ON TOTALLY UMBILICAL SUBMANIFOLDS
OF CONFORMALLY BIRECURRENT MANIFOLDS

BY

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0. Introduction. Totally umbilical submanifolds of Riemannian manifolds with certain conditions imposed on the Weyl conformal curvature tensor of an ambient space were investigated by many authors (see [6], [9]–[11], [7], [1]). In this paper we give some results concerned with this subject. First we recall some theorems of Olszak. He has proved that any totally umbilical submanifold M of a conformally recurrent manifold N is also conformally recurrent ([9], Theorems 2 and 3). Moreover, for M we have (see [10])

$$(0.1) \qquad\qquad HC_{abcd} = 0,$$

where $H = g_{rs} H^r H^s$, g_{rs} is the metric tensor of N, H^r is the mean curvature vector field of M, and C_{abcd} is the Weyl conformal curvature tensor of M. In consideration of the above result it is a natural question whether totally umbilical submanifolds of conformally birecurrent manifolds are also conformally birecurrent and satisfy relation (0.1). In the present paper we give some answer to this problem. Namely, we prove the following

THEOREM. *Let M be an analytic, non-conformally flat, totally umbilical submanifold of a conformally birecurrent manifold N. The submanifold M is conformally birecurrent if and only if the relation $H = 0$ holds on M and the tensor field $H^r \tilde{R}_{rstu} H^u B^{st}_{ab}$ is proportional to the metric tensor of M.*

1. Preliminaries. Let N be an n-dimensional Riemannian manifold with a not necessarily definite metric g_{rs}, covered by a system of coordinate neighborhoods $\{U; (x^r)\}$. We denote by $\tilde{\Gamma}^r_{st}$, $\tilde{R}^v_{rst}$, $\tilde{R}_{vt}$ and $\tilde{R}$ the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature of N, respectively. The indices p, q, r, s, t, u, v, w run over the range $\{1, 2, \ldots, n\}$. A tensor $S^{t_1 \cdots t_k}_{u_1 \cdots u_l}$ is *recurrent* [12] (resp. *birecurrent*) if the identity

$$(1.1) \qquad S^{t_1 \cdots t_k}_{u_1 \cdots u_l, w} S^{s_1 \cdots s_k}_{r_1 \cdots r_l} = S^{s_1 \cdots s_k}_{r_1 \cdots r_l, w} S^{t_1 \cdots t_k}_{u_1 \cdots u_l},$$

resp.

(1.2)
$$S^{t_1 \cdots t_k}_{u_1 \cdots u_l, vw} S^{s_1 \cdots s_k}_{r_1 \cdots r_l} = S^{s_1 \cdots s_k}_{r_1 \cdots r_l, vw} S^{t_1 \cdots t_k}_{u_1 \cdots u_l},$$

holds on N, where the comma denotes the covariant differentiation with respect to the metric of N.

A Riemannian manifold N ($n \geqslant 4$) is said to be *conformally recurrent* [2] (resp. *conformally birecurrent* [3]) if its Weyl conformal curvature tensor

(1.3)
$$\tilde{C}_{rstu} = \tilde{R}_{rstu} - \frac{1}{n-2}(g_{st}\tilde{R}_{ru} - g_{su}\tilde{R}_{rt} + g_{ru}\tilde{R}_{st} - g_{rt}\tilde{R}_{su})$$

$$+\frac{\tilde{R}}{(n-1)(n-2)}(g_{ru}g_{st} - g_{rt}g_{su})$$

is recurrent (resp. birecurrent), where $\tilde{R}_{rstu} = g_{rv}\tilde{R}^v_{stu}$.

Thus, by (1.1) (resp. (1.2)), if N is conformally recurrent (resp. conformally birecurrent) and at some point $x \in N$ the tensor $\tilde{C}_{rstu}$ is non-zero, then on some neighborhood U of x we have the relation

(1.4)
$$\tilde{C}_{rstu,v} = b_v \tilde{C}_{rstu}$$

for some vector field b_v on U (resp.

(1.5)
$$\tilde{C}_{rstu,vw} = a_{vw}\tilde{C}_{rstu}$$

for some tensor a_{vw} on U).

Let M be an m-dimensional manifold covered by a system of coordinate neighborhoods $\{V; (y^a)\}$. Suppose that M is a submanifold of N and let $x^r = x^r(y^a)$ be its local expression in N. Moreover, let the induced tensor

$$g_{ab} = g_{rs}B^{rs}_{ab}$$

be the metric tensor on the submanifold M, where

$$B^{rs}_{ab} = B_a^r B_b^s, \qquad B_a^r = \partial_a x^r, \qquad \partial_a = \partial/\partial y^a.$$

In the following we shall use the notation

$$B^{r_1 \cdots r_l}_{a_1 \cdots a_l} = B_{a_1}^{r_1} B_{a_2}^{r_2} \cdots B_{a_l}^{r_l}.$$

We denote by Γ^a_{bc}, V_a, K^a_{bcd}, K_{ad} and K the Christoffel symbols, the operator of covariant differentiation, the curvature tensor, the Ricci tensor and the scalar curvature of M with respect to g_{ab}. The indices a, b, c, d, e, f, h, i, j here and in the sequel run over the range $\{1, 2, \ldots, m\}$, $4 \leqslant m < n$.

The van der Waerden–Bortolotti covariant derivative of B_a^r is given by

(1.6)
$$V_b B_a^r = \partial_b B_a^r + \tilde{\Gamma}^r_{st} B^{st}_{ba} - B_c^r \Gamma^c_{ba}.$$

The vector field H^r defined by

$$H^r = \frac{1}{m} g^{ab} V_b B_a^r$$

is called the *mean curvature vector field* of M. Using (1.6) and the equation

$$\Gamma_{bc}^a = (\partial_c B_b^r + \tilde{\Gamma}_{st}^r B_{cb}^{st}) B_d^u g^{da} g_{ru},$$

we obtain on M the relation

(1.7) $$g_{rs} H^r B_a^s = 0.$$

The *Schouten curvature tensor* H_{ab}^r of M is defined by

(1.8) $$H_{ab}^r = V_b B_a^r.$$

If the tensor H_{ab}^r satisfies the condition

(1.9) $$H_{ab}^r = g_{ab} H^r,$$

then M is called a *totally umbilical submanifold* of N (see [14]).

