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CYCLIC ACTIONS ON S^2- AND P^2-BUNDLES OVER S^1

BY

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1. Preliminaries. This paper is a continuation of papers [8] and [9], and is based on their terminology. We work in the PL-category (our results are valid for Diff-category without any changes). We classify all effective Z_n-actions on S^2- and P^2-bundles over S^1, proving that they are standard (Definition 1.1). We use the recent results of W. Meeks III, S. Yau, and W. Thurston.

Let N denote a nonorientable S^2-bundle over S^1, B a Klein bottle, and Bs a solid Klein bottle. Let

$$S^3 = \{z_1, z_2 \in C\colon |z_1|^2 + |z_2|^2 = 1\}$$

and assume that

$$P^3 = S^3/\sim, \quad (z_1, z_2) \sim (-z_1, -z_2),$$
$$S^2 = \{z \in C,\ x \in R\colon |z|^2 + x^2 = 1\};$$
$$P^2 = S^2/\sim, \quad (z, x) \sim (-z, -x), \quad S^1 = \{z \in C\colon |z| = 1\}.$$

Let $g_{(n,h)}$ (where h is taken modulo n) be defined by

$$g_{(n,h)}\colon S^1 \to S^1, \quad g_{(n,h)}(z) = e^{2\pi ih/n} z.$$

The same notation is used for

(a) $g_{(n,h)}\colon S^3 \to S^3$, $g_{(n,h)}(z_1, z_2) = (e^{2\pi ih/n} z_1, z_2)$;

(b) $g_{(n,h)}\colon S^2 \to S^2$, $g_{(n,h)}(z, x) = (e^{2\pi ih/n} z, x)$;

(c) $g_{(n,h)}\colon P^3 \to P^3$ and $g_{(n,h)}\colon P^2 \to P^2$ are determined by (a) and (b), respectively.

Let $C\colon S^2 \to S^2$, $C(z, x) = (z, -x)$, and $C_0\colon S^2 \to S^2$, $C_0(z, x) = (\bar{z}, x)$. Let A denote the antipodal map.

1.1. Definition. Let $M = S^1 \hat{\times} F = R \times F/\sim$, $(t, y) \sim (t+1, \varphi(y))$, be an F-bundle over S^1, where F is a manifold and φ is a self-homeomorphism of F. An action of Z_n (generated by T) on M is said to be *standard* if it may be described by one of the following expressions:

1. $T(t, y) = \big(t+i/s, g_0(y)\big)$, where g_0 is a self-homeomorphism of F, s divides n, $0 < i \leqslant s$, $(i, s) = 1$ (i.e., i and s are relatively prime), $g_0^n = \varphi^{ni/s}$, and $g_0\varphi = \varphi g_0$. We shall call such an action a *standard action of type* $(1; n, s, i, g_0)_\varphi$.

2. $T(t, y) = \big(1-t, g_0(y)\big)$, where n is even, g_0 generates a Z_n-action on F, and $\varphi g_0 = g_0\varphi^{-1}$. We call such an action a *standard action of type* $(2; n, g_0)_\varphi$.

1.2. Definition. Two G-actions μ_1, μ_2: $G\times M \to M$ are called *weakly conjugate* if there exist a group automorphism A: $G \to G$ and a self-homeomorphism f: $M \to M$ (f preserves orientation if M is orientable) such that $\mu_1 = f^{-1}\mu_2(A\times f)$. If A is the identity, then μ_1 and μ_2 are conjugate.

Let $\mathrm{Iz}(M)$ denote the set of points with nontrivial isotropy group (for a given G-action on M). Let $\mathscr{A}$ denote some class of actions. Then $\xi_c(\mathscr{A})$ (respectively, $\xi_w(\mathscr{A})$) denotes the number of actions in $\mathscr{A}$ up to conjugation (respectively, up to weak conjugation). Let $\varphi(n)$ denote the cardinality of the set of all natural numbers relatively prime to n, less than n, and let $[n]$ denote the integer part of n.

