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PROCESSING A RADAR SIGNAL AND REPRESENTATIONS OF THE DISCRETE HEISENBERG GROUP

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1. Introduction. Let g be the element of $L^2(\mathbf{R})$ defined by

$$g(t) = e^{-t^2}.$$

Let α and β be fixed positive numbers, and let p and q be arbitrary integers. Define

$$g_{p,q}(t) = e^{2\pi iq\alpha t}e^{-(t-p\beta)^2}.$$

We call the functions $\{g_{p,q}\}$ the *Gabor functions* for scales α and β. The following problem arises in the theory of the reliability of interpreting a radar signal, as well as in a number of other contexts. Do the Gabor functions for scales α and β span a dense subspace of $L^2(\mathbf{R})$? The idea is that if an outgoing radar signal is of the form

$$s(t) = e^{-t^2}e^{2\pi i\omega t}$$

for ω a fixed frequency, then we may use the Gabor functions with scales α and β as standards of comparison for the incoming signal. If these Gabor functions span a dense subspace of $L^2(\mathbf{R})$, then in theory the incoming signal can be interpreted reliably. See for example [aus], [jan], or [fol].

The answer to the above question was given in 1971 independently by Perelomov ([per]) and Bargmann *et al.* ([barg]), and is as follows:

1.1. THEOREM.

(1) *If* $\gamma = \alpha\beta = 1$, *then the Gabor functions for scales* α *and* β *do span a dense subspace of* $L^2(\mathbf{R})$.

(2) *If* $\gamma = \alpha\beta < 1$, *then the Gabor functions for scales* α *and* β *do span a dense subspace of* $L^2(\mathbf{R})$.

We are indebted to W. Moran who introduced this subject to us during a visit in Boulder in the summer of 1989. Thanks are also expressed to A. Ramsay who, with W. Moran, discussed the beginnings of this research with us. Finally, the author wishes to express his gratitude to J. Packer whose helpful conversations and suggestions were most constructive.

(3) *If* $\gamma = \alpha\beta > 1$, *the Gabor functions for scales* α *and* β *do not span a dense subspace of* $L^2(\mathbf{R})$.

Both proofs of this theorem use relatively deep results from complex analysis and are specifically designed to treat the Gaussian $g(t) = e^{-t^2}$. Already in the 1940's von Neumann had asserted without proof part 1 of this theorem.

In this paper we prove a generalization of parts 1 and 3 of the above theorem that is applicable to an arbitrary function (signal) $f \in L^2(\mathbf{R})$. We also give a theorem along the lines of part 2 of Theorem 1.1, but it is not as precise as our other results, and it is not clear how to use it to derive part 2. Our method of proof uses the unitary representation theory of the non-type-I discrete Heisenberg group rather than complex function theory, and we suggest that this non-type-I kind of analysis can be used to deduce other similar results.

Let f be an element of $L^2(\mathbf{R})$, let α and β be fixed positive numbers (scales), and for integers p and q define

$$f_{p,q}(t) = e^{2\pi i q\alpha t} f(t - p\beta).$$

We say that f satisfies the *Gabor condition* for scales α and β if the set of $f_{p,q}$'s spans a dense subspace of $L^2(\mathbf{R})$. The generalized question from signal processing is then: When does f satisfy the Gabor condition for scales α and β ?

With the notation as in the preceding paragraph, our first two theorems are the following:

1.2. THEOREM. *Suppose* $\gamma = \alpha\beta = 1$. *Then* f *satisfies the Gabor condition for scales* α *and* β *if and only if*

$$\sum_{n=-\infty}^{\infty} f(t + n\beta)e^{2\pi i n s} \neq 0$$

for almost all t *and* s.

1.3. THEOREM. *If* $\gamma = \alpha\beta > 1$, *then* f *does not satisfy the Gabor condition for scales* α *and* β.

Remark. It is evident from Theorem 1.2 that whether f satisfies the Gabor condition depends explicitly on the values of the scales α and β and not simply on the product $\alpha\beta$.