Let N_y^r ($y, z = m+1, m+2, \ldots, n$) be pairwise orthogonal unit vectors normal to M. Then we have the relations

(1.10) $$g_{rs} N_y^r N_y^s = e_y, \qquad g_{rs} N_y^r N_z^s = 0 \ (y \neq z), \qquad g_{rs} N_y^r B_a^s = 0$$

and

(1.11) $$g^{rs} = B_{ab}^{rs} g^{ab} + \sum_y e_y N_y^r N_y^s,$$

where e_y is the indicator of the vector N_y^r.

For a totally umbilical submanifold M of N the Gauss and Codazzi equations can be written in the form (cf., e.g., [11])

(1.12) $$K_{abcd} = \tilde{R}_{rstu} B_{abcd}^{rstu} + H(g_{ad} g_{bc} - g_{ac} g_{bd})$$

and

(1.13) $$\tilde{R}_{rstu} B_{abc}^{rst} N_y^u = A_{ay} g_{bc} - A_{by} g_{ac},$$

where

$$H = g_{rs} H^r H^s, \qquad A_{ay} = \partial_a H_y + \sum_z e_z L_{azy} H_z, \qquad H_y = H^r N_y^s g_{rs}$$

and

$$L_{azy} = N_y^s (V_a N_z^r) g_{rs}.$$

In the sequel we need the formulas (see [9] and [10])

(1.14) $$\tilde{R}_{rstu} B_{abc}^{rst} H^u = \tfrac{1}{2}(H_a g_{bc} - H_b g_{ac}),$$

(1.15) $$V_e K_{abcd} = \tilde{R}_{rstu,v} B_{abcde}^{rstuv} + H_{abcde},$$

(1.16) $$V_a H^r = -H B_a^r + \sum_z e_z A_{az} N_z^r,$$

where $H_a = V_a H$ and

$$H_{abcde} = \frac{1}{2}[H_a(g_{bc}\,g_{de} - g_{ce}\,g_{bd}) + H_b(g_{ad}\,g_{ce} - g_{ac}\,g_{de})$$

$$+ H_c(g_{ad}\,g_{be} - g_{ae}\,g_{bd}) + H_d(g_{ae}\,g_{bc} - g_{ac}\,g_{be})] + H_e(g_{ad}\,g_{bc} - g_{ac}\,g_{bd}).$$

Let T_{rstu} be a tensor field of type (0, 4) on a Riemannian manifold N. Then we can define on N some tensor field of type (0, 6) in the following manner:

$$Q(T)_{rstuvw} = g_{rv}\,T_{wstu} - g_{rw}\,T_{vstu} - g_{sv}\,T_{wrtu} + g_{sw}\,T_{vrtu}$$

$$+ g_{tv}\,T_{wurs} - g_{tw}\,T_{vurs} - g_{uv}\,T_{wtrs} + g_{uw}\,T_{vtrs}.$$

In the following, if $T_{s_1\ldots s_k}^{r_1\cdots r_l}$ is a tensor field on N, then we shall denote by $T_{s_1\ldots s_k,[v,w]}^{r_1\cdots r_l}$ the difference

$$T_{s_1\ldots s_k,vw}^{r_1\cdots r_l} - T_{s_1\ldots s_k,wv}^{r_1\cdots r_l}.$$

Throughout this paper all manifolds are assumed to be connected Hausdorff manifolds of class C^∞. Whenever analyticity is supossed, it will concern all objects involved.

2. Preliminary results.

LEMMA 1. *Let M be a totally umbilical submanifold of a manifold N. Then the Weyl conformal curvature tensor of M*

$$(2.1) \qquad C_{abcd} = K_{abcd} - \frac{1}{m-2}(g_{ad}\,K_{bc} + g_{bc}\,K_{ad} - g_{ac}\,K_{bd} - g_{bd}\,K_{ac})$$

$$+ \frac{K}{(m-1)(m-2)}(g_{ad}\,g_{bc} - g_{ac}\,g_{bd})$$

satisfies the relation

$$(2.2) \qquad C_{abcd} = \bar{C}_{abcd} - \frac{1}{m-2}(g_{ad}\,T_{bc} + g_{bc}\,T_{ad} - g_{ac}.T_{bd} - g_{bd}\,T_{ac})$$

$$+ \frac{P}{(m-1)(m-2)}(g_{ad}\,g_{bc} - g_{ac}\,g_{bd}),$$

where

$$(2.3) \qquad \bar{C}_{abcd} = \tilde{C}_{rstu}\,B_{abcd}^{rstu},$$

$$(2.4) \qquad T_{bc} = K_{bc} - \frac{m-2}{n-2}\tilde{R}_{rs}\,B_{bc}^{rs}$$

and

$$(2.5) \qquad P = K + (m-1)(m-2)\,H - \frac{(m-1)(m-2)}{(n-1)(n-2)}\tilde{R}.$$

Proof. Adding to both sides of equation (1.12) the expression

$$-\frac{1}{m-2}(g_{ad}\,K_{bc}+g_{bc}\,K_{ad}-g_{ac}\,K_{bd}-g_{bd}\,K_{ac})+\frac{K}{(m-1)(m-2)}(g_{ad}\,g_{bc}-g_{ac}\,g_{bd})$$

and using (2.1) we obtain

$$C_{abcd}=\tilde{R}_{rstu}\,B^{rstu}_{abcd}-\frac{1}{m-2}(g_{ad}\,K_{bc}+g_{bc}\,K_{ad}-g_{ac}\,K_{bd}-g_{bd}\,K_{ac})$$

$$+\left(H+\frac{K}{(m-1)(m-2)}\right)(g_{ad}\,g_{bc}-g_{ac}\,g_{bd}).$$

But from this, by making use of (1.3), (2.3)–(2.5), it follows that relation (2.2) holds true on M. This completes the proof.