The following statement (cf. [1]) seems to be a "folklore" result (up to the Smith Conjecture):

1.3. Proposition. *Let a group G act effectively on S^3 and assume that either* (a) *or* (b) *holds*:

(a) $G = Z_{2n}$ *and* $\mathrm{Fix}(T^2) \neq \emptyset$, *where T is a generator of the action*;

(b) $G = Z_n \oplus Z_2$ *and* $\mathrm{Fix}(T^2) \neq \emptyset$, *where T is a generator of* Z_n.

Then such an action of G is conjugate to an orthogonal action.

1.4. Lemma. *Each effective action of $Z_2\oplus Z_2$ on S^3 is conjugate to an orthogonal action. That is, each action takes one of the following forms* (*for some generators T_1 and T_2*):

(i) $T_1(z_1, z_2) = (-z_1, -z_2)$, $T_2(z_1, z_2) = (z_1, \bar{z}_2)$;

(ii) $T_1(z_1, z_2) = (-z_1, -z_2)$, $T_2(z_1, z_2) = (z_1, -z_2)$;

(iii) $T_1(z_1, z_2) = (-z_1, \bar{z}_2)$, $T_2(z_1, z_2) = (-z_1, z_2)$;

(iv) $T_1(z_1, z_2) = (-z_1, \bar{z}_2)$, $T_2(z_1, z_2) = (z_1, -z_2)$;

(v) $T_1(z_1, z_2) = (-z_1, z_2)$, $T_2(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$;

(vi) $T_1(z_1, z_2) = (-z_1, z_2)$, $T_2(z_1, z_2) = (\bar{z}_1, z_2)$.

Proof. Some element T of $Z_2\oplus Z_2$ satisfies either ($*$) $\mathrm{Fix}(T) = S^1$ or ($**$) $\mathrm{Fix}(T) = S^2$. Otherwise, there exists a free action of $Z_2\oplus Z_2$ on S^3 with holes and, consequently, there exists a 3-manifold with fundamental group equal to $Z_2\oplus Z_2$, which is impossible. Now, we consider two possibilities:

(*) $\mathrm{Fix}(T) = S^1$; we divide S^3 by T. Let T_0 be a second generator of $Z_2 \oplus Z_2$. The generator T_0 determines the involution on $(S^3, S^1) = (S^3, \mathrm{Fix}(T))/T$. Involutions on S^3 are classified (see [4] and [5]). Similarly, we can classify involutions on (S^3, S^1), where S^1 is unknotted. Therefore, we complete easily the proof of Lemma 1.4 in case (*).

(**) $\mathrm{Fix}(T) = S^2$; we divide S^3 by T. Let T_0 be a second generator of $Z_2 \oplus Z_2$. The generator T_0 determines the involution on $(D^3, \partial D^3) = (S^3, \mathrm{Fix}(T))/T$. Involutions on D^3 (and $(D^3, \partial D^3)$) are classified. Using this classification, we complete the proof in case (**).

Outline of the proof of Proposition 1.3. It follows from the Smith Conjecture (just proved by W. Thurston) that T^2 is conjugate to an orthogonal action. We divide S^3 by the cyclic action generated by T^2 to obtain S^3 with an involution or with a $(Z_2 \oplus Z_2)$-action. Now we use theorems of Livesay or Lemma 1.4.

We will consider several cases. Let T preserve orientation. T determines an orientation-preserving involution T_0 on $(S^3, S^1) = (S^3, \mathrm{Fix}(T^2))/T^2$. We have the following possibilities:

(i) $\mathrm{Fix}(T) \neq \emptyset$. Then $\mathrm{Fix}(T) = \mathrm{Fix}(T^2)$ and it follows from the Smith Conjecture that $T(z_1, z_2) = (g_{(2n,h)}(z_1), z_2)$.