We can also derive part 1 of Theorem 1.1 from Theorem 1.2 by studying the trigonometric series

$$\sum_{n=-\infty}^{\infty} e^{-(t+n\beta)^2} e^{2\pi i n s}.$$

For each fixed t, this is the Fourier series of a nonvanishing function of s because the coefficients decay so rapidly (see [zyg]). Note that this series is nonvanishing independent of the size of $\alpha\beta$. The point is that when $\alpha\beta > 1$ the nonvanishing of this trigonometric series does not imply that f satisfies the Gabor condition.

It is also evident from Theorem 1.2 that any f having compact support, and nonzero everywhere on some interval of length β, will satisfy the condition in that theorem, the trigonometric series being a polynomial in that case. Hence, if $\alpha\beta = 1$, the set of f's satisfying the Gabor condition for scales α and β is dense in $L^2(\mathbf{R})$.

Before stating our final result we must introduce some additional notation. For α and β fixed positive scales, $\gamma = \alpha\beta$, t any real number, and f any function on $\mathbf{R}$, define $\eta_{\alpha,\beta,t}(f)$ to be the sequence given by

$$\eta_{\alpha,\beta,t}(f)_n = f((\langle t - n\gamma\rangle + n\gamma)/\alpha),$$

where $\langle x\rangle$ denotes the fractional part of x. We will see below that, if $f \in L^2(\mathbf{R})$, then for almost all t the sequence $\eta_{\alpha,\beta,t}(f)$ belongs to l^2.

We let $R(\alpha, \beta, t)$ denote the subspace of l^2 consisting of the sequences of the form $\eta_{\alpha,\beta,t}(f)$ for $f \in L^2(\mathbf{R})$.

Our final theorem is then:

1.4. THEOREM. *Let α and β be positive scales, suppose $\gamma = \alpha\beta < 1$, and let f be in $L^2(\mathbf{R})$. Write f_x for the translate of f by x, i.e., $f_x(y) = f(x+y)$. Then f satisfies the Gabor condition for scales α and β if and only if for almost all $0 \le t < \gamma$, the sequences $\{f^{j,t}\}$ span a dense subspace of $R(\alpha, \beta, t)$, where*

$$f^{j,t} = \eta_{\alpha,\beta,t}(f_{j\beta}).$$

Remark. The condition in Theorem 1.4 leaves much to be desired. In the first place, the subspace $R(\alpha, \beta, t)$ is in general not dense in l^2, and it does not seem simple to describe it. It is therefore difficult to verify that some set of sequences spans it. In particular, if $f(t) = e^{-t^2}$, we do not see how to verify the condition, so we cannot derive part 2 of Theorem 1.1 from Theorem 1.4.

2. Some representations of the discrete Heisenberg group.

The *discrete Heisenberg group* is the set G of all matrices of the form

$$\begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix},$$

where p, q, r are integers. The center of G is the subgroup Z determined by the equations $p = q = 0$, and we shall also have occasion to use the abelian subgroup H determined by the equation $p = 0$.

For positive scales α and β, set $\gamma = \alpha\beta$, and define operators $\rho^{\alpha,\beta}_{p,q,r}$ on $L^2(\mathbf{R})$ by

$$\rho^{\alpha,\beta}_{p,q,r} f(t) = e^{-2\pi i r\gamma} e^{2\pi i q\alpha t} f(t - p\beta).$$

For each real t, define a character $\chi^{\gamma,t}$ of the subgroup H by

$$\chi^{\gamma,t}(q,r) = e^{2\pi i q t} e^{-2\pi i r\gamma},$$

and let $\sigma^{\gamma,t}$ denote the representation $\operatorname{ind}_H^G \chi^{\gamma,t}$ of G induced from $\chi^{\gamma,t}$. That is, $\sigma^{\gamma,t}$ acts in l^2 and is given by

$$\sigma^{\gamma,t}_{p,q,r} f(n) = e^{-2\pi i r\gamma} e^{2\pi i q t} e^{2\pi i q\gamma n} f(n-p).$$

For each $\delta > 0$ define I_δ to be the interval $[0,\delta)$, and define $D^{\gamma,\delta}$ to be the direct integral representation

$$D^{\gamma,\delta} = \int_{I_\delta}^{\oplus} \sigma^{\gamma,t}\, dt,$$

and write $M^{\gamma,\delta}$ for the von Neumann algebra generated by the representation $D^{\gamma,\delta}$.

