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INVERSION IN A CLASS OF LATTICE-ORDERED ALGEBRAS

BY

ANTHONY W. HAGER (MIDDLETOWN, CONN.), PHILIP NANZETTA (ST. MARY'S CITY, MD.) AND DONALD PLANK (AMHERST, MASS.)

The Φ-algebras of M. Henriksen and D. G. Johnson form a class of lattice-ordered real algebras extensively generalizing the algebras of the form $C(X)$. We consider here the existence of rings of quotients of a special kind for uniformly closed Φ-algebras.

Can each uniformly closed Φ-algebra be embedded in another uniformly closed Φ-algebra which is closed under inversion, without changing the space of maximal ideals?

We shall give an algebraic condition on the Φ-algebra equivalent to the existence of such an embedding, and discuss the possible embeddings when one exists; we then give an example of a Φ-algebra for which there is no such embedding. This example, as well as one preliminary to it, are also examples of Φ-algebras closed under "countable composition" but not under inversion. We know of no others in the literature.

There is a similar embedding problem which we mention: as above, with "inversion" replaced by "countable composition". Eleanor Aron has shown that such embeddings always exist. We include a brief discussion in Section 4.

A familiarity with the basic ideas about rings of continuous functions, as in [3], will be assumed. We refer the reader to [3] for unexplained terminology.

1. Background on Φ-algebras. We summarize some information, mostly from [4].

A *Φ-algebra* is a real archimedean lattice-ordered algebra with a positive identity element which is a weak order unit (equivalently, a real archimedean f-algebra with identity). If A is a Φ-algebra, then $\mathfrak{M}(A)$ denotes the compact Hausdorff space obtained by equipping the collection of maximal absolutely convex ring ideals with the Stone (= hull-kernel) topology. The subspace of real ideals $\mathfrak{R}(A)$ consists of those $I \in \mathfrak{M}(A)$ for which A/I is isomorphic to the real field $\boldsymbol{R}$.

Let $\gamma\boldsymbol{R}$ denote the two-point compactification of $\boldsymbol{R}$. For X a compact Hausdorff space, let $D(X)$ denote the set of continuous $f: X \to \gamma\boldsymbol{R}$ for which $\mathfrak{R}(f) = \{x \in X : f(x) \in \boldsymbol{R}\}$ is dense. Let $f, g, h \in D(X)$. By definition, $f = g+h$ if $f(x) = g(x)+h(x)$ for all $x \in \mathfrak{R}(g) \cap \mathfrak{R}(h)$; "$f = gh$" is defined similarly. While the sum and product of $g, h \in D(X)$ need not exist in $D(X)$, $g \vee h$, $g \wedge h$ and rg ($r \in \boldsymbol{R}$) (with pointwise definitions) always exist in $D(X)$. A subset of $D(X)$ which becomes a Φ-algebra when equipped with these operations will be called a *sub-Φ-algebra* of $D(X)$.

A Φ-algebra A is isomorphic (as an algebra and lattice) to a sub-Φ-algebra A' of $D(\mathfrak{M}(A))$ such that $\mathfrak{M}(A)$ has the weak topology generated by A', and $\mathfrak{R}(A) = \bigcap \{\mathfrak{R}(f) \colon f \in A'\}$; the isomorphism carries the copy of $\boldsymbol{R}$ contained in A onto the constant functions in $D(\mathfrak{M}(A))$ ([4], 2.3). This representation of A will be called the *canonical representation*, and usually A and A' will be identified.

Also, there is a uniqueness of representation which will be used frequently to recognize $\mathfrak{M}(A)$ and $\mathfrak{R}(A)$. Suppose that A is represented as a sub-Φ-algebra of $D(X)$ (X compact Hausdorff). If A separates points of X and contains the constant function 1, then $\mathfrak{M}(A) = X$ and $\mathfrak{R}(A) = \bigcap \{\mathfrak{R}(f) : f \in A\}$ ([8], 3.2 and 3.3). Thus, any such representation deserves to be, and will be, called canonical.

