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A HILBERT TRANSFORM FOR NON-HOMOGENEOUS MARTINGALES

BY

J.-A. CHAO (CLEVELAND, OHIO)
AND MITCHELL H. TAIBLESON (SAINT LOUIS, MISSOURI)

Introduction. Let X be a locally compact Hausdorff space with a non-Archimedean metric topology and a martingale structure induced by a family of finite partitions and an associated Borel measure. Examples include: local fields, the ring of integers in a local field, Vilenkin groups, the boundary of a regular q-martingale, and the boundary of a tree induced by a transient random walk on the tree (see [4], [8]–[10], [12]). By a *Hilbert transform* we mean a "singular integral" operator: $f \to \tilde{f}$ which is bounded on L^p, $1 < p < \infty$; on BMO; and on the Hardy space H^1; and which characterizes H^1 in the sense that f is in H^1 if and only if f and $\tilde{f}$ are in L^1 and

$$\|f\|_{H^1} \sim \|f\|_1 + \|\tilde{f}\|_1.$$

H^1 is defined to be a space of functions whose martingale maximal function is in L^1 (see Section 3 below). Such operators have been studied for homogeneous isotropic martingales in a variety of settings (see [2], [6], [8], [11]). Certain examples for the non-homogeneous isotropic case were implicitly considered in [4] and [3].

In this paper* we consider the non-homogeneous, non-isotropic case. This situation arises naturally in the study of random walks on a non-homogeneous tree. (See [9] for the general case and [13] for a special case.) For simplicity of the exposition we assume that X is compact. The extension to the non-compact case is both obvious and easy.

1. Martingale structure on X. For $k = 0, 1, 2, \ldots$ and $x \in X$ there is a unique set I_k^x, called the *interval of level k*, that contains x, $x \in I_k^x \subset X$; $I_0^x = X$ for all x. For k fixed there are a finite number of intervals of level k that form a partition of X, the intervals of level $k+1$ are subsets of intervals of level k. The collection of all intervals forms a base for a compact topology on X. We assume that there is a Borel measure ν on X, $\nu(X) = 1$. For each interval I_{k-1}^x, $k = 1, 2, 3, \ldots$, there is a partition of it: $\{I_k^{x_i}\}$, $i = 1, \ldots, q$, $q \geq 2$, $q \equiv q(x, k)$, $x_i \equiv x_i(x, k)$.

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Think of the $\{x_i\}$ as representatives of the members of the partition of I_{k-1}^x into intervals of level k. Furthermore, it is convenient to think of the $\{x_i\}$ as fixed.

A metric is defined on X by

$$\Delta(x, y) = \inf\{\nu(I_k^x)\colon\ y \in I_k^x\};$$

(X, Δ) is a non-Archimedean metric space.

For $x \in X$ and $k = 1, 2, 3, \ldots$, let

$$p_i \equiv p_i(x, k) = \nu(I_k^{x_i})/\nu(I_{k-1}^x).$$

We assume that $p_i(x, k) \geqslant \delta > 0$ for all x, k, and i, where δ is a fixed positive constant. Note that

$$\sum_i p_i(x, k) = 1 \qquad \text{for all } x \text{ and } k.$$

Let $\mathscr{F}_k$ be the σ-algebra generated by the intervals of level k. We say that $f = \{f_k\}$ is a *martingale* if f_k is $\mathscr{F}_k$-measurable for all k and $\mathrm{E}[f_n|\mathscr{F}_k] = f_k$ when $n \geqslant k$. This last property reduces to

$$\int_{I_k^x} f(\xi, n)d\nu(\xi) = \int_{I_k^x} f(\xi, k)d\nu(\xi)$$

if $n \geqslant k$. This, in turn, is the same as requiring that

$$f(x, k-1) = \sum_{i=1}^{q} p_i f(x_i, k)$$

for each (x, k). We often write $f(x, k)$ for $f_k(x)$.

EXAMPLES. 1. Suppose g is an integrable function. Then

$$\big(1/\nu(I_k^x)\big) \int_{I_k^x} g d\nu$$

is a martingale and we say that $f \in L^1$ and that $f = g$.