Because the induced representation $\sigma^{\gamma,t}$ is unitarily equivalent to the induced representation $\sigma^{\gamma,t+\gamma}$, it follows immediately that $D^{\gamma,n\gamma}$ is unitarily equivalent to $n \times D^{\gamma,\gamma}$.

Recall finally that if M is a von Neumann algebra, then M' denotes its commutant.

2.1. THEOREM.

(1) *The map* $(p,q,r) \to \rho^{\alpha,\beta}_{p,q,r}$ *is a unitary representation of* G.

(2) *The unitary representation* $\rho^{\alpha,\beta}$ *is unitarily equivalent to the representation* $D^{\gamma,\gamma}$. *Hence, the unitary equivalence class of the representation* $\rho^{\alpha,\beta}$ *depends only on* γ.

(3) $D^{\gamma,1}$ *is equivalent to the representation* $\operatorname{ind}_Z^G \phi$ *of* G *induced from the character* ϕ *of the center* Z, *where* $\phi(r) = e^{-2\pi i r\gamma}$.

(4) $M^{\gamma,1}$ *has a cyclic and separating vector.*

(5) *Both* $M^{\gamma,1}$ *and* $(M^{\gamma,1})'$ *have finite traces, whence are finite von Neumann algebras.*

(6) $D^{\gamma,\delta}$ *is unitarily equivalent to* $D^{\gamma',\delta'}$ *if and only if* $\gamma = \gamma'$ *and* $\delta = \delta'$.

(7) *For any two positive numbers* $\gamma < \delta$, $M^{\gamma,\gamma}$ *and* $M^{\gamma,\delta}$ *are isomorphic von Neumann algebras.*

Proof. One can verify part 1 directly. Next, for each positive δ, define $U^{\alpha,\beta,\delta} : L^2(\mathbf{R}) \to L^2([0,\delta) \times \mathbf{Z})$ by

$$U^{\alpha,\beta,\delta} f(t,n) = f(t/\alpha + n\beta) = f((t + n\gamma)/\alpha).$$

One easily checks that if $\delta \geq \gamma$ then $U^{\alpha,\beta,\delta}$ is 1-1. Also, if $\delta \leq \gamma$, then $U^{\alpha,\beta,\delta}$ is onto. Further,

$$U^{\alpha,\beta,\delta} \circ \rho^{\alpha,\beta}_{p,q,r} = D^{\gamma,\delta}_{p,q,r} \circ U^{\alpha,\beta,\delta},$$

and this proves part 2. We also see here an explicit realization of $\rho^{\alpha,\beta}$ as a proper subrepresentation of $D^{\gamma,\delta}$ whenever $\delta > \gamma$.

Let Q denote the subgroup of G determined by the equations $p = r = 0$. If Λ_Q denotes the regular representation of Q, then, since Q is isomorphic to $\mathbf{Z}$, we see that the representation

$$\int_{[0,1)}^{\oplus} \chi^{\gamma,t}\, dt$$

of $H = Q \times Z$ is unitarily equivalent to the representation $\Lambda_Q \times \phi$ of H. This latter representation is unitarily equivalent to the induced representation $\mathrm{ind}_Z^H \phi$, since the forming of direct integrals and the process of inducing commute. Part 3 now follows from the theorem on inducing in stages.

The induced representation $\mathrm{ind}_Z^G \phi$ acts in $L^2(\mathbf{Z}^2)$ and generates the same von Neumann algebra as does the regular ω-representation of $\mathbf{Z}^2$, where ω is the multiplier on $\mathbf{Z}^2 \times \mathbf{Z}^2$ given by

$$\omega((p_1, q_1), (p_2, q_2)) = e^{-2\pi i p_1 q_2 \gamma},$$

so that part 4 follows from the analogous results about such regular multiplier representations of discrete abelian groups. See, for example, [klep]. Furthermore, part 5 is a consequence of Theorem 9 of that same reference.