Let (f_n) be a sequence of elements of the Φ-algebra A; (f_n) is a Cauchy sequence if, for each real $\varepsilon > 0$, there is an n_0 such that $|f_n - f_m| \leqslant \varepsilon$, whenever $n, m \geqslant n_0$; (f_n) converges to $f \in A$ if, for each real $\varepsilon > 0$, there is an n_0 such that $|f_n - f| \leqslant \varepsilon$, whenever $n \geqslant n_0$. The Φ-algebra A is *uniformly closed* if each Cauchy sequence in A converges in A. If A is a uniformly closed Φ-algebra, then it follows from the Stone-Weierstrass theorem that the sub-Φ-algebra A^* of bounded elements in A is $C(\mathfrak{M}(A))$, since A^* is also uniformly closed. Conversely, if A^* is uniformly closed, then so is A ([4], 3.7).

The Φ-algebra A is *closed under inversion* if, whenever $f \in A, f \notin I$ for each $I \in \mathfrak{R}(A)$ (i. e., f is never 0 on $\mathfrak{R}(A)$), and f is not a zero-divisor of A, then $\langle f \rangle = A$, where $\langle f \rangle$ denotes the smallest absolutely convex ideal in A containing f. If A is uniformly closed, then $\langle f \rangle = A$ iff f has an inverse $f^{-1} \in A$([4], 3.3 and 3.4).

2. The condition for embedding. Suppose that A and B are Φ-algebras, and that a is an isomorphism of A into B. If we view B as canonically represented on $\mathfrak{M}(B)$, then $a(A)$ is a representation of A on $\mathfrak{M}(B)$. We write $\mathfrak{M}(B) = \mathfrak{M}(A)$ if $a(A)$ is a canonical representation. Notice that this is defined with respect to a particular embedding of A in B.

We now state precisely the question of the introduction. Let A be a uniformly closed Φ-algebra.

2.1. *Can A be embedded in a uniformly closed Φ-algebra B which is closed under inversion such that $\mathfrak{M}(B) = \mathfrak{M}(A)$?*

Given A, we speak of a B which satisfies 2.1 as a "solution to 2.1". From the remarks above, it is clear that a solution to 2.1 may be regarded as a sub-Φ-algebra of $D(\mathfrak{M}(A))$ which contains A. We will do this frequently in the sequel. The problem of embedding A as in 2.1 thus becomes the problem of "enlarging" A within $D(\mathfrak{M}(A))$ in a certain way. We describe a candidate for such an enlargement.

Let M be the set of non-zero-divisors of A which lie in no real ideal. It is easy to see that f is a zero-divisor of A iff f is 0 on some non-empty open set in $\mathfrak{M}(A)$. Now, clearly, each real ideal of A is of the form $\{f \in A: f(x) = 0\}$, for some $x \in \mathfrak{R}(A)$. Thus, $g \in M$ iff $\operatorname{coz} g$ ($= \{x \in \mathfrak{M}(A) : g(x) \neq 0\}$) is dense, and $\operatorname{coz} g \supset \mathfrak{R}(A)$. M is closed under multiplication, so associated with M is a ring of quotients Q of A, which consists of equivalence classes of quotients f/g, where $f \in A$ and $g \in M$ ([9], p. 46). *Q is isomorphic as a ring to $\bigcup \{C(\operatorname{coz} g) : g \in M \cap A^*\}$ modulo identification of functions which agree on a dense set.* In this family, the operations are defined as follows. If $\hat{f}_1$ and $\hat{f}_2$ are the equivalence classes of $f_i \in C(\operatorname{coz} g_i)$, $i = 1, 2$, then $\hat{f}_1 + \hat{f}_2$ is the equivalence class of the function f defined by $f(x) = f_1(x) + f_2(x)$ for $x \in \operatorname{coz} g_1 \cap \operatorname{coz} g_2 = \operatorname{coz}(g_1 g_2)$; note that $g_1 g_2 \in M \cap A^*$. Products are analogous. The situation is similar to one discussed completely in [2], Chapter 2, so we give just a rough sketch of the isomorphism. The equivalence class f/g in Q is mapped to the class of $f/g \in C(\operatorname{coz}(|g| \wedge 1))$. It is easy to see that the map is well-defined and one-to-one. To find a pre-image for the equivalence class of $f \in C(\operatorname{coz} g)$, we write $f = f_1/f_2$, where $f_1 = f(1 + f^2)^{-1} g$, and $f_2 = (1 + f^2)^{-1} g$. It can be verified that f_1 and f_2 have continuous extensions over $\mathfrak{M}(A)$ into $\gamma \boldsymbol{R}$, which are in A; the equivalence class f_1/f_2 in Q is the desired pre-image.