(ii) $\mathrm{Fix}(T) = \emptyset$. Then T and T_0 act freely on $\mathrm{Fix}(T^2)$ and S_0^1, respectively; we have two possibilities:

(j) $\mathrm{Fix}(T_0)$ is a circle. Then $T_0(z_1, z_2) = (z_1, -z_2)$ and we may assume that $S_0^1 = \{z_1, z_2 \in S^3: z_1 = 0\}$ (we analyze the involution T_0 on S^3-$\mathrm{int}(X)$, where X is an invariant regular neighborhood of S_0^1, and we use the fact that each involution on $S^1 \times D^2$ with $\mathrm{Fix}(\cdot) = S^1$ is standard [11]). Thus it follows easily that $T(z_1, z_2) = (g_{(n,h)}(z_1), -z_2)$, where n is odd.

(jj) $\mathrm{Fix}(T_0) = \emptyset$. Then $T_0(z_1, z_2) = (-z_1, -z_2)$ and we may assume that $S_0^1 = \{z_1, z_2 \in S^3: z_1 = 0\}$. Now we infer easily that $T(z_1, z_2) = (g_{(2n,h)}(z_1), z_2)$.

Similarly (but with more complications) we may proceed with other cases of Lemma 1.3.

1.5. COROLLARY. *Let T be a generator of an effective Z_n-action on $I \times P^2$. Then we have one of the following possibilities for T (up to conjugation):*

1. *T preserves the components of $\partial(I \times P^2)$: $T = \mathrm{Id} \times g_{(n,h)}$.*

2. *T reverses the components of $\partial(I \times P^2)$:*

(a) $T = A \times g_{(n,h)}$, *n is even, $(n, h) = 1$;*

(b) $T = A \times g_{(n/2,h)}$, *$n/2$ is odd, $(n/2, h) = 1$.*

Proof. Let $p: I \times S^2 \to I \times P^2$ denote the universal covering. Let T_0 be the covering transformation. We can lift T to $\tilde{T}: I \times S^2 \to I \times S^2$ and assume that $\tilde{T}$ preserves orientation (T_0 reverses orientation and $\tilde{T}$ is determined up to T_0). It is easy to see that $\tilde{T}$ commutes with T_0. Since T_0 reverses and $\tilde{T}$ preserves orientation, we infer that $\tilde{T}^n = 1$ and $\{\tilde{T}, T_0\}$ generates the action of $Z_n \oplus Z_2$ on $I \times S^2$. This action can be extended to S^3 in an obvious way. Now Corollary 1.5 follows from Proposition 1.3.

1.6. LEMMA. *Let T generate the action of Z_n on F and let (F, T) be a set of equivariant self-homeomorphisms of F (up to an equivariant isotopy). Then*:

(a) *if $F = S^2$ and T is equal either to $g_{(n,h)}$ or to $g_{(n,h)}C$ (n is even) or to $g_{(n/2,h)}C$ ($n/2$ is odd), then (S^2, T) is generated by* Id *and* C *(and if $n = 2$, we must possibly adjoin C_0 and C_0C)*;

(b) *if $F = P^2$ and $T = g_{(n,h)}$, then (P^2, T) is generated by* Id *(and C_0 if $n = 2$).*

The proof is easy and we omit it.

The recent results of Meeks III and Yau [7] imply the following

1.7. THEOREM. *Let a finite group G act effectively on $S^1 \hat{\times} F$ ($F = S^2$ or P^2). Then there exists an embedding $F \hookrightarrow S^1 \hat{\times} F$ such that $g(F) \cap F = \emptyset$ or F for each $g \in G$, F is in a general position with respect to* $\mathrm{Iz}(S^1 \hat{\times} F)$, *and F does not separate $S^1 \hat{\times} F$.*