Since the restriction to the center Z of the representation $D^{\gamma,\delta}$ is just a multiple of the character $r \to e^{-2\pi i r \gamma}$, we surely must have that $D^{\gamma,\delta}$ is unitarily equivalent to $D^{\gamma',\delta'}$ only if $\gamma = \gamma'$. To complete the proof of part 6 then, we may assume that $\gamma = \gamma'$ and that $\delta < \delta'$. Suppose first that $\delta' = 1$, and assume by way of contradiction that P is a unitary equivalence between $D^{\gamma,1}$ and $D^{\gamma,\delta}$. Then P is an isometry of the space $H(\gamma, 1)$ of the direct integral representation $D^{\gamma,1}$, i.e., the space $L^2([0,1), l^2)$, into itself since the space of $D^{\gamma,\delta}$ is

$$H(\gamma, \delta) = L^2([0, \delta), l^2).$$

It follows easily that P commutes with every operator $D^{\gamma,1}_{p,q,r}$, whence $P \in (M^{\gamma,1})'$. Let I denote the identity operator on $H(\gamma, 1)$. If $p \neq I$ is the projection of $H(\gamma, 1)$ onto the proper subspace $H(\gamma, \delta)$, then we see immediately that $p = PP^*$ and $I = P^*P$. Now, since $(M^{\gamma,1})'$ is a finite von Neumann algebra, the identity operator I is not equivalent (in $(M^{\gamma,1})'$) to a proper projection, so we have arrived at a contradiction.

For a general δ', let S be the map of $H(\gamma, \delta') = L^2([0, \delta'), l^2)$ onto $H(\gamma, 1)$ given by

$$Sf(t) = f(t/\delta').$$

If Q were a unitary equivalence between $D^{\gamma,\delta'}$ and $D^{\gamma,\delta}$, then $P = S \circ Q \circ S^{-1}$ would be a unitary equivalence between $D^{\gamma,1}$ and $D^{\gamma,\delta/\delta'}$, so that the general case of part 6 follows from the special case when $\delta' = 1$.

Finally, given positive numbers γ and δ, with $\gamma < \delta$, it is obvious that $D^{\gamma,\gamma}$ is a subrepresentation of $D^{\gamma,\delta}$. The restriction map is then a homomorphism of $M^{\gamma,\delta}$ onto $M^{\gamma,\gamma}$. Also, if n is an integer for which $n\gamma \geq \delta$, then $D^{\gamma,\delta}$ is a subrepresentation of $D^{\gamma,n\gamma}$. But, since $D^{\gamma,n\gamma} \equiv n \times D^{\gamma,\gamma}$, $M^{\gamma,n\gamma}$ is isomorphic to $M^{\gamma,\gamma}$. Again, the restriction map is a homomorphism of $M^{\gamma,n\gamma}$ onto $M^{\gamma,\delta}$, and the proof of part 7 is complete.

Remark 1. The connection between the signal processing problem mentioned in the introduction and the representations of the discrete Heisenberg group is this. An element f of $L^2(\mathbf{R})$ satisfies the Gabor condition for scales α and β if and only if f is a cyclic vector for the unitary representation $\rho^{\alpha,\beta}$, i.e., the linear span of the functions $\rho^{\alpha,\beta}_{p,q,r} f$ is dense in $L^2(\mathbf{R})$.

Remark 2. From the point of view of representation theory, it is parts 6 and 7 of the preceding theorem that are perhaps of most interest. Indeed, for a fixed γ, the representations $D^{\gamma,\delta}$ form an uncountable family of pairwise inequivalent unitary representations, but their von Neumann algebras are all isomorphic. Moreover, if γ is irrational, these von Neumann algebras are type II factors.

3. Proof of Theorems 1.2, 1.3, and 1.4. Let α and β be positive numbers, and suppose first that $\gamma = \alpha\beta \leq 1$. By Theorem 2.1, $\rho^{\alpha,\beta}$ is equivalent to $D^{\gamma,\gamma}$, and it is obvious that $D^{\gamma,\gamma}$ is a subrepresentation of $D^{\gamma,1}$. Also, by Theorem 2.1, $D^{\gamma,1}$ is cyclic, whence so is the subrepresentation $\rho^{\alpha,\beta}$. However, we are specifically interested here in knowing exactly which functions f are cyclic vectors. To determine this, we introduce another representation of G.