LEMMA 2. *Let M be a totally umbilical submanifold of a manifold N. Let the condition*

$$(2.6)\qquad \tilde{C}_{rstu,[v,w]}=c_{vw}\,\tilde{C}_{rstu}+\bar{A}Q\,(\tilde{C})_{rstuvw}$$

be satisfied on N, where c_{vw} is a tensor field and $\bar{A}$ is a function on N. Then the relation

$$(2.7)\qquad C_{abcd,[e,f]}=c_{ef}\,C_{abcd}+(\bar{A}-H)\,Q\,(C)_{abcdef}$$

holds true on M, where

$$(2.8)\qquad c_{ef}=c_{vw}\,B^{vw}_{ef}.$$

Proof. By the Ricci identity, the tensor field $\bar{C}_{abcd,[e,f]}$ satisfies the equation

$$\bar{C}_{abcd,[e,f]}=(-\bar{C}_{ibcd}\,K_{jaef}+\bar{C}_{iacd}\,K_{jbef}-\bar{C}_{idab}\,K_{jcef}+\bar{C}_{icab}\,K_{jdef})\,g^{ij}.$$

Applying the equation (1.12) to the above identity we obtain

$$\bar{C}_{abcd,[e,f]}=(-\bar{C}_{ibcd}\,\bar{R}_{jaef}+\bar{C}_{iacd}\,\bar{R}_{jbef}-\bar{C}_{idab}\,\bar{R}_{jcef}$$

$$+\bar{C}_{icab}\,\bar{R}_{jdef})\,g^{ij}-HQ\,(\bar{C})_{abcdef},$$

where $\bar{R}_{jaef}=\tilde{R}_{rstu}\,B^{rstu}_{jaef}$. From the last equality, using (1.11) and the definitions of the tensors $\bar{C}_{abcd}$ and $\bar{R}_{abcd}$, it follows that

$$\bar{C}_{abcd,[e,f]}=(-\tilde{C}_{pstu}\,\tilde{R}_{qrvw}+\tilde{C}_{prtu}\,\tilde{R}_{qsvw}-\tilde{C}_{purs}\,\tilde{R}_{qtvw}$$

$$+\tilde{C}_{ptrs}\,\tilde{R}_{quvw})\left(g^{pq}-\sum_{y}e_{y}\,N_{y}{}^{p}\,N_{y}{}^{q}\right)B^{rstuvw}_{abcdef}-HQ\,(\bar{C})_{abcdef}.$$

Now, the right-hand side of the above equation, by the Ricci identity, (2.6), (2.8) and (2.3), takes the form

(2.9) $\quad \bar{C}_{abcd,[e,f]} = c_{ef}\,\bar{C}_{abcd} + (\bar{A} - H)\,Q\,(\bar{C})_{abcdef} - \sum_y e_y\,N_y{}^p\,N_y{}^q\,B^{rstuvw}_{abcdef}$

$$\times(-\tilde{C}_{pstu}\,\tilde{R}_{qrvw} + \tilde{C}_{prtu}\,\tilde{R}_{qsvw} - \tilde{C}_{purs}\,\tilde{R}_{qtvw} + \tilde{C}_{ptrs}\,\tilde{R}_{quvw}).$$

The following equation is an immediate consequence of (1.3), (1.13) and (1.10):

$$\sum_y e_y\,N_y{}^p\,N_y{}^q\,B^{rstuvw}_{abcdef}\,\tilde{C}_{pstu}\,\tilde{R}_{qrvw} = g_{bc}\,g_{ae}\,L_{df} - g_{bd}\,g_{ae}\,L_{cf} - g_{bc}\,g_{af}\,L_{de}$$

$$+ g_{bd}\,g_{af}\,L_{ce},$$

where

$$L_{dc} = \sum_y e_y\,A_{cy}\left(A_{dy} - \frac{1}{n-2}\,\tilde{R}_{rs}\,N_y{}^r\,B_d{}^s\right).$$

Substituting this into (2.9) we get easily

(2.10) $\quad \bar{C}_{abcd,[e,f]} = c_{ef}\,\bar{C}_{abcd} + (\bar{A} - H)\,Q\,(\bar{C})_{abcdef}$

$$+ (g_{bc}\,g_{de} - g_{bd}\,g_{ce})\,L_{af} - (g_{bc}\,g_{df} - g_{bd}\,g_{cf})\,L_{ae} - (g_{ca}\,g_{de} - g_{ad}\,g_{ce})\,L_{bf}$$

$$+ (g_{ce}\,g_{df} - g_{da}\,g_{cf})\,L_{be} + (g_{ad}\,g_{be} - g_{bd}\,g_{ae})\,L_{cf} - (g_{ad}\,g_{bf} - g_{bd}\,g_{af})\,L_{ce}$$

$$+ (g_{ac}\,g_{bf} - g_{bc}\,g_{af})\,L_{de} - (g_{ac}\,g_{be} - g_{bc}\,g_{ae})\,L_{df}.$$

From (2.2), by contraction with g^{bc}, we obtain

$$\bar{C}_{abcd}\,g^{bc} = T_{ad} + \frac{1}{m-2}(T - P)\,g_{ad}.$$

Now, contracting (2.10) with g^{bc} and using the above equality and (2.2), we find

(2.11) $\quad T_{ad,[e,f]} = c_{ef}\,T_{ad} + \frac{1}{m-2}(T - P)\,c_{ef}\,g_{ad}$

$$+ (\bar{A} - H)(g_{ae}\,T_{fd} - g_{af}\,T_{ed} + g_{de}\,T_{fa} - g_{df}\,T_{ea})$$

$$+ (m - 2)(g_{de}\,L_{af} - g_{df}\,L_{ae} + g_{ae}\,L_{df} - g_{af}\,L_{de}) + 2g_{ad}(L_{ef} - L_{fe}),$$

whence, by contraction with g^{ad}, we get

(2.12) $\quad -c_{ef}\left(\frac{2}{m-2}\,T - \frac{m}{(m-1)(m-2)}\,P\right) = 4(L_{ef} - L_{fe}).$

Applying the relations (2.2) and (2.11) to (2.10), we obtain, after straight-forward calculations,

$$C_{abcd,[e,f]} + \left(2\frac{T-P}{(m-2)^2} c_{ef} + \frac{4}{m-2}(L_{ef} - L_{fe}) \right)(g_{ad}g_{bc} - g_{ac}g_{bd})$$

$$= c_{ef} C_{abcd} + (\bar{A} - H) Q(C)_{abcdef} - c_{ef}\frac{P}{(m-1)(m-2)}(g_{ad}g_{bc} - g_{ac}g_{bd}).$$

But the last equation, together with (2.12), leads immediately to (2.7). Our lemma is thus proved.