2. If μ is a complex Borel measure, then $f(x, k) = \mu(I_k^x)/\nu(I_k^x)$ is a martingale and we say that $f = \mu$.

3. Suppose h is a distribution, that is, a finitely additive functional on the space of test functions: the linear span of characteristic functions of intervals. Then

$$f(x, k) = h(\chi_{I_k^x})/\nu(I_k^x)$$

is a martingale and we say that $f = h$. In fact, the class of martingales is in one-to-one correspondence with the space of distributions: the martingale property of $f(\cdot, k)$ implies that the linearization of $h(\chi_{I_k^x}) = \nu(I_k^x)f(x, k)$ is a distribution.

The *martingale difference sequence* of $f = \{f_k\}$ is $\{df(x, k)\}$, where

$$df(x, k) = f(x, k) - f(x, k-1).$$

(By convention, $f(x, -1) \equiv 0$.) Since f is a martingale,

$$\sum_i p_i df(x_i, k) = 0.$$

Let

$$d_k f(x) = (df(x_1, k), \ldots, df(x_q, k))$$

and $p \equiv p(x, k) = (p_1, \ldots, p_q)$. These are vectors in C^q.

NOTATION. If $u, v \in C^q$, let

$$u \# v = (u_1 v_1, \ldots, u_q v_q).$$

Let $U \equiv U^q$ denote the subspace of C^q of vectors whose components add to zero.

If f is a martingale, then

$$p(x, k) \# d_k f(x) \in U^{q(x,k)}.$$

Conversely, if $\{f(\cdot, k)\}$ is a sequence of functions and $f(\cdot, k)$ is constant on intervals of level k for each k, and

$$p(x, k) \# d_k f(x) \in U^{q(x,k)} \quad \text{for all } x \text{ and } k,$$

then $\{f(\cdot, k)\}$ is a martingale.

2. Construction of singular integral transforms on X. We construct a variant of the singular integral transforms introduced by Janson [8]. For each (x, k) we select a matrix $M \equiv M(x, k)$ such that $M = (m_{ij})$ is a $(q \times q)$-matrix that sends U into U, and has Euclidean norm 1. If $f = \{f(\cdot, k)\}$ is a martingale, we define another martingale $\tilde{f} = \{\tilde{f}(\cdot, k)\}$ as follows: $\tilde{f}(x, 0) \equiv 0$. For $k \geqslant 1$ let

$$p \# d_k \tilde{f}(x) = M(p \# d_k f(x));$$

that is,

$$d\tilde{f}(x_i, k) = \sum_{j=1}^{q} m_{ij} \frac{p_j}{p_i} df(x_j, k).$$

Then set

$$\tilde{f}(x, k) = \sum_{n=1}^{k} d_n \tilde{f}(x).$$

Since $p \# d_k \tilde{f} \in U$ and $\tilde{f}(\cdot, k)$ is constant on intervals of level k, we see that $\{\tilde{f}(\cdot, k)\}$ is a martingale.

From [8], Section 3, we see that $f \to \tilde{f}$ is bounded on L^p, $1 < p < \infty$; on BMO; and on H^1. Janson's proof uses $p_i \equiv 1/q(x, k)$, but the extension to the non-isotropic case is trivial. Observe that

$$\min\{p_i(x, k)\} \leqslant 1/q(x, k),$$

so $q(x, k) \leqslant 1/\delta$ for all (x, k); so the q's are bounded.

In the following sections we show that if the matrices M are properly chosen, then $f \to \tilde{f}$ characterizes H^1. For now we show how to realize $f \to \tilde{f}$ by a "singular" kernel. We have

$$d\tilde{f}(x_i, k) = \sum_{j=1}^{q} m_{ij}\frac{p_j}{p_i}f(x_j, k) - \left(\sum_{j=1}^{q} m_{ij}\frac{p_j}{p_i}\right)f(x, k-1).$$

CONVENTION. If f is a distribution and $\{f(\cdot, k)\}$ is its associated martingale, we write

$$\int f\varphi dv \equiv f(\varphi)$$

and note that

$$\int f\varphi = \lim_{k\to\infty} \int f(x, k)\varphi(x)dv(x),$$

and that if φ is $\mathscr{F}_n$-measurable, then

$$\int f\varphi dv = \int f(x, n)\varphi(x)dv(x).$$

If f is an integrable function, then $\int f\varphi dv$ can be taken literally.