Consider the representation T^γ, acting in $L^2([0,1) \times [0,1))$, that is given by

$$T^{\gamma}_{p,q,r} f(t,s) = e^{-2\pi i r\gamma} e^{2\pi i q t} e^{2\pi i p s} f(\langle t - p\gamma \rangle, s),$$

where again $\langle x \rangle$ denotes the fractional part of x.

3.1. THEOREM. *Let α and β be arbitrary positive scales, and write γ for $\alpha\beta$.*

(1) *T^γ is unitarily equivalent to $D^{\gamma,1}$.*

(2) *A function $f(t,s)$ is a cyclic vector for T^γ if and only if for almost all $0 \leq t < 1$ the sequence of functions $\{f^{n,t}(s)\}$ spans a dense subspace of $L^2([0,1))$, where*

$$f^{n,t}(s) = e^{2\pi i n s} f(\langle t - n\gamma \rangle, s).$$

(3) *If* $\gamma = 1$, *then an* $f \in L^2(\mathbf{R})$ *satisfies the Gabor condition for scales* α *and* β *if and only if*

$$\sum_{n=-\infty}^{\infty} f(t+n\beta)e^{2\pi ins} \neq 0$$

for almost all t *and* s.

Proof. Define $V : L^2([0,1) \times \mathbf{Z}) \to L^2([0,1) \times [0,1))$ by

$$Vf(t,s) = \sum_{n=-\infty}^{\infty} f(\langle t - n\gamma \rangle, n)e^{2\pi ins}.$$

One checks directly that V is unitary and that

$$V \circ D^{\gamma,1}_{p,q,r} \circ V^{-1} = T^{\gamma}_{p,q,r}.$$

This shows part 1.

Suppose next that $f \in L^2([0,1) \times [0,1))$ and that $h \in L^2([0,1) \times [0,1))$ is orthogonal to each function $T^{\gamma}_{p,q,r}f$. Fixing p and r we then have

$$0 = \int_0^1 e^{2\pi iqt}\Big[\int_0^1 e^{2\pi ips} f(\langle t - p\gamma\rangle, s)\bar{h}(t,s)\,ds\Big]\,dt,$$

which implies that for almost all $0 \leq t < 1$ we have

$$0 = \int_0^1 e^{2\pi ips} f(\langle t - p\gamma\rangle, s)\bar{h}(t,s)\,ds = \int_0^1 f^{p,t}(s)\bar{h}(t,s)\,ds.$$

Now, f is a cyclic vector for T^{γ} if and only if the only vector h that is orthogonal to each $T^{\gamma}_{p,q,r}f$ is the zero vector, which we see is so if and only if the functions $\{f^{n,t}\}$ span a dense subspace of $L^2([0,1))$. This completes the proof of part 2.

Finally, if $\gamma = 1$, then $\rho^{\alpha,\beta}$ is equivalent to T^1. An element f of $L^2(\mathbf{R})$ is then a cyclic vector for $\rho^{\alpha,\beta}$ if and only if the function $V(U^{\alpha,\beta,1}(f))$ is cyclic for T^1. By part 2, f is cyclic for $\rho^{\alpha,\beta}$ if and only if for almost all $0 \leq t < 1$ the sequence $\{f^{n,t}\}$ spans a dense subspace of $L^2([0,1))$, where

$$f^{n,t}(s) = e^{2\pi ins} \sum_{j=-\infty}^{\infty} f((t+j)/\alpha)e^{2\pi ijs}.$$

Clearly these functions span a dense subspace if and only if

$$\sum_{n=-\infty}^{\infty} f(t+n\beta)e^{2\pi ins} \neq 0$$

for almost all s, and this completes the proof.