It is now clear that Q can be made into a Φ-algebra containing A. For example, the supremum of two members of Q is the equivalence class of a function in some $C(\operatorname{coz} g)$, $g \in M \cap A^*$. While Q is closed under inversion, in general $\mathfrak{M}(Q) \neq \mathfrak{M}(A)$. The reason is this: as above, each member of Q can be regarded as a continuous function on a dense subset of $\mathfrak{M}(A)$, but these functions need not be continuously extendible over $\mathfrak{M}(A)$ (into $\gamma \boldsymbol{R}$). We shall see that 2.1 has a solution exactly when such extensions exist, and we now formulate a condition which ensures this.

We shall say that the uniformly closed Φ-algebra A is *closed under bounded quotients* provided that

2.2. *If $f \in A$, $g \in M$, and f/g is a bounded function on $\operatorname{coz} g$, then f/g has a continuous (real-valued) extension over $\mathfrak{M}(A)$.*

When such extensions exist, they lie in A, because $C(\mathfrak{M}(A)) = A^*$ (see Section 1). Thus, it is readily seen that 2.2 is equivalent to the following algebraic condition on A:

2.2′. *If $f \in A$, $g \in M$, and $|f| \leqslant |g|$, then the equation $f = ag$ has a solution $a \in A$.*

Clearly, a uniformly closed Φ-algebra A which is closed under inversion is closed under bounded quotients; the converse fails, as the following simple example shows.

Let $A = \{f \in D(\beta\boldsymbol{R})$: there is a polynomial p on $\boldsymbol{R}$ with $|f| \leqslant |p|\}$; $\mathfrak{M}(A) = \beta\boldsymbol{R}$, and $\mathfrak{R}(A) = \boldsymbol{R}$. Then $\exp(-x^2) \in A$, but $\exp(x^2) \notin A$, so A is not closed under inversion. Consider f, g as in 2.2. Then $\operatorname{coz} g \supset \boldsymbol{R}$, so $\operatorname{coz} g$ is C^*-embedded in $\beta\boldsymbol{R}$ (it means that every bounded continuous function on $\operatorname{coz} g$ has an extension on $\beta\boldsymbol{R}$). Hence f/g extends over $\beta\boldsymbol{R}$. Here there is a solution to 2.1, namely $C(\boldsymbol{R})$. This is, of course, a special case of the following.

2.3. THEOREM. *Let A be a uniformly closed Φ-algebra. The following conditions are equivalent*:

(a) *There is a solution to* 2.1.

(b) *A is closed under bounded quotients.*

(c) *Each dense cozero-set in $\mathfrak{M}(A)$ which contains $\mathfrak{R}(A)$ is C^*-embedded in $\mathfrak{M}(A)$.*

Proof. (a) implies (b). Let B be a solution to 2.1. Then $A^* = C(\mathfrak{M}(A)) = B^*$, because A and B are uniformly closed. Let f, g be as in 2.2. Then $\operatorname{coz} g \supset \mathfrak{R}(A) \supset \mathfrak{R}(B)$, so that $g^{-1} \in B$. Hence $fg^{-1} \in B^* = A^*$.

(b) implies (c). Assume (b). Let S be a dense cozero-set in $\mathfrak{M}(A)$ which contains $\mathfrak{R}(A)$, and let $h \in C^*(S)$. We can write $S = \operatorname{coz} g$, with $g \in M \cap A^*$. Extend $(g \mid S)h$ to $f \in C(\mathfrak{M}(A))$ by assigning the value 0 off S. Apply 2.2 to f and g. The extension of f/g is the desired extension of h.