Let $S_k(x, \xi) = S_{k1}(x, \xi) - S_{k2}(x, \xi)$, where

$$S_{k1}(x_i, \xi) = \sum_{j=1}^{q} m_{ij}\frac{p_j}{p_i}\frac{1}{v(I_k^{x_j})}\chi_{I_k^{x_j}}(\xi) = \frac{1}{v(I_k^{x_i})}\sum_{j=1}^{q} m_{ij}\chi_{I_k^{x_j}}(\xi),$$

$$S_{k2}(x_i, \xi) = \left(\sum_{j=1}^{q} m_{ij}p_j\right)\frac{1}{p_i}\frac{1}{v(I_{k-1}^{x})}\chi_{I_{k-1}^{x}}(\xi) = \frac{1}{v(I_k^{x_i})}\left(\sum_{j=1}^{q} m_{ij}p_j\right)\chi_{I_{k-1}^{x}}(\xi).$$

We see that $d_k\tilde{f}(x_i) = \int f S_k(x_i, \cdot)dv$, $S_k(x, \cdot)$ is supported on I_{k-1}^x, and

$$S_k(x, \xi) \leqslant 2/\delta v(I_k^x).$$

It follows that

$$\tilde{f}(x, k) = \int \sum_{n=0}^{k} S_n(x, \cdot)f dv.$$

In the special case $p_i(x, k) \equiv 1/q(x, k)$ and $q(x, k)$ odd for all (x, k), M can be chosen so that its row sums are zero and its diagonal terms are zero. In this case $S_{k2} \equiv 0$ and $S_{k1}(x_i, k)$ is supported on $I_{k-1}^x \backslash I_k^{x_i}$, and the kernel takes on a more classical appearance.

3. Characterization of the Hardy space H^1. We have assumed that $p_i(x, k) \geqslant \delta$ for δ a fixed positive constant, and therefore the $\{q(x, k)\}$ are bounded from above. We have also assumed that the matrices $M(x, k)$ map $U^{q(x,k)}$ into itself. We now make the key assumption that $M(x, k)$ does not have any real eigenvectors in $U^{q(x,k)}$. We also assume that there are only a finite number of different matrices M for each q. We make the temporary assumption that $q \geqslant 3$. In Section 4 we will prove the following

PROPOSITION. *There exists an* s_0, $0 < s_0 < 1$, *such that if* $s \geqslant s_0$, *then* $|(f(x, k), \tilde{f}(x, k))|^s$ *is a submartingale. That is,*

$$(1) \qquad |(f(x, k-1), \tilde{f}(x, k-1))|^s \leqslant \sum_{i=1}^{q} p_i |(f(x_i, k), \tilde{f}(x_i, k))|^s$$

for all $x \in X$, $k = 1, 2, 3, \ldots$

At this point, standard machinery takes over. If

$$\sup_k \int |f(x, k)| \, dv(x) = A < \infty,$$

we say that f is L^1-bounded and that its L^1-bound is A. If f and $\tilde{f}$ are in L^1, then they are both L^1-bounded,

$$\sup_k \int |f(x, k)| \, dv(x) = \|f\|_1 \quad \text{and} \quad \sup_k \int |\tilde{f}(x, k)| \, dv(x) = \|\tilde{f}\|_1.$$

Since $|(f(x, k), \tilde{f}(x, k))|^s$ is a submartingale, we can construct a majorant g in L^1 of $(f(x, k), \tilde{f}(x, k))$ with $\|g\|_1 \sim \|f\|_1 + \|\tilde{f}\|_1$. If we set

$$f^*(x) = \sup_k |f(x, k)|,$$

it follows that $f^* \in L^1$ and $\|f\|_1 \sim \|f\|_1 + \|\tilde{f}\|_1$. (See [1], [8], [12] for specific instances of this argument.) Recall the definition of H^1 for martingales:

DEFINITION. $H^1 = \{f \colon f^* \in L^1\}$, $\|f\|_{H^1} = \|f^*\|_1$.

See [9] where the theory of such spaces in the setting of non-homogeneous martingales is worked out.

As we pointed out in Section 2, if $f \in H^1$, then

$$\tilde{f} \in H^1 \subset L^1 \quad \text{and} \quad \|\tilde{f}\|_1 \leqslant \|\tilde{f}\|_{H^1} \leqslant C \|f^*\|_1 = C \|f\|_{H^1}.$$

Consequently,

THEOREM. *A function f is in H^1 if and only if f and $\tilde{f}$ are in L^1. Furthermore*

$$\|f\|_{H^1} \sim \|f\|_1 + \|\tilde{f}\|_1.$$

In Section 5 we show how to get the extension to the case $q(x, k) \geqslant 2$.