Let M be a totally umbilical submanifold of a manifold N. Define on M the following tensor fields:

$$(2.13) \qquad S_{abcdef} = \tilde{R}_{rstu,v} \, \nabla_f (B^{rstuv}_{abcde}),$$

$$(2.14) \qquad D_{abcd} = \tilde{R}_{rstu,v} \, B^{rstv}_{abcd} \, H^u$$

and

$$(2.15) \qquad D_{ad} = g^{bc} D_{abcd}.$$

By virtue of (1.8), (1.9) and (2.14), we obtain from (2.13) the identity

$$(2.16) \qquad S_{abcdef} = g_{fa} D_{dcbe} + g_{fb} D_{cdae} + g_{fc} D_{bade} + g_{fd} D_{abce} + g_{ef} X_{abcd},$$

where $X_{abcd} = \tilde{R}_{rstu,v} B^{rstu}_{abcd} H^v = D_{abcd} - D_{abdc}$.

Relation (1.14), by covariant differentiation and using (1.8), (1.9), (1.12), (1.13) and (1.16), yields

$$(2.17) \qquad D_{abcd} = \tfrac{1}{2}(g_{bc} \nabla_d H_a - g_{ac} \nabla_d H_b) - g_{da} H^r H^u B^{st}_{bc} \tilde{R}_{rstu} + H K_{abcd}$$

$$+ g_{bd} H^r H^u \tilde{R}_{rstu} B^{st}_{ac} - H^2 (g_{ad} g_{bc} - g_{ac} g_{bd})$$

$$- \sum_y e_y A_{dy}(g_{bc} A_{ay} - g_{ac} A_{by}).$$

If we put

$$(2.18) \qquad A_{bc} = H^r H^u B^{st}_{bc} \tilde{R}_{rstu}$$

and $E_{ad} = \tfrac{1}{2} \nabla_d H_a - \sum_y e_y A_{dy} A_{ay}$, then (2.17) takes the form

$$(2.19) \qquad D_{abcd} = g_{bc} E_{ad} - g_{ac} E_{bd} - g_{ad} A_{bc} + g_{bd} A_{ac} + H K_{abcd}$$

$$- H^2 (g_{ad} g_{bc} - g_{ac} g_{bd}).$$

LEMMA 3. *Let M be a totally umbilical submanifold of a manifold N and let*

$$(2.20) \qquad T_{abcdef} = S_{abcdef} - \frac{1}{m-2}(g_{ad} S_{bcef} - g_{ac} S_{bdef} + g_{bc} S_{adef} - g_{bd} S_{acef})$$

$$+ \frac{1}{(m-1)(m-2)} S_{ef}(g_{ad} g_{bc} - g_{ac} g_{bd}).$$

If at some point $x \in M$ we have $H(x) = 0$, then the tensor T_{abcdef} satisfies at x the equation

$$(2.21) \qquad T_{abcdef} = g_{fa}(g_{ce} A_{db} - g_{de} A_{cb}) + g_{fb}(g_{de} A_{ca} - g_{ce} A_{da})$$

$$+ g_{fc}(g_{ae} A_{bd} - g_{be} A_{ad}) + g_{fd}(g_{be} A_{ac} - g_{ae} A_{bc})$$

$$+ \frac{2}{m-2} g_{ef}(g_{ad} A_{bc} + g_{bc} A_{ad} - g_{ac} A_{bd} - g_{bd} A_{ac})$$

$$- \frac{1}{m-2} \{ g_{bc} [g_{fa}(A_{de} - Ag_{de}) + g_{fd}(A_{ae} - Ag_{ae})$$

$$+ g_{ae} A_{fd} + g_{de} A_{fa}]$$

$$- g_{bd} [g_{fa}(A_{ce} - Ag_{ce}) + g_{fc}(A_{ae} - Ag_{ae}) + g_{ae} A_{fc} + g_{ce} A_{fa}]$$

$$+ g_{ad} [g_{fb}(A_{ce} - Ag_{ce}) + g_{fc}(A_{be} - Ag_{be}) + g_{be} A_{fc} + g_{ce} A_{fb}]$$

$$- g_{ac} [g_{fb}(A_{de} - Ag_{de}) + g_{fd}(A_{be} - Ag_{be}) + g_{be} A_{fd} + g_{de} A_{fb}] \}$$

$$+ \frac{4}{(m-1)(m-2)} (A_{fe} - Ag_{fe})(g_{ad} g_{bc} - g_{ac} g_{bd}),$$

where $S_{adef} = g^{bc} S_{abcdef}$, $S_{ef} = g^{ad} S_{adef}$ and $A = g^{bc} A_{bc}$.

Proof. Since $H(x) = 0$, equation (2.19) yields

$$D_{abcd} = g_{bc} E_{ad} - g_{ac} E_{bd} - g_{ad} A_{bc} + g_{bd} A_{ac}.$$

Substituting this into (2.16) we get

$$(2.22) \qquad S_{abcdef} = g_{fa}(g_{ce} A_{db} - g_{de} A_{cb} + E_{de} g_{cb} - E_{ce} g_{bd})$$

$$+ g_{fb}(g_{de} A_{ca} - g_{ce} A_{da} + E_{ce} g_{da} - E_{de} g_{ca})$$

$$+ g_{fc}(g_{ae} A_{bd} - g_{be} A_{da} + E_{be} g_{da} - E_{ae} g_{bd})$$

$$+ g_{fd}(g_{be} A_{ac} - g_{ae} A_{bc} + E_{ae} g_{bc} - E_{be} g_{ac})$$

$$+ g_{ef}(g_{bd} A_{ac} - g_{ad} A_{bc} + g_{bc} E_{ad} - g_{ac} E_{bd}$$

$$- g_{bc} A_{ad} + g_{ac} A_{bd} - g_{bd} E_{ac} + g_{ad} E_{bc}).$$

Contracting the last equation with g^{bc}, we obtain

$$(2.23) \qquad S_{adef} = (m-2)(g_{af} E_{de} + g_{fd} E_{ae} + g_{ef} E_{ad}) + 2E_{ef} g_{ad}$$

$$- m A_{ad} g_{ef} + g_{fa}(A_{de} - Ag_{de}) + g_{fd}(A_{ae} - Ag_{ae})$$

$$+ g_{ae} A_{fd} + g_{de} A_{fa} + (E - A) g_{ef} g_{ad},$$

which, by contraction with g^{ad}, gives

$$(2.24) \qquad S_{ef} = 4(m-1) E_{ef} + 2(m-1)(E - A) g_{ef} + 4(A_{fe} - Ag_{fe}),$$

where $E = g^{bc} E_{bc}$. Substituting now relations (2.22)–(2.24) into (2.20), we obtain (2.21), completing the proof.