(c) implies (a). Assume (c). Let $B = \{f \in D(\mathfrak{M}(A)) : \mathfrak{R}(f) \supset \mathfrak{R}(A)\}$. Obviously, $A \subset B$. For $f, g \in B$, $\mathfrak{R}(f)$, $\mathfrak{R}(g)$, and $\mathfrak{R}(f) \cap \mathfrak{R}(g)$ are dense cozero-sets containing $\mathfrak{R}(A)$, and therefore are C^*-embedded in $\mathfrak{M}(A)$. So, for example, $f+g$ considered as a real-valued function on $\mathfrak{R}(f) \cap \mathfrak{R}(g)$ has a continuous extension $h \in D(\mathfrak{M}(A))$ ([3], 6.4). $\mathfrak{R}(h) \supset \mathfrak{R}(A)$, so $h \in B$. This shows sums are in B; the other operations go similarly, and B is a Φ-algebra. B is shown to be closed under inversion in a similar way. The obvious equality $B^* = C(\mathfrak{M}(A))$ shows first that B^*, and hence B, is uniformly closed (see Section 1), and second that the representation of B is canonical. Thus B is a solution to 2.1.

2.4. Remark. The condition in 2.3 (c) is due to Isbell. In [6], 1.17, he essentially observes that this condition holds if A is closed under inversion. The condition in 2.3 (b) is based on a suggestion made to one of us by S. G. Mrówka in another context (see 3.3).

2.5. Remark. The problem of finding an A for which 2.1 has no solution is equivalent to that of finding a compact Hausdorff space X with a filter $\mathfrak{F}$ of dense C^*-embedded cozero-sets such that, for some dense cozero-set $S \supset \bigcap \mathfrak{F}$, S is not C^*-embedded in X. For, if A is such that 2.1 has no solution, take $X = \mathfrak{M}(A)$ and $\mathfrak{F} = \{\mathfrak{R}(f) : f \in A\}$; each member of $\mathfrak{F}$ is C^*-embedded by [4], 3.5. By 2.3 (c), there is an S as above. Conversely, given X, $\mathfrak{F}$, and S as described, let $A = \{f \in D(X) : \mathfrak{R}(f) \in \mathfrak{F}\}$. It is not hard to see that A is a uniformly closed Φ-algebra; S violates 2.3 (c), so 2.1 has no solution.

We continue our discussion of the ring of quotients Q. By $Q \subset D(\mathfrak{M}(A))$ we mean, of course, that each quotient f/g has an extension in $D(\mathfrak{M}(A))$.

2.6. PROPOSITION. 2.1 *has a solution iff* $Q \subset D(\mathfrak{M}(A))$. *When this occurs,* Q *is a solution.*

Proof. If 2.1 has a solution, then 2.3 (c) shows that each quotient has an extension in $D(\mathfrak{M}(A))$ (using [3], 6.4). Conversely, if $Q \subset D(\mathfrak{M}(A))$, then $Q^* = C(\mathfrak{M}(A))$, so Q^* and Q are uniformly closed (see Section 1); thus, Q is a solution to 2.1. $\mathfrak{M}(Q) = \mathfrak{M}(A)$ because the representation of Q is canonical.

Now we shall show that when 2.1 has solutions, Q is naturally distinguished among them by each of the properties: (i) Q is the smallest solution (as a subset of $D(\mathfrak{M}(A))$), (ii) Q is the only solution with the same real ideal space as A.

We shall need the following maximality property:

2.7. LEMMA. *If* B *is a uniformly closed* Φ*-algebra which is closed under inversion, then* $B = \{f \in D(\mathfrak{M}(B)) : \mathfrak{R}(f) \supset \mathfrak{R}(B)\}$.

Proof. Let B' denote the family described. Clearly, $B \subset B'$. For $f \in B'$, let $f^+ = f \vee 0$. Define $g(x) = (f^+(x)+1)^{-1}$ for $x \in \mathfrak{R}(f^+)$, and $g(x) = 0$ if $x \notin \mathfrak{R}(f^+)$. g is continuous because f^+ is. (Recall that $D(\mathfrak{M}(B))$ is a lattice.) Thus $g \in C(\mathfrak{M}(B)) = B^*$. Put $Z(g) = \{x \in \mathfrak{M}(B) : g(x) = 0\}$. Since $Z(g) \cap \cap \mathfrak{R}(f) = \emptyset$, it follows that g is not a zero-divisor of B, and $Z(g) \cap \mathfrak{R}(B) = \emptyset$. So g^{-1}, and hence f^+, are in B. Similarly, $f \wedge 0 = f^- \in B$, so that $f = f^+ + f^- \in B$.