4. Proof of the Main Lemma. We want to show that there is an s, $0 < s < 1$, such that $|(f(\cdot, k), \tilde{f}(\cdot, k))|^s$ is a submartingale; which is to say that (1) is satisfied for an s independent of $x \in X$ and $k = 1, 2, 3, \ldots$

Let

$$a_0 = f(x, k-1), \quad a_1 = \tilde{f}(x, k-1), \quad a = (a_0, a_1);$$

$$v_{0i} = df(x_i, k), \quad v_{1i} = d\tilde{f}(x_i, k),$$

$$v_0 = (v_{01}, \ldots, v_{0q}), \quad v_1 = (v_{11}, \ldots, v_{1q}), \quad \text{and} \quad v_{.i} = (v_{0i}, v_{1i}).$$

We have $p \# v_1 = M(p \# v_0)$, $p \# v_0 \in U$, $|M| = 1$, M sends U into U, $\sum_i p_i = 1$, and $p_i \geqslant \delta$. Note that

$$(f(x, k-1), \tilde{f}(x, k-1)) = \sum_i p_i (f(x, k-1) + df(x_i, k), \tilde{f}(x, k-1) + d\tilde{f}(x_i, k)),$$

which is to say that

$$a = \sum_{i=1}^{q} p_i (a + v_{.i}). \tag{2}$$

In terms of this notation we want to show that

$$|a|^s \leqslant \sum_{i=1}^{q} p_i |a + v_{.i}|^s. \tag{3}$$

Since there are only a finite number of different matrices M, the Proposition will follow from the following result which is a generalization of a result of Janson [8], which was a generalization of a result of the authors [6], [1].

LEMMA. *Suppose M is a $(q \times q)$-matrix that maps U into U, M has no real eigenvectors in U, and M has Euclidean norm 1. Suppose $p = (p_1, \ldots, p_q)$ is a probability vector and $p_i \geqslant \delta > 0$, $i = 1, 2, 3, \ldots, q$; $a = (a_0, a_1) \in C^2$, $p \# v_1 = M(p \# v_0)$. Then there is a constant $s_0 \equiv s_0(M, \delta)$ that depends only on M and δ, $0 < s_0 < 1$, such that (3) is satisfied whenever $s \geqslant s_0$.*

Proof. Since (2) is satisfied, (3) is valid for all $s \geqslant 1$, so we may assume that $0 < s < 1$. Since (3) is trivial if $a = 0$, we may assume that $a \neq 0$. Let $u_0 = p \# v_0$ and $u_1 = p \# v_1$. We recall that $u_0, u_1 \in U$. Fix a probability vector p.

We evaluate the right-hand side of (3):

$$\begin{aligned}
\sum_i p_i |a + v_{.i}|^s &= \sum_i p_i |(a_0 + v_{0i}, a_1 + v_{1i})|^s \\
&= \sum_i p_i (|a_0|^2 + 2\Re \bar{a}_0 v_{0i} + |v_{0i}|^2 + |a_1|^2 + 2\Re \bar{a}_1 v_{1i} + |v_{1i}|^2)^{s/2} \\
&= \sum_i p_i (|a|^2 + \sum_j 2\Re \bar{a}_j v_{ji} + |v_{.i}|^2)^{s/2} \\
&= |a|^s \sum_i p_i \left\{ 1 + \frac{2\Re(\sum_j \bar{a}_j v_{ji}) + |v_{.i}|^2}{|a|^2} \right\}^{s/2} \\
&= |a|^s \sum_i p_i \left\{ 1 + \frac{s}{2} \frac{2\Re(\sum_j \bar{a}_j v_{ji})}{|a|^2} + \frac{s}{2} \frac{|v_{.i}|^2}{|a|^2} \right. \\
&\qquad \left. + \frac{1}{2} \frac{s}{2} \left(\frac{s}{2} - 1 \right) \left(\frac{2\Re \sum_j \bar{a}_j v_{ji}}{|a|^2} \right)^2 + O\left(\left(\frac{|v_0|}{|a|} \right)^3 \right) \right\}
\end{aligned} \tag{4}$$

$$= |a|^s\left\{1+\frac{s}{2}\sum_i p_i|v_{.i}|^2/|a|^2\right.$$

$$\left.+\frac{s}{2}(s-2)\sum_i p_i\left(\Re\sum_i \bar{a}_j v_{ji}/|a|^2\right)^2+O\left(\left(\frac{|v_0|}{|a|}\right)^3\right)\right\},$$

