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ON THE SZ.-NAGY BOUNDEDNESS CONDITION ON ABELIAN INVOLUTION SEMIGROUPS

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Following Szafraniec [7] we consider positive definite forms over (S, X), where S is an Abelian involution semigroup and X is a Baire topological linear space. It is shown that under suitable assumptions such forms satisfy the boundedness condition if and only if they satisfy the weak boundedness condition.

1. Preliminaries. In what follows $\boldsymbol{F}$ is either $\boldsymbol{R}$ or $\boldsymbol{C}$, endowed with the natural involution $-$ and the Euclidean norm $|\cdot|$ ($|a|^2 = a\bar{a}$, $a \in \boldsymbol{F}$).

Let X be a linear space over $\boldsymbol{F}$ and let $(S, +, *)$ be an involution Abelian semigroup. We have in mind a semigroup without zero, but formally this shall not be required. Fix once for all an element $u \in S$. Following Szafraniec [7] we say that a map $f\colon S \times X \times X \to \boldsymbol{F}$ is a *form over* (S, X) if, for every $s \in S$, $f(s; \cdot, -)$ is a sesquilinear form over X. A form f is called *positive definite* (in short, PD) if and only if

$$(1)\quad \sum_{i,j=1}^{n} f(s_i^* + s_j; x_j, x_i) \geqslant 0, \qquad s_1, \ldots, s_n \in S,\ x_1, \ldots, x_n \in X,\ n = 1, 2, \ldots,$$

and

$$(2)\qquad f(s^* + t; x, y)^- = f(t^* + s; y, x), \qquad s, t \in S,\ x, y \in X.$$

Notice that if $\boldsymbol{F} = \boldsymbol{C}$, then (2) follows from (1). Interesting examples of such forms can be found in [7].

The boundedness condition has been discussed in a series of papers in connection with general dilation theorems (cf. [8], [1], [3]–[5], [7]). The aim of this paper is to propose a new form of it and to discuss relationship between these two versions of the boundedness condition.

A PD form f is said to satisfy the *boundedness condition* (BC) at u if there exists a number $c \geqslant 0$ (depending only on u) such that $cf - f^u$ is PD, f^u being a form over (S, X) defined by

$$f^u(s; x, y) = f(s + u^* + u; x, y), \qquad x, y \in X,\ s \in S.$$

A PD form f is said to satisfy the *weak boundedness condition* (WBC) at u if for every $x \in X$ the form f_x over $(S, \boldsymbol{F})$ defined by

$$f_x(s; a, b) = f(s; x, x)a\bar{b}, \quad a, b \in \boldsymbol{F}, \ s \in S,$$

satisfies BC at u.

It is known (cf. [7] and [5]) that a PD form f satisfies BC at u if and only if all the forms f_x, $x \in X$, satisfy BC at u with the same constant c. This means that BC always implies WBC. Below we present an example of a PD form which satisfies WBC but not BC.

Example. The set $\boldsymbol{Z}_+ = \{0, 1, 2, \ldots\}$ with the addition as a semigroup operation and the identity map as an involution becomes an Abelian involution semigroup with zero. Let $(H, (\cdot, -))$ be a separable Hilbert space over $\boldsymbol{F}$ with an orthonormal basis $(e_n)_{n=1}^{\infty}$. Denote by X the linear span of the set $\{e_n: n = 1, 2, \ldots\}$. Then $(X, (\cdot, -))$ is a pre-Hilbert space. Consider the form f over $(\boldsymbol{Z}_+, X)$ defined by the formula

$$f(n; x, y) = \sum_{k=1}^{\infty} a_{k,n}(x, e_k)\overline{(y, e_k)}, \quad x, y \in X,$$

where

$$a_{k,n} = \int_0^k t^n e^{-t} dt, \quad k = 1, 2, \ldots, n \in \boldsymbol{Z}_+.$$