2.8. PROPOSITION. *If there is a solution to* 2.1, *then there is a smallest solution, and this is* Q. $\mathfrak{R}(Q) = \mathfrak{R}(A)$, *and* Q *is the only solution to* 2.1 *with this property*

Proof. Let B_1 be the intersection of all solutions to 2.1. It is trivial that B_1 is a solution, and the smallest solution. We shall show that $\mathfrak{R}(B_1) = \mathfrak{R}(A)$, and that B_1 is the only solution with this property. Since $\mathfrak{R}(Q) = \mathfrak{R}(A)$ (because $\mathfrak{R}(f/g) \supset \mathfrak{R}(f) \cap \operatorname{coz} g \supset \mathfrak{R}(A)$), the proof then will be complete.

If B is any solution, then $B' = \{f \in B : \mathfrak{R}(f) \supset \mathfrak{R}(A)\}$ is a solution also, with $B' \subset B$ and $\mathfrak{R}(B') = \mathfrak{R}(A)$. Therefore, $B_1' = B_1$ and $\mathfrak{R}(B_1) = \mathfrak{R}(A)$. Applying 2.7, $B_1 = \{f \in D(\mathfrak{M}(A)) : \mathfrak{R}(f) \supset \mathfrak{R}(A)\}$. But if B is any solution with $\mathfrak{R}(B) = \mathfrak{R}(A)$, then 2.7 yields exactly the same description of B.

We conclude this section with two comments. (i) Solutions to 2.1 need not be unique. Let $A = C(\beta \boldsymbol{R})$. Since $\mathfrak{R}(A) = \beta \boldsymbol{R} = \mathfrak{M}(A)$, A is closed under inversion and is itself a solution to 2.1. Let Y be any space with $\boldsymbol{R} \subset Y \subset \beta\boldsymbol{R}$. Then $\mathfrak{M}(C(Y)) = \beta\boldsymbol{R}$, and $C(Y)$ is a solution to 2.1. There are infinitely many such $C(Y)'$s. (ii) There can be solutions to 2.1, but no largest solution. An example is presented in 3.6.

3. The examples. In 3.5, we give an example of a uniformly closed Φ-algebra A for which 2.1 has no solution. The example is rather well-behaved in other respects. It is a Φ-algebra of real-valued functions ([4], 4) — i. e., $\mathfrak{R}(A)$ is dense in $\mathfrak{M}(A)$, so that A may be viewed as a sub-Φ-algebra of $C(\mathfrak{R}(A))$. And A is closed under countable composition; we explain what this means.

Let $\boldsymbol{R}^\infty$ denote the product of countably many real lines. The Φ-algebra A is *closed under countable composition* if whenever $g \in C(\boldsymbol{R}^\infty)$ and (f_n) is a sequence from A, there is $h \in A$ such that

$$h(x) = g(f_1(x), f_2(x), \ldots) \quad \text{for } x \in \bigcap_{n=1}^{\infty} \mathfrak{R}(f_n).$$

Since $\bigcap_{n=1}^{\infty} \mathfrak{R}(f_n)$ is dense (by the Baire Category Theorem), such an h is unique.

This definition is from [5]. In [5], 2.1, it is shown that if A is closed under countable composition, then A is uniformly closed and "almost closed under inversion". We know of no previous example of a Φ-algebra closed under countable composition but not inversion. 3.5 must be such an example, of course, since it is not even closed under bounded quotients. We also give a simpler example of this (3.4), which is preliminary to 3.5.

Actually, it is quite natural that the example for which 2.1 has no solution be closed under countable composition, for the following reason. One can (and we did) consider the problem obtained by replacing in 2.1 the word "inversion" by "countable composition". Eleanor Aron has shown recently that, for each uniformly closed A, this problem has a solution (see 4.2). Thus, any A for which 2.1 has no solution can be "enlarged" to one closed under countable composition for which 2.1 has no solution.

Prior to the examples, we prove two clarifying theorems. The first is due to Isbell, and shows how Φ-algebras closed under countable compositon arise naturally. The result was essentially formulated in [6], 1.30, and was completed using a result of Corson and Isbell as is discussed in an oblique way in [5], 2.6.