where the "big oh" is uniform in s and p. Furthermore,

$$\sum_i p_i(\Re\sum_j \bar{a}_j v_{ji})^2 \leqslant \sum_i p_i\left|\sum_j \bar{a}_j v_{ji}\right|^2 = \sum_i p_i \sum_{jk} \bar{a}_j v_{ji} a_k \bar{v}_{ki} \tag{5}$$

$$= \sum_{ijk} p_i^{1/2}(\bar{a}_j\bar{v}_{ki})(p_i^{1/2}a_k v_{ji}) \leqslant \sum_{ijk} p_i|a_j|^2|v_{ki}|^2$$

$$= \sum_j |a_j|^2 \sum_{ik} p_i|v_{ki}|^2 = |a|^2\sum_i p_i|v_{.i}|^2.$$

Let

$$K_{1p} = \{(a, v_0)\in C^{2+q}: |a| = 1,\ u_0\in U,\ u_1 = Mu_0,\ \sum_i p_i|v_{.i}|^2 = 1\}.$$

Since $p_j \geqslant \delta$, we have $|v_{ji}| \leqslant 1/\sqrt{\delta}$ for all i, j; so K_{1p} is compact. Equation (5) reduces to

$$\sum_i p_i(\Re\sum_j \bar{a}_j v_{ji})^2 \leqslant 1 \quad \text{if } (a, v_0)\in K_{1p}. \tag{6}$$

If there is equality in (6), then we have

$$\sum_i p_i(\Re\sum_j \bar{a}_j v_{ji})^2 = 1. \tag{7}$$

If (7) is satisfied, then the derivation of (6) shows that $\sum_j \bar{a}_j v_{ji}$ is real for all i and that $a_j v_{ki} = \lambda a_k v_{ji}$ for all i, j, k, where λ is non-negative. Then

$$a_j v_{ki} = \lambda a_k v_{ji} = \lambda^2 a_j v_{ki} \quad \text{for all } i, j, k.$$

At least one of the products is non-zero (else by (5) the left-hand side of (7) is zero and this contradicts (7)). Thus $\lambda = 1$, and so $a_j v_{ki} = a_k v_{ji}$ for all i, j, k and, consequently, we have $a_0 u_1 = a_1 u_0$. We know that $a_0 \neq 0$; for if not, then $a_1 \neq 0$, and so $u_0 = 0$, and hence $u_1 = 0$. Consequently, v_0 and v_1 are both zero, which implies that the left-hand side of (7) is zero, and that is not possible.

Now from $a_0 u_1 = a_1 u_0$ and $a_0 \neq 0$ we see that

$$Mu_0 = u_1 = (a_1/a_0)u_0.$$

Thus u_0 is an eigenvector of M, and therefore so is u_0/a_0. However,

$$u_0/a_0 = (\sum_j \bar{a}_j a_j)u_0/a_0 = \sum_j \bar{a}_j u_j \in \boldsymbol{R}^q.$$

That is, u_0/a_0 is a real eigenvector of M, and that is a contradiction.

Therefore we never have equality in (6).

Let

$$\alpha(p) = \sup\left\{\sum_i p_i(\Re\sum_j \bar{a}_j v_{ji})^2:\ (a, v_0)\in K_{1p}\right\} \leqslant 1.$$

We claim that $\alpha(p) < 1$, for if $\alpha(p) = 1$, then, since K_{1p} is compact, there is a pair $(a, v_0)\in K_{1p}$ for which (7) is satisfied, which leads to a contradiction, as above.

We now claim that $\alpha = \sup\{\alpha(p):\ p_i \geqslant \delta\} < 1$. Let

$$K_1 = \{(a, v_0, p):\ (a, v_0)\in K_{1p},\ p_i \geqslant \delta\}.$$

K_1 is a compact subset of C^{2q+2}. Since K_{1p} is compact, there is a pair $(a, v_0)\in K_{1p}$ for which

$$\sum_i (\Re\sum_j \bar{a}_j v_{ji})^2 = \alpha(p).$$

If $\alpha = 1$, there is a triple $(a, v_0, p)\in K_1$ such that (7) is satisfied and this is, once again, a contradiction. Thus $\alpha < 1$. By homogeneity,

$$\sum_i p_i(\Re\sum_j \bar{a}_j v_{ji})^2 \leqslant \alpha|a|^2\sum_i p_i|v_{.i}|^2 \tag{8}$$

for all a, v_0, and p, for an α satisfying $0 < \alpha < 1$.

