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COLLECTIONWISE NORMALITY AND FRAGMENTED COLLECTIONWISE NORMALITY

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Smith [5] characterizes collectionwise normality (among normal spaces) as the existence of open refinements of various kinds that one usually associates with paracompactness, although there this existence is demanded only of what he calls weak $\bar{\theta}$-covers and not demanded of all open covers as in the case of paracompactness. He derives all his theorems from a characterization of collectionwise normality he attributes to Zenor [8] (Theorem 1.3 of [5]), and finishes with questions that include one that asks whether we have similarly nice characterizations if normality is not assumed (P 1045). Here, answering P 1045, we give a general characterization of collectionwise normality in the genre of Smith's, in terms of the existence of σ-cushioned open refinements for covers well within the class of weak $\bar{\theta}$-covers, thus subsuming Smith. It turns out that our theorem is equivalent to Zenor's.

From our characterization of collectionwise normality we obtain a factorization of the concept into (ordinary) normality and a component which we are to call *fragmented collectionwise normality* and which does not by itself imply normality, a result that tells, like Bing's [1] Theorem 8 and Traylor's [7] extension of it, by how much a normal Moore space is short of being metrizable, at a time when it is largely (CH, to be precise) resolved that Moore's Conjecture is false (see [2] and [3]).

Definition 1. Given, on a topological space X, a discrete family $\mathscr{C}$ of closed subsets, we can form, if for every $C \in \mathscr{C}$ we put $|C = \bigcup \mathscr{C} \setminus C$, the open cover

$$\{X \setminus (|C) \colon C \in \mathscr{C}\} \cup \{X \setminus \bigcup \mathscr{C}\},$$

to be denoted by $\sim\mathscr{C}$. We say that $\sim\mathscr{C}$ is an open cover *derived* from $\mathscr{C}$.

THEOREM 1. *A topological space X is collectionwise normal if and only if every open cover $\sim\mathscr{C}$ derived from some discrete family $\mathscr{C}$ of closed sets has a σ-cushioned open refinement.*

Proof. Let $\mathscr{C}$ be a discrete family of closed subsets on X. We are to construct a family of disjoint neighbourhoods for the members of $\mathscr{C}$. By hypothesis, there is a σ-cushioned open refinement for $\sim\mathscr{C}$ in which there is a subfamily $\{C_n\colon C\in\mathscr{C},\ n\in N\}$ such that

(i) $C\subset\bigcup_{n\in N} C_n\subset X\setminus(|C)$ for all $C\in\mathscr{C}$,

(ii) $\{C_n\colon C\in\mathscr{C}\}$ is cushioned in $\{X\setminus(|C)\colon C\in\mathscr{C}\}$ for all $n\in N$.

We would have constructed a family of disjoint neighbourhoods for members of $\mathscr{C}$ if we let

$$\tilde{C}_n = C_n\setminus\bigcup_{j\leqslant n}\mathrm{Cl}\bigcup\{D_j\colon D\neq C,\ D\in\mathscr{C}\}\qquad\text{for all } C\in\mathscr{C},\ n\in N,$$

and

$$\hat{C} = \bigcup_{n\in N}\tilde{C}_n\qquad\text{for all } C\in\mathscr{C}.$$

Indeed, $\{\hat{C}\colon C\in\mathscr{C}\}$ is such a family, as the following shows.

Clearly, $C\subset C$ for all $C\in\mathscr{C}$. The family is disjoint because otherwise there is a point $x\in X$ such that $x\in\tilde{E}_m$, $\tilde{F}_n$ for some distinct E, $F\in\mathscr{C}$ and some m, $n\in N$, $m\leqslant n$; even though

$$x\in\tilde{F}_n\Rightarrow x\notin\bigcup\{D_m\colon D\neq F\}\Rightarrow x\notin E_m\Rightarrow x\notin\tilde{E}_m.$$

Conversely, let X be collectionwise normal. Let $\sim\mathscr{C}$ be an open cover derived from a discrete family $\mathscr{C}$ of closed subsets on X. Assume that for every $C\in\mathscr{C}$ there is an open neighbourhood $\tilde{C}$ such that the family $\{\tilde{C}\colon C\in\mathscr{C}\}$ is disjoint. By the normality of X, there are open sets A and B such that

$$\bigcup\mathscr{C}\subset A\subset\mathrm{Cl}\,A\subset B\subset\mathrm{Cl}\,B\subset\bigcup\{\tilde{C}\colon C\in\mathscr{C}\}.$$

Clearly, $C\subset B\cap\tilde{C}\subset X\setminus(|C)$ for all $C\in\mathscr{C}$, and $X\setminus\mathrm{Cl}\,A\subset X\setminus\bigcup\mathscr{C}$. Moreover, the family $\{B\cap\tilde{C}\colon C\in\mathscr{C}\}\cup\{X\setminus\mathrm{Cl}\,A\}$ covers X and is cushioned in the family $\{X\setminus(|C)\colon C\in\mathscr{C}\}\cup\{X\setminus\bigcup\mathscr{C}\}$.

Remarks. It was learnt well after the submission of this paper that Smith and Telgársky [6] had, at around the same time, come to a result similar to our Theorem 1: A space X is collectionwise normal iff every weak $\bar{\theta}$-cover of X has an order-cushioned open refinement. It is interesting to see how, on the one hand, their result led them to compare collectionwise normality and countable paracompactness in terms of the availability of an order-cushioned refinement; and how, on the other hand, Theorem 1 led us to the notions of *fragmented collectionwise normality*, the factorization of collectionwise normality into a normal and a non-normal components and a simultaneous strengthening of the results of Bing and Traylor.