LEMMA 4. *Let M be a totally umbilical submanifold of a conformally birecurrent manifold N. If $\tilde{C}_{rstu}(x) \neq 0$ at a certain point $x \in M$, then on some neighborhood $V \subset M$ of x the relation*

$$(2.25) \qquad \nabla_f \, \nabla_e \, C_{abcd} - a_{ef} \, C_{abcd} = T_{abcdef}$$

is satisfied, where

$$(2.26) \qquad a_{ef} = a_{vw} B_{ef}^{vw}.$$

Moreover, if M is also conformally birecurrent and $C_{abcd}(x) \neq 0$, then at x the equation

$$(2.27) \qquad P_{ij} \, T_{abcdef} = P_{ef} \, T_{abcdij}$$

is fulfilled, where $P_{ef} = l_{ef} - a_{ef}$ and l_{ef} is the tensor of birecurrency of C_{abcd}.

Proof. Since $\tilde{C}_{rstu}(x) \neq 0$, we have on some open neighborhood $U \subset N$ of x the relation (1.5). Substitution (1.3) into (1.5) leads to

$$(2.28) \quad \tilde{R}_{rstu,vw} - a_{vw} \, \tilde{R}_{rstu} = P_{vw} (g_{ru} g_{st} - g_{rt} g_{su})$$

$$+ \frac{1}{n-2} (g_{ru} \, M_{vwst} - g_{rt} \, M_{vwsu} + g_{st} \, M_{vwru} - g_{su} \, M_{vwrt}),$$

where

$$M_{vwst} = \tilde{R}_{st,vw} - a_{vw} \, \tilde{R}_{st} \quad \text{and} \quad P_{vw} = \frac{1}{(n-1)(n-2)} (\tilde{R} a_{vw} - \tilde{R}_{,vw}).$$

Equation (1.15), by covariant differentiation, yields

$$(2.29) \qquad \nabla_f \, \nabla_e \, K_{abcd} = (\tilde{R}_{rstu,vw}) B_{abcdef}^{rstuvw} + S_{abcdef} + H_{abcdef},$$

where $H_{abcdef} = \nabla_f H_{abcde}$. Applying the relations (2.28), (2.26) and (1.12) in (2.29) we find on $V = U \cap M$ (if $V \neq \emptyset$) the equality

$$(2.30) \quad \nabla_f \, \nabla_e \, K_{abcd} - a_{ef} \, K_{abcd} = \bar{P}_{ef} (g_{ad} g_{bc} - g_{ac} g_{bd}) + S_{abcdef}$$

$$+ \frac{1}{n-2} (g_{ad} \, M_{efbc} - g_{ac} \, M_{efbd} + g_{bc} \, M_{efad} - g_{bd} \, M_{efac}) + H_{abcdef}$$

$$- H a_{ef} (g_{ad} g_{bc} - g_{ac} g_{bd}),$$

where $M_{efbc} = M_{vwrs} B_{efbc}^{vwrs}$ and $\bar{P}_{ef} = P_{vw} B_{ef}^{vw}$. Relations (2.29) and (1.12) lead to

$$\nabla_f \, \nabla_e \, K_{abcd} - a_{ef} \, K_{abcd} = S_{abcdef} + H_{abcdef} - H a_{ef} (g_{ad} g_{bc} - g_{ac} g_{bd})$$

$$+ (\tilde{R}_{rstu,vw} - a_{vw} \, \tilde{R}_{rstu}) B_{abcdef}^{rstuvw}.$$

Contracting the above equation with g^{bc} and making use of (1.11) we get

$$(2.31) \quad V_f\, V_e\, K_{ad} - a_{ef}\, K_{ad} = M_{efad} - \sum_y e_y\, N_y{}^s\, N_y{}^t (\tilde{R}_{rstu,vw} - a_{vw}\, \tilde{R}_{rstu})\, B_{adef}^{ruvw}$$

$$-(m-1)\, Ha_{ef}\, g_{ad} + S_{adef} + H_{adef},$$

where $H_{adef} = g^{bc}\, H_{abcdef}$. On the other hand, from (2.28), by transvection with $\sum_y e_y\, N_y{}^s\, N_y{}^t\, B_{adef}^{ruvw}$ and using (1.10), it follows that

$$\sum_y e_y\, N_y{}^s\, N_y{}^t\, B_{adef}^{ruvw} (\tilde{R}_{rstu,vw} - a_{vw}\, \tilde{R}_{rstu})$$

$$= \frac{1}{n-2} [g_{ad}(\sum_y e_y\, N_y{}^s\, N_y{}^t)\, B_{ef}^{vw}\, M_{vwst} + (n-m)\, M_{efad}] + (n-m)\, \bar{P}_{ef}\, g_{ad}.$$

Substituting the last equation into (2.31), we obtain

$$(2.32) \quad V_f\, V_e\, K_{ad} - a_{ef}\, K_{ad} = \frac{m-2}{n-2} M_{efad} - \frac{1}{n-2} g_{ad}(\sum_y e_y\, N_y{}^s\, N_y{}^t)\, B_{ef}^{vw}\, M_{vwst}$$

$$-(n-m)\, \bar{P}_{ef}\, g_{ad} - (m-1)\, Ha_{ef}\, g_{ad} + H_{adef} + S_{adef}.$$

Hence, by contraction with g^{ad}, we obtain

$$M_{vwru} \sum_y e_y\, N_y{}^r\, N_y{}^u\, B_{ef}^{vw} = \frac{1}{2(m-1)} \left[m - 2 + \frac{m(n-m)}{n-1} \right] (\tilde{R}_{,vw} - \tilde{R}a_{vw})\, B_{ef}^{vw}$$

$$- \frac{n-2}{2(m-1)}(V_f\, V_e\, K - Ka_{ef}) - \frac{1}{2}m(n-2)\, Ha_{ef} + \frac{1}{2}(n-2)(m+2)\, V_f\, H_e$$

$$+ \frac{n-2}{2(m-1)} S_{ef}.$$

Substituting the last equation into (2.32), we get

$$M_{efad} = \frac{n-2}{m-2} \left[V_f\, V_e\, K_{ad} - a_{ef}\, K_{ad} - S_{adef} \right.$$

$$\left. - \frac{1}{2(m-1)} g_{ad}(V_f\, V_e\, K - Ka_{ef} - S_{ef}) \right]$$

$$- \frac{n-2}{2} [g_{ad}\, \bar{P}_{ef} + g_{ad}(V_f\, H_e - Ha_{ef})$$

$$+ g_{ae}\, V_f\, H_d + g_{de}\, V_f\, H_a].$$

Finally, substituting this into (2.30), after straightforward calculations, we obtain (2.25).