Substituting (8) into (4) we have

$$\sum_i p_i|a+v_{.i}|^s \geqslant |a|^s\left\{1+\frac{s}{2}(1+\alpha(s-2))\sum_i p_i|v_{.i}|^2/|a|^2 + O\left(\left(\frac{|v_0|}{|a|}\right)^3\right)\right\}.$$

Consequently, there is an $\varepsilon > 0$ (independent of s and p) for which

$$|a|^s \leqslant \sum_i p_i|a+v_{.i}|^s \quad \text{if } |v_0| \leqslant \varepsilon|a| \text{ and } s \geqslant \alpha. \tag{9}$$

Now that we have found α and ε let

$$K_{2p} = \{(a, v_0):\ u_0\in U,\ u_1 = Mu_0, \sum_i p_i|a+v_{.i}| = 1,\ |v_0| \geqslant \varepsilon|a|\}.$$

We see that

$$|a| = |\sum_i p_i(a+v_{.i})| \leqslant \sum_i p_i|a+v_{.i}| = 1,$$

and so

$$\sum_i p_i|v_{.i}| \leqslant |a| + \sum_i p_i|a+v_{.i}| \leqslant 2.$$

Consequently, $|v_{.i}| \leqslant 2/\delta$ for all i. It follows that K_{2p} is compact. Let

$$\beta(p) = \sup\{|a|:\ (a, v_0)\in K_{2p}\}.$$

It is clear that $\beta(p) \leqslant 1$. We claim that $\beta(p) < 1$. If not, there is a pair

$(a, v_0) \in K_{2p}$ for which

(10) $$|a| = \sum_i p_i |(a_0 + v_{0i}, a_1 + v_{1i})|.$$

Since

$$|a| = \sum_i p_i (a_0 + v_{0i}, a_1 + v_{1i}),$$

equality in (10) implies that there are λ_i, $\lambda_i \geqslant 0$, for which $p_i(a_j + v_{ji}) = \lambda_i a_j$ for all i and j. That is,

$$p_i v_{ji} = (\lambda_i - p_i) a_j.$$

If we set $\lambda = (\lambda_1, \ldots, \lambda_q)$, we have $u_j = a_j(\lambda - p)$. We claim that $a_0 \neq 0$. Otherwise $u_0 = 0$, and so $v_0 = 0$, which contradicts $|v_0| \geqslant \varepsilon |a| \neq 0$. Thus $u_0/a_0 = \lambda - p$. Notice that $\lambda - p \neq 0$, else $u_0 = 0$, which leads to a contradiction. We have shown that $\lambda - p$ is real and non-zero,

$$M(\lambda - p) = M(u_0/a_0) = u_1/a_0 = (a_1/a_0)(\lambda - p),$$

and so $\lambda - p$ is a real eigenvector of M, a contradiction.

Thus $\beta(p) < 1$, $|a| \leqslant \beta(p)$ for all $(a, v_0) \in K_{2p}$.

Now we remove the dependence on p. Since K_{2p} is compact, there is a pair $(a, v_0) \in K_{2p}$ such that $|a| = \beta(p)$. Let

$$K_2 = \{(a, v_0, p)\colon (a, v_0) \in K_{2p}, p_i \geqslant \delta\}.$$

We argue as above. Let

$$\beta = \sup\{\beta(p)\colon p_i \geqslant \delta\}.$$

Note that $\beta \leqslant 1$. Since K_2 is compact, if $\beta = 1$ there is a triple $(a, v_0, p) \in K_2$ such that

$$|a| = \sum_i p_i |a + v_{.i}|.$$

But this is not possible, and so $\beta < 1$. Thus,

(11) $$|a|^s \leqslant \Big(\beta \sum_i p_i |a + v_{.i}|\Big)^s \leqslant \sum_i p_i |a + v_{.i}|^s,$$

provided

$$|v_0| \geqslant \varepsilon |a| \quad \text{and} \quad s \geqslant 1/(1 + \log\beta/\log\delta).$$

The second inequality in (11) follows from the fact that the form $\sum_i p_i |c_i|^s$ is concave if $0 < s < 1$ so that the maximum of $\beta \sum_i p_i |c_i|^s$ occurs at an extreme point; that is, a point where all the $c_i = 0$ except for one term.

It follows from (8) and (11) that

$$|a|^s \leqslant \sum_i p_i |a + v_{.i}|^s \quad \text{if } s \geqslant s_0 = \max\{\alpha, 1/(1 + \log\beta/\log\delta)\}.$$

This completes the proof of the Lemma.

