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ON PREPONDERANT MAXIMA

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Let λ_n (respectively, λ_n^*) stand for Lebesgue (respectively, outer Lebesgue) measure on the Euclidean space E_n. We denote by $B(x, r)$ the closed ball of centre x and radius r. Let $A \subset E_n$ be an arbitrary set. The number

$$\bar{D}_x(A) = \overline{\lim_{h\to 0_+}} \lambda_n^*(A\cap B(x, h))/\lambda_n(B(x, h))$$

is called the *outer upper symmetric density* of A at x.

Let f be an arbitrary (possibly infinite-valued) function on E_n and let $0 < a \leqslant 1$. Following Foran [2] put

$$\bar{M}_a(f) = \{x: \bar{D}_x(\{t: f(t) \geqslant f(x)\}) < a\}.$$

Foran [2] proved that $\bar{M}_a(f)$ is a set of measure zero for $a = 2^{-n}$ and any f. He raised the problem (P 1019) whether $a = 2^{-n}$ can be improved.

It is natural to say that f has a *preponderant maximum* at x if $x \in \bar{M}_{1/2}(f)$ and we will use this terminology. Since for any linear non-constant function f on E_n and $1/2 < a \leqslant 1$ we have obviously $\bar{M}_a(f) = E_n$, the following theorem solves completely the Foran problem.

THEOREM. *Let f be an arbitrary function on E_n. Then the set $\bar{M}_{1/2}(f)$ of all points at which f has a preponderant maximum is of Lebesgue measure zero.*

Proof. Let $u(x)$ be the upper measurable boundary of f defined by Blumberg in [1]. The upper boundary

$$u(x) = \inf\{t: \bar{D}_x(\{y: f(y) > t\}) = 0\}$$

is a measurable (possibly infinite-valued) function. Modifying slightly the proof in [3], p. 504, it is easy to show that $\lambda_n(\bar{M}_{1/2}(u)) = 0$ implies $\lambda_n(\bar{M}_{1/2}(f)) = 0$. Thus it is sufficient to prove the Theorem only for an arbitrary measurable (possibly infinite-valued) function f.

For integers $m > 0$ and $k > 0$ define

$$A_{m,k} = \{x: \lambda_n(\{y: f(y) \geqslant f(x)\} \cap B(x, r)) < (2^{-1} - m^{-1})\lambda_n(B(x, r)) \text{ for any } 0 < r < k^{-1}\}.$$

Since obviously

$$\overline{M}_{1/2}(f) = \bigcup_{m,k=1}^{\infty} A_{m,k},$$

it is sufficient to show that all sets $A_{m,k}$ are of measure zero. Suppose on the contrary that for some m, k we have $\lambda_n^*(A_{m,k}) > 0$. Then we can choose $a \in A_{m,k}$ which is a point of the outer density for $A_{m,k}$. Choose $\varepsilon > 0$ such that

(1) $$(1 - \varepsilon)(1/2 + 1/m) > 1/2.$$

Further choose $\delta > 0$ such that

(2) $$\lambda_n^*(A_{m,k} \cap B(a, \Delta))/\lambda_n(B(a, \Delta)) > 1 - \varepsilon \quad \text{for } 0 < \Delta \leqslant \delta.$$

Finally, it follows from (1) that we can choose $r > 0$ such that

(3) $$r < \min(\delta, 1/k) \quad \text{and} \quad (1 - \varepsilon)(1/2 + 1/m)(\delta^n + (\delta - r)^n) > \delta^n.$$

Define the subset C of $E_{2n} = E_n \times E_n$ as

$$C = \{(x, y): x \in B(a, \delta),\ y \in B(x, r)\}$$

and put $S = C \cap \{(x, y): f(x) > f(y)\}$. The sets C and S are obviously measurable and by the Fubini theorem we have

$$\lambda_{2n}(S) = \int_{B(a,\delta)} g(x)\, d\lambda_n(x),$$

where $g(x) = \lambda_n(\{y \in B(x, r): f(x) > f(y)\})$. Let V_n denote the volume of the unit ball in E_n. By the definition of $A_{m,k}$, for $x \in A_{m,k} \cap B(a, \delta)$ we have $g(x) \geqslant (1/2 + 1/m) V_n r^n$. Therefore, using (2) we obtain

(4) $$\lambda_{2n}(S) \geqslant (1/2 + 1/m)(1 - \varepsilon) V_n^2 \delta^n r^n.$$

Further, put

$$T = \{(x, y): y \in B(a, \delta - r),\ x \in B(y, r),\ f(x) < f(y)\}.$$

Obviously, $T \subset C$, $T \cap S = \emptyset$, and T is measurable. By the Fubini theorem we have

$$\lambda_{2n}(T) = \int_{B(a,\delta - r)} h(y)\, d\lambda_n(y),$$

where $h(y) = \lambda_n\{x \in B(y, r): f(x) < f(y)\}$.

Similarly as above we obtain

$$\lambda_{2n}(T) \geqslant (1/2+1/m)(1-\varepsilon)V_n^2(\delta-r)^n r^n. \tag{5}$$

Using (4) and (5) we get

$$\begin{aligned} V_n^2\delta^n r^n &= \lambda_{2n}(C) \geqslant \lambda_{2n}(S)+\lambda_{2n}(T) \\ &\geqslant V_n^2 r^n(1/2+1/m)(1-\varepsilon)\big(\delta^n+(\delta-r)^n\big), \end{aligned}$$

which contradicts (3).

REFERENCES

[1] H. Blumberg, *The measurable boundaries of an arbitrary function*, Acta Mathematica 65 (1935), p. 263-282.

[2] J. Foran, *On the density maxima of a function*, Colloquium Mathematicum 37 (1977), p. 245-254.

[3] R. J. O'Malley, *Strict essential minima*, Proceedings of the American Mathematical Society 33 (1972), p. 501-504.

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