Let X be a completely regular Hausdorff space, and X_0 a subspace. Let $C(X) \mid X_0$ denote the Φ-algebra of restrictions $f \mid X_0$, where $f \in C(X)$. If $A = C(X) \mid X_0$, then clearly, $A^* = C(\mathrm{cl}_{\beta X} X_0) \mid X_0$; this shows $\mathfrak{M}(A) = \mathrm{cl}_{\beta X} X_0$. Also, $\mathfrak{R}(A) = \mathrm{cl}_{\varepsilon X} X_0$, εX denoting the Hewitt real compactification of X. To see this, first note that $\mathrm{cl}_{\varepsilon X} X_0 = \varepsilon X \cap \mathrm{cl}_{\beta X} X_0$. Now,

$x \in \beta X \sim \varepsilon X$ iff there is $f \in C(X)$ whose extension $f^*: \beta X \to \gamma \boldsymbol{R}$ satisfies $f^*(x) = +\infty$ ([3], 8B). For $x \in \mathfrak{M}(A)$, this is equivalent to $x \notin \mathfrak{R}(A)$.

3.1. THEOREM (Isbell). *For a Φ-algebra A, the following are equivalent:*

(a) *A is a Φ-algebra of real-valued functions which is closed under countable composition.*

(b) *A is isomorphic to some $C(X) \mid X_0$.*

(c) *A is isomorphic to some $C(X) \mid X_0$, with X realcompact and X_0 closed.*

Proof. (a) implies (b). In the usual way, embed $\mathfrak{R}(A)$ into the product X of copies of $\boldsymbol{R}$ indexed by A, and let X_0 be the image. The details follow [5], 2.6.

(b) implies (c). This follows from the natural isomorphism between $C(X) \mid X_0$ and $C(\varepsilon X) \mid \mathrm{cl}_{\varepsilon X} X_0$.

(c) implies (a). Trivial.

We now show exactly where to look for a $C(X) \mid X_0$ for which 2.1 has no solution. The subspace X_0 of X is *z-embedded* in X if each zero-set of X_0 is the intersection of X_0 and a zero-set of X.

3.2. THEOREM. *Let X_0 be a closed subspace of the realcompact space X.*

(a) *X_0 is C^*-embedded in X iff X_0 is z-embedded in X and $C(X) \mid X_0$ is closed under bounded quotients.*

(b) *X_0 is C-embedded in X iff X_0 is z-embedded in X and $C(X) \mid X_0$ is closed under inversion.*

Proof. In each case, the necessity of the conditions is clear. We turn to the converses.

(a) We shall use ([3], 1.17): X_0 is C^*-embedded in X iff disjoint zero-sets of X_0 are completely separated in X. Suppose that X_0 is z-embedded in X, and $C(X) \mid X_0$ is closed under bounded quotients. If Z_1 and Z_2 are disjoint zero-sets of X_0, choose $f_1, f_2 \in C(X)$ with $Z(f_i) \cap X_0 = Z_i$ for $i = 1, 2$. Since $X_0 \cap Z(f_1^2 + f_2^2) = \emptyset$, there exists $g \in C(X)$ such that $g(x) = f_1^2(x)\big(f_1^2(x) + f_2^2(x)\big)^{-1}$ for $x \in X_0$; clearly then, g completely separates Z_1 and Z_2. So X_0 is C^*-embedded. (This argument follows [4], 5.2.)

(b) We shall use ([3], 1.18): If X_0 is C^*-embedded in X, then X_0 is C-embedded in X iff X_0 is completely separated from every disjoint zero-set of X. Suppose that X_0 is z-embedded in X, and $C(X) \mid X_0$ is closed under inversion. By (a), X_0 is C^*-embedded in X. Let $f \in C(X)$ with $Z(f) \cap X_0 = \emptyset$. Then there exists $g \in C(X)$ such that $g(x) = f(x)^{-1}$ for $x \in X_0$; clearly, fg completely separates X_0 and $Z(f)$.

3.3. Remark. 3.2 (a) was pointed out by Mrówka, 3.2 also appears in his paper [7]. Another version of 3.2, and some applications, are in [1].