Of course, our characterization of collectionwise normality offers also a comparison between countable paracompactness and collectionwise normality: A space is collectionwise normal and countably paracompact if and only if every open cover, be it countable or be it derived from some discrete family of closed sets, admits a σ-cushioned open refinement.

Definition 2. A topological space is *fragmented collectionwise normal* if, given any discrete family $\mathscr{C}$ of closed sets,

($*$) for every $C \in \mathscr{C}$ and $n \in N$ there exists a closed set C_n contained in an open neighbourhood $\tilde{C}_n$ such that $\bigcup_{n \in N} C_n = C$ for all $C \in \mathscr{C}$ and such that the family $\{\tilde{C}_n\colon C \in \mathscr{C}\}$, $n \in N$, is disjoint.

Remarks. Trivially, if $|\mathscr{C}| \leqslant \omega$, then ($*$) is always true. Spaces on which every discrete closed family is at most countable are therefore always fragmented collectionwise normal. In particular, the Tychonoff plank is fragmented collectionwise normal, although it is not normal.

THEOREM 2. *A topological space is collectionwise normal if* (*and only if*) *it is fragmented collectionwise normal and normal.*

Proof. Let $\mathscr{C}$ be a discrete family of closed sets on a topological space X. We are to construct a family of disjoint open neighbourhoods for members of $\mathscr{C}$. To this end, it suffices, as in the proof of Theorem 1, to construct a family of open sets $\{C_n^*\colon C \in \mathscr{C},\ n \in N\}$ such that

(i) $C \subset \bigcup_{n \in N} C_n^* \subset X \setminus (|C)$ for all $C \in \mathscr{C}$,

(ii) $\{C_n^*\colon C \in \mathscr{C}\}$ is cushioned in $\{X \setminus (|C)\colon C \in \mathscr{C}\}$ for all $n \in N$.

If X is fragmented collectionwise normal, for every $C \in \mathscr{C}$ and $n \in N$ there is a closed set C_n, contained in an open neighbourhood $\tilde{C}_n$, such that $\bigcup_{n \in N} C_n = C$ for all $C \in \mathscr{C}$, $\tilde{C}_n \subset X \setminus (|C)$ for all $C \in \mathscr{C}$ and $n \in N$, and $\{\tilde{C}_n\colon C \in \mathscr{C}\}$ is disjoint for every $n \in N$. If further X is normal, for every $n \in N$ there is an open set A_n such that

$$\bigcup \{C_n\colon C \in \mathscr{C}\} \subset A_n \subset \operatorname{Cl} A_n \subset \bigcup \{\tilde{C}_n\colon C \in \mathscr{C}\}.$$

Clearly,

(i) $C \subset \bigcup_{n \in N} (A_n \cap \tilde{C}_n) \subset X \setminus (|C)$ for all $C \in \mathscr{C}$,

(ii) $\{A_n \cap \tilde{C}_n\colon C \in \mathscr{C}\}$ is cushioned in $\{X \setminus (|C)\colon C \in \mathscr{C}\}$ for all $n \in N$.

With the concept of fragmented collectionwise normality in place, an old theorem of F. B. Jones is so much more immediate that a generalization of it is in order in Theorem 3 below, especially when one is mindful of the characterizations of developable spaces given in [4].

Definition 3. A topological space X is a *chrysanthemum* if $X = \bigcup \{A_\xi\colon \xi < \alpha\}$ for some ordinal α such that

(i) there is a closed subset $A \subset X$ such that $A_\xi \setminus A$ is open for all $\xi < \alpha$ and $A_\xi \cap A_\eta = A$ for all $\xi \neq \eta$, ξ, $\eta < \alpha$;

(ii) on every A_ξ, no discrete families of closed subsets have cardinality in excess of ω.

Remarks. All hereditarily separable spaces and, in particular, separable metric spaces are chrysanthemums, as are all Lindelöf spaces and all countably compact spaces. Kowalsky's hedgehogs are also chrysanthemums.

THEOREM 3. *All normal Moore chrysanthemums are metrizable; and all metrizable spaces are subspaces of countable products of normal Moore chrysanthemums.*

Maybe it should be remarked that, among developable spaces, indeed, among spaces on which all closed subsets are G_δ-sets, there is fragmented collectionwise normality if, given any discrete family $\mathscr{C}$ of closed sets,

(#) for every $C \in \mathscr{C}$ and $n \in N$ there exists an open set $\tilde{C}_n$ such that $\bigcup_{n \in N} \tilde{C}_n \supset C$ for all $C \in \mathscr{C}$ and such that the family $\{\tilde{C}_n: C \in \mathscr{C}\}$, $n \in N$, is disjoint.

Furthermore, to ensure fragmented collectionwise normality, we need only to demand (#) of discrete families of *nowhere dense* closed sets because non-empty interiors of closed sets, if there are any, can always be culled from them. A simultaneous strengthening of results of Bing and Traylor that normal Moore spaces are metrizable if (and only if) they are either collectionwise normal or screenable on boundaries is therefore in order:

THEOREM 4. *A normal Moore space X is metrizable if (and only if) every discrete family of nowhere dense closed subsets of X can be covered by a σ-disjoint family of open sets such that no members of the latter intersect more than one member of the former.*

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