Now we prove that relation (2.27) holds at x. Since M is conformally birecurrent and $C_{abcd}(x) \neq 0$, relation (2.25) takes at x the form

(2.33) $P_{ef} C_{abcd} = T_{abcdef}.$

Multiplying this by P_{ij} we obtain

$$P_{ij} P_{ef} C_{abcd} = P_{ij} T_{abcdef}.$$

But from the last equation we get easily (2.27). Our lemma is thus proved.

LEMMA 5 (cf. [6], Theorem 2). *Let M be a totally umbilical submanifold of a conformally birecurrent manifold N. If $\tilde{C}_{rstu}(x) = 0$ at a certain point x of M, then $C_{abcd}(x) = 0$ at x.*

LEMMA 6. *Let M be a totally umbilical submanifold of a conformally birecurrent manifold N. Moreover, let M be also conformally birecurrent. If $C_{abcd}(x) \neq 0$ and $H(x) = 0$ at some point $x \in M$, then the relation*

(2.34) $$A_{ef} - \frac{A}{m} g_{ef} = 0$$

holds at x.

Proof. First of all we notice that in view of (2.21) the equations

$$T_{abcdef} = T_{abcdfe} \quad \text{and} \quad g^{ef} T_{abcdef} = 0$$

hold at x. Therefore, from (2.33) we get the relations

$$P^i_{\ i} C_{abcd} = 0 \quad \text{and} \quad (P_{ef} - P_{fe}) C_{abcd} = 0,$$

where $P^i_{\ i} = g^{ij} P_{ij}$. Since $C_{abcd}(x) \neq 0$, the above equations give

(2.35) $P^i_{\ i}(x) = 0$

and

(2.36) $(P_{ef} - P_{fe})(x) = 0.$

Contracting now equality (2.27) with g^{ai} and g^{cj} and using (2.21), (2.35) and (2.36), we obtain

(2.37) $2P_{ef} A_{bd} - g_{de} Z_{fb} - g_{fb} Z_{ed} - g_{be} Z_{fd} - g_{fd} Z_{eb}$

$$+ Z(g_{fb} g_{de} + g_{fd} g_{be}) + \frac{2}{m-2} g_{ef}(Z_{db} + Z_{bd} - Zg_{bd})$$

$$- \frac{1}{m-2} [(A_{de} - Ag_{de}) P_{fb} + (A_{be} - Ag_{be}) P_{df} + A_{fd} P_{eb} + A_{fb} P_{de}$$

$$- 2g_{bd}(Z_{fe} - AP_{fe}) + g_{bf}(Z_{de} - AP_{de}) + g_{fd}(Z_{be} - AP_{be})$$

$$-2g_{bd}Z_{ef}+g_{be}Z_{df}+g_{de}Z_{bf}]+\frac{4}{(m-1)(m-2)}(A_{fe}-Ag_{fe})\,P_{bd}$$

$$=\frac{m(m+1)(m-3)}{m-1}P_{ef}\left(A_{bd}-\frac{A}{m}g_{bd}\right),$$

where

(2.38)
$$Z_{bd}=P^i_b\,A_{id},\qquad P^i_b=g^{ia}\,P_{ab},\qquad P^{ib}=g^{bd}\,P^i_d,\qquad Z=g^{bd}\,Z_{bd}=P^{bc}\,A_{bc}.$$

Contracting (2.37) with g^{bf} and using (2.35), (2.36) and (2.38) we get

(2.39) $(m-1)Zg_{ed}-Z_{de}-(m^2-m-1)Z_{ed}+(m-1)AP_{ed}=0.$

Alternating now the indices e and d in the last equation we obtain

(2.40) $Z_{ed}=Z_{de}.$

Thus relation (2.39) takes the form

(2.41) $Z_{ed}=\frac{Z}{m}g_{ed}+\frac{A}{m}P_{ed}.$

On the other hand, alternating (2.37) in pairs of indices $(e,\,b)$ and $(f,\,d)$ and making use of (2.40), we find

(2.42) $P_{ef}\left(A_{bd}-\frac{A}{m}g_{bd}\right)=P_{bd}\left(A_{ef}-\frac{A}{m}g_{ef}\right).$

Assume that the condition

(2.43) $\left(A_{bd}-\frac{A}{m}g_{bd}\right)\neq 0$

holds at the point $x\in M$. Thus, by (2.42) and (2.43), we obtain at x

(2.44) $P_{ef}=F\left(A_{ef}-\frac{A}{m}g_{ef}\right),$

where F is a non-zero number. We prove now that at x the following relations hold:

(2.45) $P^{ef}P_{ef}=FZ,$

(2.46) $P^{ef}Z_{ef}=\frac{1}{m}FAZ,$

(2.47) $P^i_k\,P_{if}=\frac{1}{m}FZg_{kf}$

and

$$(2.48) \qquad P^i{}_k Z_{if} = \frac{Z}{m}\left(P_{kf} + \frac{1}{m} AF g_{kf}\right).$$

Indeed, from (2.44), by transvection with P^{ef} and application of (2.38) and (2.35), we obtain (2.45). Transvecting (2.41) with P^{ed} and making use of (2.35) and (2.45) we get (2.46). Further, transvecting (2.44) and (2.41) with $P^e{}_k$ and applying (2.38) and (2.41) we find (2.47) and (2.48). Now, the transvection of (2.37) with P^{ef} and the use of (2.38) and (2.44)–(2.48) give, after straightforward calculations,

$$(2.49) \qquad \alpha(m) Z P_{bd} = 0,$$

where $\alpha(m) = m^4 - 4m^3 - 3m^2 + 22m - 16$. By (2.43) and (2.44) we have

$$(2.50) \qquad P_{bd}(x) \neq 0.$$

Thus equality (2.49) together with the last relation gives for $m \geqslant 4$

$$(2.51) \qquad Z = 0.$$

By (2.41), (2.51) and (2.44), from (2.37) it follows that

$$(2.52) \qquad \beta(m) P_{ef} P_{bd} + 2(m-1)(P_{bf} P_{de} + P_{df} P_{be}) = 0,$$

where $\beta(m) = m^4 - 4m^3 - m^2 + 12m - 8$. Alternating in (2.52) the indices b and f we get $P_{ef} P_{bd} = P_{eb} P_{fd}$. Applying the last result in (2.52) we obtain $P_{ef}(x) = 0$. But this is a contradiction with (2.50). Thus, at x we have (2.34), which completes the proof.