3.4. Example[1]. *A realcompact space X with a closed subspace X_0 which is C^*-embedded but not C-embedded.*

While this pair (X, X_0) is principally for use in the further construction of an A for which 2.1 has no solution, the Φ-algebra $C(X) \mid X_0$ has some interesting properties. It is a simple example of a Φ-algebra of real-valued functions which is closed under countable composition but not inversion (X_0 is z-embedded but not C-embedded, so 3.2 (b) applies). And, while we easily saw in Section 2 that a Φ-algebra closed under bounded quotients need not be closed under inversion, it is not so obvious that there is a $C(X) \mid X_0$ with this property, that is, that closure under bounded quotients and countable composition does not imply closure under inversion.

We begin with the space Π of [3], 6 Q, which is constructed briefly as follows. A discrete subset D of cardinal $2^{\aleph_0}$ is constructed in $\beta\boldsymbol{N} \sim \boldsymbol{N}$ in such a way that there is a continuous one-to-one mapping of $\boldsymbol{N} \cup D$ onto $\boldsymbol{R}$; let $\Pi = \boldsymbol{N} \cup D$. By [3], 8.18, Π is hereditarily realcompact.

Let $X = D^* \times \boldsymbol{N}^* \sim \{(\infty, \infty)\}$ (the asterisk denoting one-point compactification), and define $X_0 = \{(\infty, n) : n \in \boldsymbol{N}\}$ (the right-hand edge) and $X_1 = \{(p, \infty) : p \in D\}$ (the top edge). Topologize X by means of the following enlargement of the product topology. For $p \in D$ and $E \subset \boldsymbol{N}$ with $p \in \mathrm{cl}_{\beta\boldsymbol{N}} E$, define $V_p(E)$ to be $\{(p, \infty)\} \cup (\{p\} \times E)$; then let the family $\{V_p(E) : E \subset \boldsymbol{N}$ and $p \in \mathrm{cl}_{\beta\boldsymbol{N}} E\}$ serve as a neighborhood base for the point $(p, \infty) \in X_1$ in the enlarged topology (i. e., the set $\{p\} \times \boldsymbol{N}^*$ is given the topology of the subspace $\boldsymbol{N} \cup \{p\}$ of $\beta\boldsymbol{N}$). It is easy to check that this gives a completely regular topology on X and that X_0 is closed and C^*-embedded.

To show that X_0 is not C-embedded in X, we note that any $f \in C(X)$ satisfies the condition that $f(p, \cdot) = f(\infty, \cdot)$ for all but a countable number of $p \in D$. This is clear, since $\{p \in D^* : f(p, n) = f(\infty, n)$ for all $n \in \boldsymbol{N}\}$ is a G_δ in D^* containing ∞, hence has a countable complement in D^*. Clearly then, the function $g \in C(X_0)$, defined by $g(\infty, n) = n$, cannot be extended continuously over X.

To show that X is realcompact, we consider the mapping θ of X onto Π defined as follows. For $(p, \infty) \in X_1$, let $\theta(p, \infty) = p$, and, for $(p, n) \in X \sim X_1$, let $\theta(p, n) = n$ (θ collapses each horizontal line except X_1 to a point). It is easy to see that θ is continuous; in fact, one can show that Π has the quotient topology induced by θ. Furthermore, $\theta^{\leftarrow}(p)$ is compact for each $p \in \Pi$; therefore, since Π is hereditarily realcompact, X is realcompact by [3], 8.17.

We now present the main example. According to 3.2, if X_0 is a closed z-embedded subspace of the realcompact space X, and X_0 is not C^*-embedded, then $C(X) \mid X_0$ is not closed under bounded quotients.

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[1] We are indebted to Leonard Gillman for helpful advice concerning Example 3.4.

3.5. Example. *A realcompact space X with a closed subspace X_0 which is z-embedded but not C^*-embedded. Hence, for $A = C(X) \mid X_0$, 2.1 has no solution.*

Let (Y, Y_0, Y_1) and (Z, Z_0, Z_1) be two copies of the triple (X, X_0, X_1) of 3.4. We construct a new space X by taking the disjoint union of Y and Z and identifying corresponding points of Y_1 and Z_1 — that is, identifying $(p, \infty) \in Y_1$ with $(p, \infty) \in Z_1$. It is easy to verify that X is completely regular. Let $X_1 = Y_1 = Z_1$ and $X_0 = Y_0 \cup Z_0$, a closed countable discrete subset of X. Since X_0 is countable, it is Lindelöf. According to a theorem of Jerison [4], 5.3, a Lindelöf space is z-embedded in any completely regular superspace.