Thus the Theorem has been proved provided $q \geqslant 3$.

5. Extension to the case $q \geqslant 2$. There is a simple approach that works if some of the $q(x, k)$ are equal to 2. It is based on an idea introduced in [2] to deal with dyadic martingales. Let

$$\mathrm{E}f = \{\mathrm{E}f(\cdot, k)\}, \qquad \mathrm{E}f(x, k) = f(x, 2k).$$

Then $\mathrm{E}f$ is a martingale with respect to $\mathscr{F}_{2k}$. We will say that a martingale with respect to $\{\mathscr{F}_k\}$ is a *martingale* while a martingale with respect to $\{\mathscr{F}_{2k}\}$ is an *even martingale.*

Let

$$Mf(x) = \sup\{|f(\xi, k)|:\ \xi \in I^x_{k-1}\}.$$

The function Mf is called the *non-tangential maximal function* of f. From [9], Lemma 2, $f^* \in L^1$ if and only if

$$Mf \in L^1 \qquad \text{and} \qquad \|f^*\|_1 \sim \|Mf\|_1.$$

That is, $\|f\|_{H^1} \sim \|Mf\|_1$. Observe that for the even martingale $\mathrm{E}f$ we have

$$M(\mathrm{E}f)(x) = \sup\{|f(\xi, 2k)|:\ \xi \in I^x_{2k-2}\}.$$

We claim that:

f is L^1-bounded if and only if $\mathrm{E}f$ is L^1-bounded.

$f \in L^1$ if and only if $\mathrm{E}f \in L^1$.

$f \in H^1$ if and only if $\mathrm{E}f \in H^1$.

Furthermore their "norms" in these spaces are equivalent. The direction $f \Rightarrow \mathrm{E}f$ is trivial in each case. For $\mathrm{E}f \Rightarrow f$:

L^1-bounded.

$$f(x, 2k+1) = \sum_i p_i f(x_i, 2k+2), \qquad p_i \equiv p_i(x, 2k+2),$$

so

$$|f(x, 2k+1)| \leqslant \sum_i p_i |f(x_i, 2k+2)|.$$

Thus

$$\begin{aligned}
\int_{I^x_{2k+1}} |f(\xi, 2k+1)|\,d\nu(\xi) &= \nu(I^x_{2k+1})|f(x, 2k+1)| \\
&\leqslant \sum_i p_i \nu(I^x_{2k+1})|f(x_i, 2k+2)| \\
&= \sum_i \int_{I^{x_i}_{2k+2}} |f(\xi, 2k+2)|\,d\nu(\xi) \\
&= \int_{I^x_{2k+1}} |f(\xi, 2k+2)|\,d\nu(\xi).
\end{aligned}$$

L^1. We note that $f \in L^1$ if and only if f is L^1-bounded and some subsequence $\{f(\cdot, k_n)\}$ is Cauchy in L^1; and if $f \in L^1$, then $\|f\|_1$ is equal to the L^1-bound of the martingale f.

H^1.

$$|f(x, 2k+1)| \leqslant \sum_i p_i |f(x_i, 2k+2)|$$

$$\leqslant \max_i \{|f(x_i, 2k+2)|\} \leqslant M(\mathrm{E}f)(x).$$

Thus, $f^* \leqslant M(\mathrm{E}f)(x)$, and so $\|f\|_{H^1} \leqslant C\|\mathrm{E}f\|_{H^1}$.

Since the partitions in $\{\mathscr{F}_k\}$ always have at least two members, the partitions in $\{\mathscr{F}_{2k}\}$ always have at least four. Thus we may construct a Hilbert transform that characterizes H^1 for even martingales: $\mathrm{E}f \to \mathrm{E}\tilde{f}$.