2. Actions of Z_n on $S^1 \hat{\times} S^2$.

2.1. THEOREM. *Each effective action of Z_n on $M = S^1 \hat{\times} S^2$ (generated by T) takes one of the following forms (up to conjugation). Each of the cases* I.1(a)-II(i) *describes exactly one class of weakly conjugate actions (unless otherwise specified).*

I. *Actions on $S^1 \times S^2$.*

1. *Actions which preserve orientation*:

(a) *Actions of type $(1;\, n, s, i, g_{(n,h)})_{\mathrm{Id}}$, where $0 < h \leqslant j = n/s$, $(j, h) = 1$. Two such actions, for $s = s', s''$, $i = i', i''$, and $h = h', h''$, are weakly conjugate iff $s'' = s'$ and there exists a natural number a such that*

$$i'' \equiv ai' \text{ or } a(s-i') \pmod s \quad \text{and} \quad h'' \equiv ah' \text{ or } a(j-h') \pmod j.$$

If $a = 1$, then the actions are conjugate. For given s we have

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(s)+1}{2}\right]\left[\frac{\varphi(j)+1}{2}\right], \quad \xi_w\big((\mathrm{a})\big) = \left[\frac{\varphi\big(\mathrm{g.c.d.}(s, j)\big)+1}{2}\right].$$

Each class of actions (up to weak conjugation) contains $\xi_c\big((\mathrm{a})\big)/\xi_w\big((\mathrm{a})\big)$ actions, up to conjugation. Moreover,

$$M^* = M/Z_n = S^1 \times S^2, \quad \mathrm{Iz}(M) = \begin{cases} \emptyset & \text{if } s = n, \\ \mathrm{Fix}(T^s) = S^1 \mathring{\cup} S^1 & \text{if } s < n. \end{cases}$$

(b) *Actions of type* $(1; n, s, i, C_0C)_{\sigma(2,1)}$, *where* $n = 2s$. *Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$ *or* $s - i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(s)+1}{2}\right], \quad M^* = S^1 \times S^2, \quad \mathrm{Iz}(M) = \mathrm{Fix}(T^s) = S^1.$$

(c) *Actions of type* $(2; n, g_{(n,h)}C)_{\mathrm{Id}}$, *where* n *is even and* $(n, h) = 1$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$ *or* $n - h'$; *hence*

$$\xi_c\big((\mathrm{c})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = \emptyset,$$

$$\mathrm{Iz}(M) = \begin{cases} \emptyset & \text{if } n = 2, \\ \mathrm{Fix}(T^2) = S^1 \mathring{\cup} S^1 & \text{if } n > 2, \end{cases} \quad M^* = P^3 \# P^3.$$

(d) *Actions of type* $(2; n, g_{(n/2,h)}C)_{\mathrm{Id}}$, *where* $n/2$ *is odd and* $(n/2, h) = 1$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$ *or* $n/2 - h'$; *hence*

$$\xi_c\big((\mathrm{d})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} S^1 \mathring{\cup} S^1 & \text{if } n = 2, \\ \emptyset & \text{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = S^1 \mathring{\cup} S^1 & \text{if } n = 2, \\ \mathrm{Fix}(T^{n/2}) \cup \mathrm{Fix}(T^2) = S^1 \mathring{\cup} S^1 \mathring{\cup} S^1 \mathring{\cup} S^1 & \text{if } n > 2, \end{cases}$$

$$M^* = S^3.$$

(e) *Actions of type* $(2; n, g_{(n,h)}C)_{\sigma(2,1)}$, *where* n *is even and* $(n, h) = 1$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h', n - h', h' \mp n/2$ *or* $n - (h' \mp n/2)$; *hence*

$$\xi_c\big((\mathrm{e})\big) = \left[\frac{\varphi(n/2)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} S^1 & \text{if } n = 2, \\ \emptyset & \text{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = S^1 & \text{if } n = 2, \\ \mathrm{Fix}(T^{n/2}) \cup \mathrm{Fix}(T^2) = S^1 \mathring{\cup} S^1 \mathring{\cup} S^1 & \text{if } n > 2,\ n/2 \text{ is odd}, \\ \mathrm{Fix}(T^2) = S^1 \mathring{\cup} S^1 & \text{if } n \text{ is a multiple of } 4, \end{cases}$$

$$M^* = \begin{cases} P^3 & \text{if } n/2 \text{ is odd}, \\ P^3 \# P^3 & \text{if } n \text{ is a multiple of } 4. \end{cases}$$