3.2. COROLLARY. *Let α and β be positive, and suppose $\gamma = \alpha\beta < 1$. Then a function $f(t) \in L^2(\mathbf{R})$ is a cyclic vector for the representation $\rho^{\alpha,\beta}$ (i.e., satisfies the Gabor condition for scales α and β) if and only if for almost all $0 \le t < \gamma$, the set of sequences $\{f^{n,t}\}$ spans a dense subspace of $R(\alpha,\beta,t)$, where*

$$f^{n,t} = \eta_{\alpha,\beta,t}(f_{n\beta}).$$

Proof. Since $\gamma < 1$, the map $U^{\alpha,\beta,1}$, defined in the proof of Theorem 2.1, is a unitary equivalence between the representation $\rho^{\alpha,\beta}$ and a subrepresentation of $D^{\gamma,1}$. We may easily adapt the proof of part 2 of Theorem 3.1 to show that f is a cyclic vector for $\rho^{\alpha,\beta}$ if and only if for almost all $0 \le t < 1$ the sequence of functions $\{[V(U^{\alpha,\beta,1}(f))]^{n,t}\}$ spans a dense subspace of the set of all functions $\phi \in L^2([0,1))$ such that $\phi(s) = V(U^{\alpha,\beta,1}(h))(t,s)$ for some $h \in L^2(\mathbf{R})$. For each integer n and each $0 \le t < 1$, let $\{a_j^{n,t}\}$ denote the sequence defined by

$$a_j^{n,t} = U^{\alpha,\beta,1}f(\langle t - j\gamma\rangle, j - n) = f((\langle t - j\gamma\rangle + j\gamma - n\gamma)/\alpha) = \eta_{\alpha,\beta,t}(f_{n\beta}).$$

We see then that f is a cyclic vector for $\rho^{\alpha,\beta}$ if and only if for almost all $0 \le t < 1$ the set of sequences $\{a^{n,t}\}$ spans a dense subspace of the set of all sequences $\{a_j\} \in l^2$ for which there exists an element $h \in L^2(\mathbf{R})$ such that

$$a_j = U^{\alpha,\beta,1}h(\langle t - j\gamma\rangle, j) = h((\langle t - j\gamma\rangle + j\gamma)/\alpha) = \eta_{\alpha,\beta,t}(h)_j).$$

That is, f is a cyclic vector for $\rho^{\alpha,\beta}$ if and only if for almost all $0 \le t < 1$ the set of sequences $\{\eta_{\alpha,\beta,t}(f_{n\beta})\}$ spans a dense subspace of $R(\alpha,\beta,t)$.

Finally, if $0 \le t < 1-\gamma$, then we see that the sequences $\{\eta_{\alpha,\beta,t}(f_{n\beta})\}$ span a dense subspace of $R(\alpha,\beta,t)$ if and only if the sequences $\{\eta_{\alpha,\beta,t+\gamma}(f_{n\beta})\}$ span a dense subspace of $R(\alpha,\beta,t+\gamma)$. This clearly reduces to the assertion of the corollary.

Remark. Of course, part 3 of Theorem 3.1 implies Theorem 1.2. The corollary implies Theorem 1.4.

3.3. THEOREM. *Let α and β be positive numbers, and suppose that $\gamma = \alpha\beta > 1$. Then the representation $\rho^{\alpha,\beta}$ is not a cyclic representation, i.e., no function $f \in L^2(\mathbf{R})$ is a cyclic vector for $\rho^{\alpha,\beta}$.*

Proof. Assume false. Then the von Neumann algebra $M^{\gamma,\gamma}$ has a cyclic vector. Also, each operator $\rho^{\alpha,\beta}_{p,q,r}$ commutes with the representation $\rho^{1/\beta,1/\alpha}$. Therefore, the von Neumann algebra $(M^{1/\gamma,1/\gamma})'$ has a cyclic vector. Since $1/\gamma < 1$, we know that the representation $\rho^{1/\beta,1/\alpha}$ also has a cyclic vector, whence the von Neumann algebra $M^{1/\gamma,1/\gamma}$ has both a cyclic vector and a separating vector.

Now the two von Neumann algebras $M^{1/\gamma,1/\gamma}$ and $M^{1/\gamma,1}$ both have cyclic and separating vectors, and, by Theorem 2.1, they are isomorphic von

Neumann algebras. It then follows from Corollaries 1.13 and 1.14 of Chapter V of [tak] that these von Neumann algebras are spatially isomorphic, whence the representations $D^{1/\gamma,1/\gamma}$ and $D^{1/\gamma,1}$ would be unitarily equivalent. This contradicts Theorem 2.1, and the proof is complete.

Remark. Of course this theorem implies Theorem 1.3.

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