It is plain that f is a PD form which satisfies WBC at each n, $n \in \boldsymbol{Z}_+$. Since

$$\sup\{a_{k,n}: k = 1, 2, \ldots\} = \int_0^{\infty} t^n e^{-t} dt = n!, \quad n \in \boldsymbol{Z}_+,$$

one can prove that for each $n \in \boldsymbol{Z}_+$ the sesquilinear form $f(n; \cdot, -)$ is bounded and jointly continuous in consequence. Notice that if $x = e_k$ ($k = 1, 2, \ldots$), then, in virtue of Remark 2 in [7], the form f_x satisfies BC at n with the minimal constant c_x equal to

$$\begin{aligned}\lim_{m \to \infty} f_x\big(m(2n)\big)^{1/m} &= \lim_{m \to \infty} \Big(\int_0^k (t^{2n})^m e^{-t} dt\Big)^{1/m} \\ &= \operatorname{ess\,sup}\{t^{2n}: t \in [0, k]\} = k^{2n}.\end{aligned}$$

This implies that f does not satisfy BC at any $n \geqslant 1$.

2. Main result. Now we show that under additional topological assumptions about X the situation described by the Example cannot happen.

Theorem. *Let f be a PD form over (S, X), which satisfies WBC at u. Suppose that*

(i) *X is a Baire topological linear space over $\boldsymbol{F}$,*

(ii) *for each $k = 1, 2, \ldots$ the function $f\big(k(u^* + u); \cdot, \cdot\big)$ is lower semicontinuous.*

Then f satisfies BC at u and the form $f(ku+lu^*; \cdot, -)$ is jointly continuous for all $k, l \in \boldsymbol{Z}_+$ such that $k+l > 1$.

Proof. Define the function q_n: $X \to \boldsymbol{R}_+$ by

$$q_n(x) = f\big(n(u^*+u); x, x\big)^{1/2}, \qquad x \in X.$$

The assumptions imply that q_n is a lower semicontinuous seminorm on X for each $n = 1, 2, \ldots$ Since f_x satisfies BC at u, Lemma 1 (cf. Appendix) guarantees that

$$\sup\{q_n(x)^{2/n}: n = 1, 2, \ldots\} < +\infty \qquad \text{for every } x \in X.$$

Thus by Lemma 2 (cf. Appendix) there is an open neighborhood V of 0 such that condition (ii) of Lemma 2 is satisfied with $a_n = n/2$. Now, if $x \in X$, then there is $a \in \boldsymbol{F}$ such that $a \neq 0$ and $ax \in V$. Therefore

$$\lim_{n\to\infty} f\big(n(u^*+u); x, x\big)^{1/n} \leqslant \sup\{q_n(ax)^{2/n}: n = 1, 2, \ldots\}$$
$$\leqslant \sup\{(\sup_{y\in V} q_n(y))^{2/n}: n = 1, 2, \ldots\} < +\infty.$$

Thus, in virtue of Lemma 1, f satisfies BC at u.

Since f is PD, there exist a Hilbert space H over $\boldsymbol{F}$ and a family $\{D(s): s \in S\}$ of linear maps from X into H such that

$$f(s^*+t; x, y) = \big(D(t)x, D(s)y\big)_H, \qquad s, t \in S, \; x, y \in X. \tag{3}$$

In particular, we have

$$f\big(k(u^*+u); x, x\big) = \|D(ku)x\|^2 = \|D(ku^*)x\|^2, \qquad x \in X, \; k = 1, 2, \ldots \tag{4}$$

Since $f\big(k(u^*+u); \cdot, \cdot\big)^{1/2}$ is a lower semicontinuous seminorm on a Baire topological linear space X, it is automatically continuous. Thus, in virtue of (4), the operators $D(ku)$ and $D(ku^*)$ are continuous for each $k = 1, 2, \ldots$ Now the Theorem follows from (3).

As the Example shows the assumption (i) of the Theorem must not be omitted, even if X is a pre-Hilbert space. Also the assumption (ii) of the Theorem must not be omitted. To see this it is enough to modify the Example by extending the form f over $(\boldsymbol{Z}_+, X)$ to a form $\hat{f}$ over $(\boldsymbol{Z}_+, H)$ as follows:

$$\hat{f}(n; h, k) = f(n; Ph, Pk), \qquad h, k \in H, \; n \in \boldsymbol{Z}_+,$$

where P is a noncontinuous projection of H onto X.