2. *Actions which reverse orientation*:

(a) *Actions of type* $(1; n, s, i, g_{(n,h)}C)_{\mathrm{Id}}$, *where* $0 < h \leqslant n/s = j$, $(j, h) = 1$, *and* n *is even. Two such actions, for* $s = s', s''$, $i = i', i''$, *and*

$h = h', h''$, are weakly conjugate iff $s'' = s'$ and there exists a natural number a such that

$$i'' \equiv ai' \text{ or } a(s-i') \pmod s \quad \text{and} \quad h'' \equiv ah' \text{ or } a(j-h') \pmod j.$$

If $a = 1$, then the actions are conjugate. We have

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(s)+1}{2}\right]\left[\frac{\varphi(j)+1}{2}\right] \quad \text{and} \quad \xi_w\big((\mathrm{a})\big) = \left[\frac{\varphi\big(\text{g.c.d.}(s,j)\big)+1}{2}\right].$$

Each class of actions (up to weak conjugation) contains $\xi_c\big((\mathrm{a})\big)/\xi_w\big((\mathrm{a})\big)$ actions, up to conjugation. Moreover,

$$\mathrm{Iz}(M) = \begin{cases} \emptyset & \text{if } j = 1 \text{ or } (j = 2 \text{ and } s \text{ is odd}),\\ \mathrm{Fix}(T^s) = S^1 \mathring{\cup} S^1 & \text{if } j > 1 \text{ and } s \text{ is even},\\ \mathrm{Fix}(T^{2s}) = S^1 \mathring{\cup} S^1 & \text{if } j > 2 \text{ and } s \text{ is odd},\end{cases}$$

$$M^* = \begin{cases} N & \text{if } s \text{ is even},\\ S^1 \times P^2 & \text{if } s \text{ is odd}.\end{cases}$$

(b) *Actions of type $(1;\, n, s, i, C_0)_{g(2,1)}$, where $n = 2s$ and i is odd. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$ or $s-i'$; hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(s)+1}{2}\right], \qquad \mathrm{Fix}(T) = \emptyset,$$

$$\mathrm{Iz}(M) = \mathrm{Fix}(T^s) = \begin{cases} S^1 \mathring{\cup} S^1 & \text{if } s \text{ is even},\\ B & \text{if } s \text{ is odd},\end{cases}$$

$$M^* = \begin{cases} N & \text{if } s \text{ is even},\\ Bs & \text{if } s \text{ is odd}.\end{cases}$$

(c) *Actions of type $(1;\, n, s, i, g_{(n/2,h)}C)_{\mathrm{Id}}$, where s and $n/2s$ are odd, $0 < h \leqslant n/2s = j/2$, and $(j/2, h) = 1$. Two such actions, for $s = s', s''$, $i = i', i''$, and $h = h', h''$, are weakly conjugate iff $s'' = s'$ and there exists a natural number a such that*

$$i'' \equiv ai' \text{ or } a(s-i') \pmod s \quad \text{and} \quad h'' \equiv ah' \text{ or } a(j/2-h') \pmod{j/2}.$$

If $a = 1$, then the actions are conjugate. We have

$$\xi_c\big((\mathrm{c})\big) = \left[\frac{\varphi(s)+1}{2}\right]\left[\frac{\varphi(j)+1}{2}\right] \quad \text{and} \quad \xi_w\big((\mathrm{c})\big) = \left[\frac{\varphi\big(\text{g.c.d.}(s,j)\big)+1}{2}\right].$$