3. Main results.

THEOREM 1. *Let M be a totally umbilical submanifold of a manifold N. Moreover, let M be a conformally birecurrent manifold. If on N the condition (2.6) is satisfied, then the relation*

$$(3.1) \qquad (\bar{A} - H) C_{abcd} = 0$$

holds on M.

Proof. Assume that at some point $x \in M$ we have

$$(3.2) \qquad C_{abcd}(x) \neq 0.$$

Since M is conformally birecurrent, (2.7) yields

$$(3.3) \qquad V_{ef} C_{abcd} = (\bar{A} - H) Q(C)_{abcdef},$$

where $V_{ef} = l_{ef} - l_{fe} - c_{ef}$, l_{ef} is the tensor of birecurrency of C_{abcd}, and c_{ef} is given by (2.8). From (3.3) it follows immediately that

$$V_{ef} C_{abcd} + V_{ab} C_{cdef} + V_{cd} C_{efab} = 0.$$

But the last equation gives ([13], pp. 154-155)

$$V_{ef} C_{abcd} = 0.$$

Thus (3.3) can be reduced to

$$(\bar{A} - H) Q (C)_{abcdef} = 0.$$

Hence by contraction with g^{af} we obtain (3.1). Our theorem is thus proved.

Theorem 1 implies

COROLLARY 1. *Let M be a totally umbilical submanifold of a conformally birecurrent manifold N. If M is also conformally birecurrent, then relation (0.1) holds on M.*

As is known [9], every totally umbilical submanifold of a conformally recurrent manifold is also conformally recurrent. Using this fact and Corollary 1 we obtain

COROLLARY 2 (cf. [10]). *Let M be a totally umbilical submanifold of a conformally recurrent manifold N. Then on M relation (0.1) is satisfied.*

THEOREM 2. *Let M be a totally umbilical submanifold of a conformally birecurrent manifold N. Let the conditions $\tilde{C}_{rstu}(x) \neq 0$, $H(x) = 0$ and (2.34) hold at $x \in M$. Then at x the equation*

(3.4) $$V_f V_e C_{abcd} = a_{ef} C_{abcd}$$

is satisfied, where a_{ef} is defined by (2.26).

Proof. Applying the relation (2.34) in (2.21), we obtain

$$T_{abcdef}(x) = 0.$$

But the last equation together with (2.25) gives (3.4), which completes the proof.

The next result follows immediately from Theorem 2 and Lemma 5.

THEOREM 3. *Let M be a totally umbilical submanifold of a conformally birecurrent manifold N. If the relations*

(3.5) $$H = 0$$

and (2.34) hold on M, then M is conformally birecurrent.

As an immediate consequence of Lemma 6, Corollary 1 and Theorem 3, we obtain

THEOREM 4. *Let M be an analytic, non-conformally flat, totally umbilical submanifold of a conformally birecurrent manifold N. Then M is conformally birecurrent if and only if relations (3.5) and (2.34) hold on M.*

4. Examples. In this section we give examples of conformally birecurrent totally umbilical submanifolds of a conformally birecurrent manifold satis-

fying (0.1) or (2.34) and (3.5). We define the metric $\bar{g}_{ab}$ in R^q ($q \geqslant 4$) by the formula

(4.1)
$$\bar{g}_{ab} = \begin{cases} -2 & \text{if } a = b = 1, \\ \exp(F_a) & \text{if } a+b = q+1, \\ 0 & \text{otherwise,} \end{cases}$$

where

(4.2)
$$F_a(x^1, \ldots, x^q) = F_{n+1-a}(x^1, \ldots, x^q) = \begin{cases} G(x^2) + B(x^2) & \text{if } a = 2, \\ G(x^2) & \text{otherwise,} \end{cases}$$

(4.3)
$$G(x^2) = klx^2, \qquad B(x^2) = -\frac{(kl)^2+1}{kl}x^2,$$

k and l are constants such that

(4.4)
$$k \in (0, 1) \quad \text{and} \quad l^2 = -\frac{2}{k(k-1)}$$

and $a, b, c, d, e \in \{1, \ldots, q\}$. The reciprocal $\bar{g}^{ab}$ of $\bar{g}_{ab}$ is clearly of the form

(4.5)
$$\bar{g}^{ab} = \begin{cases} 2\exp(-2F_1) & \text{if } a = b = q, \\ \exp(-F_a) & \text{if } a+b = q+1, \\ 0 & \text{otherwise.} \end{cases}$$

The only components of the Christoffel symbols $\bar{\Gamma}^a_{bc}$, the curvature tensor

$$\bar{R}_{abcd} = \bar{g}_{ae}\bar{R}^e{}_{bcd} = \bar{g}_{ae}(\partial_d \bar{\Gamma}^e_{bc} - \partial_c \bar{\Gamma}^e_{bd} + \bar{\Gamma}^f_{bc}\bar{\Gamma}^e_{fd} - \bar{\Gamma}^f_{bd}\bar{\Gamma}^e_{fc}), \qquad \partial_d = \partial/\partial x^d,$$

the Ricci tensor $\bar{R}_{ad} = \bar{g}^{bc}\bar{R}_{badc}$, the Weyl conformal curvature tensor $\bar{C}_{abcd}$ and its covariant derivative $\bar{\nabla}_e \bar{C}_{abcd}$ not identically equal to zero are those related to (see [4])