Though the assumption (ii) of the Theorem must not be omitted generally, we are able to replace it by a weaker one, provided X has an additional property.

COROLLARY. *Let f be a PD form over (S, X), which fulfils WBC at u. Suppose that X is a Baire metrizable topological linear space over $\boldsymbol{F}$ and*

$f(u^*+u; x, \cdot)$ is continuous for each $x \in X$. Then f satisfies BC at u and the forms $f(s+u; \cdot, -)$, $s \in S$, are jointly continuous.

Proof. Since X is a Baire metrizable topological linear space over $\mathbf{F}$, each separately continuous sesquilinear form over X is jointly continuous (cf. [9], Exercise 41.5). In particular, $f(s^*+t; \cdot, -)$ is jointly continuous if and only if it is separately continuous or, equivalently, by (2), if $f(s^*+t; x, \cdot)$ and $f(t^*+s; x, \cdot)$ are continuous for each $x \in X$ $(s, t \in S)$. Since the form $f(u^*+u; \cdot, -)$ has the above-mentioned property, it is jointly continuous. This implies the continuity of the operators $D(u)$ and $D(u^*)$ (use (4)). Thus, by (3), $f(u+s; x, \cdot)$ and $f(u^*+s^*; x, \cdot)$ are continuous for all $x \in X$ and $s \in S$. This gives us the joint-continuity of $f(s+u; \cdot, -)$ and $f(s+u^*; \cdot, -)$, $s \in S$. Now the Corollary follows from the Theorem.

3. Appendix. Here we formulate two lemmas needed in the proof of the Theorem. The first one is a slightly modified result of Szafraniec (cf. [7], Remark 2).

LEMMA 1. *Let f be a PD form over (S, X). Then f satisfies BC at u if and only if there exists a number $d \geqslant 0$ (depending only on u) such that*

$$\lim_{n\to\infty} f(n(u^*+u); x, x)^{1/n} \leqslant d, \qquad x \in X.$$

Proof. Similarly as in the proof of Proposition 1 in [6] one can prove that the sequence $a_n := f(n(u^*+u); x, x)$, $n = 1, 2, \ldots$, satisfies the inequality $a_n^2 \leqslant a_k a_m$ with $k+m = 2n$. Thus $\lim\limits_{n\to\infty} a_n^{1/n}$ always exists (finite or not).

The "if" part of Lemma 1 is due to Szafraniec (cf. [7], Remark 2). The "only if" part of Lemma 1 can be proved as follows. First notice that if f satisfies BC at u, then there exists a positive function c defined on the involution semigroup $S(u)$ generated by u such that $c(s+t) \leqslant c(s)c(t)$, $s, t \in S(u)$, and

$$f(t^*+s^*+s+t; x, x) \leqslant c(s) f(t^*+t; x, x), \qquad s \in S(u),\ t \in S,\ x \in X.$$

Thus

$$\begin{aligned} f(n(u^*+u); x, x) &= f(u^*+(n-1)u^*+(n-1)u+u; x, x) \\ &\leqslant c(u)^{n-1} f(u^*+u; x, x), \end{aligned}$$

which implies

$$\lim_{n\to\infty} f(n(u^*+u); x, x)^{1/n} \leqslant c(u), \qquad x \in X.$$

This completes the proof.

The next lemma can be deduced from the Osgood theorem (cf. [2], Theorem 6.1.2).

LEMMA 2. *Suppose X is a Baire topological linear space over $\mathbf{F}$ and $(a_n)_{n=1}^{\infty}$*

is a sequence of positive numbers such that

$$\inf\{a_n\colon n = 1, 2, \ldots\} > 0.$$

Let $(q_n)_{n=1}^{\infty}$ *be a sequence of lower semicontinuous seminorms on* X *such that*

(i) $$\sup\{q_n(x)^{1/a_n}\colon n = 1, 2, \ldots\} < +\infty, \quad x \in X.$$

Then there exists an open neighborhood V *of* 0 *in* X *such that*

(ii) $$\sup\{(\sup_{x\in V} q_n(x))^{1/a_n}\colon n = 1, 2, \ldots\} < +\infty.$$

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