To show that X_0 is not C^*-embedded, consider the function $g \in C^*(X_0)$ defined to be 0 on Y_0 and 1 on Z_0. If g had a continuous extension over X, then it would have to take on each of the values 0 and 1 on all but a countable number of points of X_1, which is absurd.

To show that X is realcompact, we define the mapping θ of X onto Π just as in 3.4. Again θ is continuous and $\theta^{\leftarrow}(p)$ is compact for each $p \in \Pi$. Hence, by [3], 8.17, X is realcompact.

The following comment is probably in order. There are several examples similar to 3.4 and 3.5 in [3], but in these, either X is not realcompact or X_0 is not closed; see the discussion in [3], 1.18 and 8.21. Both these properties are crucial for our requirements.

3.6. Example. *A uniformly closed Φ-algebra A for which there is a solution to* 2.1, *but no largest solution.*

Let B be the Φ-algebra $C(X) \mid X_0$ of 3.5, and let $A = C(\mathfrak{M}(B))$. For $f \in B$, let $B_f = C(\mathfrak{N}(f))$. B_f is closed under inversion; by [4], 3.5, $\mathfrak{N}(f)$ is C^*-embedded in $\mathfrak{M}(B)$, so $B_f^* = C(\mathfrak{M}(B))$ and $\mathfrak{M}(B_f) = \mathfrak{M}(B) = \mathfrak{M}(A)$. So B_f is a solution to 2.1. View B_f as a subset of $D(\mathfrak{M}(B))$. If there were a largest solution B', then $B' \supset \bigcup \{B_f : f \in B\} \supset B$. But then B' would be a solution to 2.1 *for* B, and none exists.

4. Some remarks.

4.1. We consider briefly the relationships between the following properties that a uniformly closed Φ-algebra might possess: closure under (IN) inversion, (CC) countable composition, and (BQ) bounded quotients. Obviously, (IN) implies (BQ); (IN) implies (CC), from [5], 2.1. No other implications hold. 3.4 shows that (BQ) and (CC) together do not imply (IN). 3.5 shows that (CC) does not imply (BQ). The example immediately after 2.2′ shows that (BQ) does not imply (CC): let A be this Φ-algebra, let $e(x) = x$, and let $f \in C(\boldsymbol{R}) \sim A$; then $e \in A$, but $f \circ e \notin A$, so that A is not even closed under composition with functions on $\boldsymbol{R}$.

4.2. Let X be a compact Hausdorff space, and $U(X) = \{f \in D(X): \mathfrak{N}(f)$ is C^*-embedded in $X\}$. In [8], Section 6, it is shown that $U(X)$

is a uniformly closed canonically represented sub-Φ-algebra of $D(X)$ and contains every such Φ-algebra. In particular, $\mathfrak{M}(U(X)) = X$.

If for every A, 2.1 had a solution, then, for each $U(X)$, 2.1 would have a solution. The maximal property of $U(X)$ then would imply that each $U(X)$ is closed under inversion. Conversely, given A, if $U(\mathfrak{M}(A))$ is closed under inversion, then $U(\mathfrak{M}(A))$ is a solution to 2.1. It follows that if A is the Φ-algebra of 3.5, then $U(\mathfrak{M}(A))$ is not closed under inversion. (But with A and B as in 3.6, 2.1 has a solution for A, while $U(\mathfrak{M}(A))$ is not closed under inversion.)

Now consider the problem, mentioned before, obtained by replacing in 2.1 the word "inversion" by "countable composition". As above, one shows that, for each A, this problem has a solution iff each $U(X)$ is closed under countable composition. As was noted in the introduction to Section 3, Eleanor Aron has shown that the problem in question always has a solution; hence each $U(X)$ is closed under countable composition, and the argument in [8], 9.5, can be replaced by a reference to [5], 2.7.

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WESLEYAN UNIVERSITY, MIDDLETOWN, CONNECTICUT
ST. MARY'S COLLEGE OF MARYLAND, ST. MARY'S CITY, MARYLAND
AMHERST COLLEGE, AMHERST, MASSACHUSETTS

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