If $f \in H^1$, then $f \in L^1$, and so

$$\mathrm{E}f \in L^1 \quad \text{and} \quad \mathrm{E}\tilde{f} \in H^1 \subset L^1.$$

Conversely, if f and $\mathrm{E}\tilde{f}$ are in L^1, then $\mathrm{E}f$ and $\mathrm{E}\tilde{f}$ are L^1-bounded. Furthermore, $|(\mathrm{E}f, \mathrm{E}\tilde{f})|^s$ is subharmonic for some s, $0 < s < 1$. Then, as above,

$$\mathrm{E}f \in H^1 \quad \text{and} \quad \|\mathrm{E}f\|_{H^1} \sim \|\mathrm{E}f\|_1 + \|\mathrm{E}\tilde{f}\|_1.$$

Given an even martingale there is a unique way to fill in the odd-numbered steps so that one has an (original) martingale. For $\mathrm{E}\tilde{f}$ call the extension $\tilde{f}$. (Observe that the notation "$\mathrm{E}\tilde{f}$" is consistent.) The transformation $f \to \tilde{f}$ can be realized by a kernel as before. The values of $d_{2k+1}\tilde{f}(x)$ and $d_{2k+2}\tilde{f}(x)$ are determined by matrices acting on $d_{k+1}\mathrm{E}f(x)$. Since

$$\|\mathrm{E}f\|_{H^1} \sim \|f\|_{H^1}, \quad \|\mathrm{E}f\|_1 \sim \|f\|_1 \quad \text{and} \quad \|\mathrm{E}\tilde{f}\|_1 \sim \|\tilde{f}\|_1,$$

we have $\|f\|_{H^1} \sim \|f\| + \|\tilde{f}\|_1$.

This completes the proof. However, if the number of places where $q(x, k) = 2$ is rather sparse, then the argument is rather clumsy. If one wishes, the correction can be applied locally. For example, let $\{I^{x_i}_{2k+1}\}$ be the partition for $\{I^x_{2k}\}$ and let $\{I^{x_{ij}}_{2k+2}\}$ be the partitions for the $\{I^{x_i}_{2k+1}\}$. In case one of the numbers $\{q(x, 2k), \{q(x_i, 2k+1)\}$ is equal to two proceed as in Section 5 (above) at I^x_{2k}, otherwise proceed as in Section 4.

6. Extensions.

6.1. *An F. and M. Riesz theorem.* It is a standard fact that a (regular) martingale is a finite Borel measure if and only if it is L^1-bounded. In our proof we did not use the fact that f and $\tilde{f}$ were in L^1 to get the conclusion that f was in H^1, we only used that they were L^1-bounded. Thus we have established an F. and M. Riesz theorem:

THEOREM. *If f and $\tilde{f}$ are both finite Borel measures, then they are both* *absolutely continuous.*

6.2. *Systems of conjugate operators.* It is occasionally convenient to work with a system of conjugate operators instead of just one conjugate operator. Fix a positive integer t. For each (x, k) we have a system of matrices $\{M_1, \ldots, M_t\}$ which have no real eigenvector in common in $U^{q(x,k)}$, and are otherwise as described in Section 4 or 5. Let $\{T_1, \ldots, T_t\}$ be the system of operators induced by these matrices as in Section 2. Then f is in H^1 if and only if $f \in L^1$ and $Tf_j \in L^1$, $j = 1, 2, \ldots, t$. The proofs above go through with only trivial changes. See [8] and [5] for the homogeneous isotropic case.

6.3. *Harmonic functions.* Suppose T is a non-homogeneous tree and P is a transition operator on T associated with a transient nearest neighbor random walk. (We assume that P is regular in the sense of [9].) Let X be the boundary of T and endow X with the hitting measure induced by the random walk. This gives X a martingale structure in the sense of Section 1. The vertices of the tree are in one-to-one correspondence with the family of intervals $\{I_k^x\}$. The value of k is the geodesic distance of the vertex from the starting place of the random walk.

The main idea of [9] is that there is a one-to-one correspondence between martingales on X and P-harmonic functions on T. (A function g on T is *P-harmonic* if $Pg(u) = g(u)$ for all vertices $u \in T$.) In fact, F is P-harmonic on T if and only if it is the Poisson integral of a martingale on X, where the Poisson integral is defined in [9].

For F a P-harmonic function on T, and f its associated martingale on X, we let $\tilde{F}$ be the P-harmonic function on T associated with $\tilde{f}$, the Hilbert transform of f. $\tilde{F}$ is, in a natural sense, a conjugate harmonic function to F. Using the ideas of this paper together with the constructions of [9] it can be shown that F is the Poisson integral of an $f \in H^1$ if and only if F and $\tilde{F}$ are L^1-bounded. This is worked out in the setting of q-martingales (which is the homogeneous isotropic case) in [7] using the constructions of [13].

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CLEVELAND STATE UNIVERSITY
CLEVELAND, OHIO 44115

WASHINGTON UNIVERSITY
SAINT LOUIS, MISSOURI 63130

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