(4.6)
$$\bar{\Gamma}^1_{12} = \bar{\Gamma}^\lambda_{2\lambda} = \bar{\Gamma}^q_{2q} = \tfrac{1}{2}kl, \qquad \bar{\Gamma}^2_{22} = -\frac{1}{kl},$$

$$\bar{\Gamma}^{q-1}_{1,q} = -\tfrac{1}{2}kl\exp(F_1 - F_2), \qquad \bar{\Gamma}^q_{12} = kl\exp(-F_1), \qquad \bar{\Gamma}^{q-1}_{\lambda,q+1-\lambda},$$

(4.7)
$$\bar{R}_{1212} = -\tfrac{1}{2}(kl)^2, \qquad \bar{R}_{122q} = -\tfrac{1}{4}[(kl)^2+2]\exp G,$$

$$\bar{R}_{2\lambda2,q+1-\lambda} = -\bar{R}_{122q},$$

(4.8)
$$\bar{R}_{22} = -\frac{q-2}{4}[(kl)^2+2],$$

(4.9)
$$\bar{C}_{1212} = 1, \qquad \bar{\nabla}_2\bar{C}_{1212} = -\frac{(kl)^2-2}{kl},$$

where $\bar{\nabla}_a$ denotes the covariant derivative with respect to $\bar{g}_{ab}$.

In formulas (4.6) and (4.7) we adopt the convention that the Greek index λ ranges over the set $\{3, ..., n-2\}$ (empty for $q = 4$) and that repeated indices are not be summed over. It is easy to verify that the relation $\bar{V}_e \bar{C}_{abcd}$ $= \varphi_e \bar{C}_{abcd}$ holds on R^q, where $\varphi_e = \bar{V}_e \varphi$, $\varphi = -3G - 2B$. Thus R^q with the metric given by (4.1)–(4.4) is conformally recurrent (cf. [4], Theorem 1).

Now we define in R^n the metric g_{rs} by

(4.10)
$$g_{rs} = \begin{cases} \bar{g}_{ab} & \text{if } r = a \text{ and } s = b, \\ \sigma \overset{*}{g}_{\alpha\beta} & \text{if } r = \alpha \text{ and } s = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{g}_{ab}$ is given by (4.1)–(4.4),

(4.11)
$$\sigma(x^1, ..., x^q) = \exp(-lx^2),$$

(4.12)
$$\overset{*}{g}_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$r, s, t, u, v, w \in \{1, ..., n\}, \qquad a, b, c, d, e, f \in \{1, ..., q\}$$

and

$$\alpha, \beta, \gamma, \delta \in \{q+1, ..., n\}, \qquad q \geq 4, \ n-q \geq 3.$$

For simplicity, we denote by N the space R^n with the above defined metric. We prove that N is a conformally recurrent manifold. By (4.10), the Christoffel symbols Γ^r_{st} of N satisfy the relations

(4.13)
$$\Gamma^r_{st} = \begin{cases} \bar{\Gamma}^a_{bc} & \text{if } r = a, \ s = b, \ t = c, \\ -\tfrac{1}{2}\overset{*}{g}_{\alpha\beta}\bar{g}^{2,q-1}\sigma_2 & \text{if } r = q-1, \ s = \alpha, \ t = \beta, \\ (1/2\sigma)\sigma_2 \delta^\alpha_\beta & \text{if } r = \alpha, \ s = 2, \ t = \beta, \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma_2 = \partial_2\sigma$. The only components of the curvature tensor R_{rstu}, the Ricci tensor R_{ru}, the Weyl conformal curvature tensor C_{rstu} and its covariant derivative $\nabla_v C_{rstu}$ which are not identically equal to zero are those related to

(4.14)
$$R_{1212} = \bar{R}_{1212}, \qquad R_{122q} = \bar{R}_{122q}, \qquad R_{2\lambda 2,q+1-\lambda} = -R_{122q},$$

$$R_{\alpha 22\beta} = -\frac{kl^2-2}{4k}\overset{*}{g}_{\alpha\beta}\,\sigma,$$

(4.15)
$$R_{22} = \bar{R}_{22} - \frac{n-q}{4}\frac{kl^2-2}{k},$$

(4.16)
$$C_{1212} = 1$$

and

(4.17)
$$V_2 C_{1212} = \frac{2(2k-1)}{k(k-1)l}.$$

It is easy to verify that the equation $V_v C_{rstu} = \psi_v C_{rstu}$ holds on N, where

$$\psi_v = V_v \psi, \qquad \psi = \frac{2(2k-1)}{k(k-1)l} x^2.$$

Thus N is a conformally recurrent manifold.

The submanifold V_q of N defined by

$$x^1 = y^1, \qquad \ldots, \qquad x^q = y^q, \qquad x^{q+1} = C^*_{q+1}, \qquad \ldots, \qquad x^n = C_n$$

is a totally geodesic submanifold of N and the submanifold V_{n-q} of N defined by

$$x^1 = C_1, \qquad \ldots, \qquad x^q = C_q, \qquad x^{q+1} = u^1, \qquad \ldots, \qquad x^n = u^{n-q}$$

is a totally umbilical submanifold of N ([8], Theorem 1), where $C_1, \ldots, C_n$ are constants. The submanifold V_q ($q \geqslant 4$) with the induced metric $\bar{g}_{ab}$ from the metric g_{rs} is conformally recurrent ([5], Theorem 1). Since V_q is a totally geodesic submanifold of N, the equalities (3.5) and (2.34) hold on V_q. Thus we have

THEOREM 5. *There exist conformally recurrent totally umbilical submanifolds of a conformally recurrent manifold satisfying* (3.5) *and* (2.34).

From the above theorem we obtain

COROLLARY 3. *There exist conformally birecurrent totally umbilical submanifolds of a conformally birecurrent manifold satisfying* (3.5) *and* (2.34).

The submanifold V_{n-q} with the induced metric $\sigma(C_1, \ldots, C_q)\overset{*}{g}_{\alpha\beta}$ is a flat manifold. It is clear that on V_{n-q} relation (0.1) is satisfied. Thus we have

THEOREM 6. *There exist conformally recurrent totally umbilical submanifolds of a conformally recurrent manifold satisfying* (0.1).

From this theorem it follows that

COROLLARY 4. *There exist conformally birecurrent totally umbilical submanifolds of a conformally birecurrent manifold satisfying* (0.1).

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