1, \ldots, a_k$ be integers such that the following conditions hold:*

(1) $|a_i| < p^{s_i}$ *for* $i = 1, \ldots, k$.

(2) $a_1 + \ldots + a_k = 0$.

(3) *There is an i_0 such that $a_{i_0} \neq 0$.*

Then $(p^{s_1} + a_1, \ldots, p^{s_k} + a_k) \equiv 0 \pmod{p}$.

4. An example of a d-prime d-ideal which is not quasi-prime. Let

$$T = Z_2[X_1, X_2, \ldots, Y_1, Y_2, \ldots]$$

be a ring of polynomials of variables $X_1, X_2, \ldots, Y_1, Y_2, \ldots$ over the field Z_2, and let

$$A = (X_1^2, X_2^2, \ldots, Y_1^2, Y_2^2, \ldots)$$

be an ideal in T generated by squares of all variables. Moreover, let $R = T/A$ and let $x_n = X_n + A$ and $y_n = Y_n + A$ for all $n \in N$. The ring R is a local ring with a maximal ideal

$$M = (x_1, x_2, \ldots, y_1, y_2, \ldots).$$

For an arbitrary element $r \in R$, if $r \in M$, then $r^2 = 0$, and if $r \notin M$, then $r^2 = 1$. The set of all elements from R taking the form

$$x_{i_1} \dots x_{i_k} y_{j_1} \dots y_{j_l},$$

where $1 \leqslant i_1 < \dots < i_k$ and $1 \leqslant j_1 < \dots < j_l$, is a basis of the vector space M over Z_2.

Let $d\colon R \to R$ be a derivation of R such that $d(x_n) = x_{n+1}$ and $d(y_n) = y_{n+1}$ for every $n \in N$. Therefore, R is a d-ring and M is a d-ideal.

Under these assumptions the following two lemmas hold.

LEMMA 4.1. *Every quasi-prime ideal of R is equal to M.*

Proof. Let P be a quasi-prime ideal in R and S a multiplicative subset of R such that P is maximal among d-ideals in R disjoint from S. If $P \subsetneqq M$, then there is an element s of R such that $s \in M \cap S$, and so $0 = s^2 \in M \cap S$. This contradicts the fact that $0 \notin S$.

LEMMA 4.2. *Let $i_1, \dots, i_k$ be natural numbers such that $i_1 < \dots < i_k$ and let s be a natural number such that $i_k < 2^s$. Moreover, let*

$$n = 2^{s+1} + \dots + 2^{s+k}, \qquad t_1 = 2^{s+1} + i_1, \dots, t_k = 2^{s+k} + i_k.$$

Then in the basic representation of elements $d^n(x_{i_1} \dots x_{i_k})$ a coefficient of $x_{t_1} \dots x_{t_k}$ is equal to 1.

Proof. From Lemma 3.1 it follows that

$$d^n(x_{i_1} \dots x_{i_k}) = \sum_{j_1 + \dots + j_k = n} (j_1, \dots, j_k) x_{i_1 + j_1} \dots x_{i_k + j_k}. \tag{4.1}$$

In particular, putting $j_1 = 2^{s+1}, \dots, j_k = 2^{s+k}$, we get in the right-hand side of (4.1) a summand

$$(2^{s+1}, \dots, 2^{s+k}) x_{t_1} \dots x_{t_k}.$$

By Corollary 3.1 we have

$$(2^{s+1}, \dots, 2^{s+k}) \equiv 1 \pmod 2.$$

In the right-hand side of (4.1) the term $x_{t_1} \dots x_{t_k}$ appears $k!$ times and we have there a summand

$$\big(\sigma(t_1) - i_1, \dots, \sigma(t_k) - i_k\big) x_{\sigma(t_1)} \dots x_{\sigma(t_k)}$$

for any permutation σ of the set $\{t_1, \dots, t_k\}$. Put

$$A_\sigma = \big(\sigma(t_1) - i_1, \dots, \sigma(t_k) - i_k\big).$$

We shall show that if σ is not the identity permutation, then $A_\sigma \equiv 0 \pmod 2$. To this aim put the numbers $\sigma(t_1) - i_1, \dots, \sigma(t_k) - i_k$ in an increasing order. Then

$$A_\sigma = \big(t_1 - \tau(i_1), \dots, t_k - \tau(i_k)\big),$$

where τ is a non-identity permutation of $\{i_1, \ldots, i_k\}$. Put

$$a_1 = i_1 - \tau(i_1), \ldots, a_k = i_k - \tau(i_k).$$

Then

$$A_\sigma = (2^{s+1} + a_1, \ldots, 2^{s+k} + a_k).$$

The integers $a_1, \ldots, a_k$ fulfil all the assumptions of Corollary 3.2 for $p = 2$. Therefore $A_\sigma \equiv 0 \pmod 2$, which completes the proof.

THEOREM 4.1. *If P is an ideal in R generated by the elements $y_1, y_2, \ldots$, then P is a d-prime d-ideal in R.*

Proof. Take $Q = (x_1, x_2, \ldots)$. Clearly, Q is a d-ideal and $P+Q = M$. Assume that A and B are d-ideals in R such that $A \subset M$, $B \subset M$, $AB \subset P$, and $A \not\subset P$. Let $a \in A$ and $a \notin P$. Since $A \subset M$, there exist elements $p \in P$ and $u \notin P$ such that $a = p + u$.

Let

$$u = \sum a_{i_1 \ldots i_s} x_{i_1} \ldots x_{i_s} \tag{4.2}$$

be a basic representation of u, where coefficients $a_{i_1 \ldots i_s}$ are in Z_2. Since $u \notin P$, some coefficient is equal to 1. Choose the shortest term from the set of all products $x_{i_1} \ldots x_{i_s}$ appearing in (4.2) with coefficients equal to 1 and denote it by $x_{i_1} \ldots x_{i_s}$. Denote by I the set of all indices i such that x_i appears essentially in (4.1) and put $J = I \setminus \{i_1, \ldots, i_s\}$. Clearly, I and J are finite. Then

$$u \prod_{j \in J} x_j = \prod_{i \in I} x_i$$

and, of course,

$$\prod_{i \in I} x_i \neq 0.$$

Consequently,

$$a \prod_{j \in J} x_j = (p+u) \prod_{j \in J} x_j = p_1 + \prod_{i \in I} x_i, \quad \text{where } p_1 \in P.$$

Thus we have shown that the d-ideal A contains an element v of the form

$$v = p + x_{i_1} \ldots x_{i_k}, \quad \text{where } p \in P,\ i_1 < \ldots < i_k.$$

Take now any element b from B. Since $B \subset M$, we have

$$b = p_1 + \sum_{j_1 < \ldots < j_s} \beta_{j_1 \ldots j_s} x_{j_1} \ldots x_{j_s}, \tag{4.3}$$

where $p_1 \in P$ and $\beta_{j_1 \ldots j_s} \in Z_2$.

Let $\{j_1, \ldots, j_t\}$ be the set of all indices of x's in (4.3) and assume that $j_1 < \ldots < j_t$. Moreover, let s be a natural number such that $2^s > \max(j_t, i_k)$ and put

$$n = 2^{s+1} + \ldots + 2^{s+k}, \quad t_1 = 2^{s+1} + i_1, \ldots, t_k = 2^{s+k} + i_k.$$

Since A is a d-ideal and $v \in A$, we have $d^n(v) \in A$. However,

$$d^n(v) = d^n(p + x_{i_1} \ldots x_{i_k}) = p_2 + d^n(x_{i_1} \ldots x_{i_k}), \quad \text{where } p_2 \in P.$$

It follows from Lemma 4.2 that in a basic representation of the element $d^n(x_{i_1} \ldots x_{i_k})$ the coefficient of $x_{t_1} \ldots x_{t_k}$ is equal to 1, so in this representation of $d^n(v)b$ there is a summand

$$\sum_{j_1 < \ldots < j_s} \beta_{j_1 \ldots j_s} x_{j_1} \ldots x_{j_s} x_{t_1} \ldots x_{t_k}. \tag{4.4}$$

Since $d^n(v)b \in P$, all coefficients $\beta_{j_1 \ldots j_s}$ of the summand (4.4) are equal to 0 and by (4.3) we get $b = p_1 \in P$. Thus $B \subset P$, and hence P is d-prime.

Theorem 4.1 and Lemma 4.1 imply that $P = (y_1, y_2, \ldots)$ is a d-prime d-ideal in R and that it is not quasi-prime in R.

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