Each class of actions (up to weak conjugation) contains $\xi_c((\mathrm{c}))/\xi_w((\mathrm{c}))$ actions, up to conjugation. Moreover,

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T^{n/2}) = S^1 \times S^1 & \text{if } n = 2s, \\ \mathrm{Fix}(T^{2s}) \cup \mathrm{Fix}(T^{n/2}) = S^1 \times S^1 \overset{\circ}{\cup} S^1 \overset{\circ}{\cup} S^1 & \text{if } n > 2s, \end{cases}$$

$$M^* = D^1 \times D^2.$$

(d) *Actions of type $(2;\ n, g_{(n,h)})_{\mathrm{Id}}$, where n is even and $(n, h) = 1$. Two such actions, for $h = h', h''$, are conjugate iff $h'' = h'$ or $n - h'$; henec*

$$\xi_c((\mathrm{d})) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = 4 \text{ points},$$

$$\mathrm{Iz}(M) = \begin{cases} 4 \text{ points} & \text{if } n = 2, \\ \mathrm{Fix}(T^2) = S^1 \overset{\circ}{\cup} S^1 & \text{if } n > 2, \end{cases}$$

$$M^* = \langle 0, 1\rangle \times S^2/\sim, \quad (i, x) \sim (i, g_{(2,1)}(x)) \quad (i = 1, 2).$$

(e) *Actions of type $(2;\ n, g_{(n/2,h)})_{\mathrm{Id}}$, where $n/2$ is odd and $(n/2, h) = 1$. Two such actions, for $h = h', h''$, are conjugate iff $h'' = h'$ or $n/2 - h'$; hence*

$$\xi_c((\mathrm{e})) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} S^2 \overset{\circ}{\cup} S^2 & \text{if } n = 2, \\ 4 \text{ points} & \text{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = S^2 \overset{\circ}{\cup} S^2 & \text{if } n = 2, \\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = (S^1 \overset{\circ}{\cup} S^1) \cup (S^2 \overset{\circ}{\cup} S^2) & \text{if } n > 2, \end{cases}$$

$$M^* = \langle 0,\ 1\rangle \times S^2.$$

(f) *Actions of type $(2;\ n, g_{(n,h)})_{g(2,1)}$, where n is even and $(n, h) = 1$. Two such actions, for $h = h', h''$, are conjugate iff $h'' = h'$, $n - h'$, $h' \mp n/2$ or $n - h' \mp n/2$; hence*

$$\xi_c((\mathrm{f})) = \left[\frac{\varphi(n/2)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} 2 \text{ points} \cup S^2 & \text{if } n = 2, \\ 4 \text{ points} & \text{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = 2 \text{ points} \cup S^2 & \text{if } n = 2, \\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = (S^1 \overset{\circ}{\cup} S^1) \cup S^2 & \text{if } n/2 > 1 \text{ and } n/2 \text{ is odd}, \\ \mathrm{Fix}(T^2) = S^1 \overset{\circ}{\cup} S^1 & \text{if } n \text{ is a multiple of } 4, \end{cases}$$

$$M^* = \begin{cases} [0, 1] \times S^2/\sim,\ (1, x) \sim (1, g_{(2,1)}(x)), & \text{if } n/2 \text{ is odd}, \\ [0, 1] \times S^2/\sim,\ (i, x) \sim (i;\ g_{(2,1)}(x))\ (i = 1, 2), & \text{if } n \text{ is a multiple of } 4. \end{cases}$$

II. *Actions of* Z_n *on* $S^1 \hat{\times} S^2 = N$:

(a) *Actions of type* $(1;\ n, s, i, g_{(n,h)}C)_C$, *where* s *and* i *are odd,* $0 < h \leqslant n/s = j$, *and* $(h, j) = 1$. *Two such actions, for* $s = s', s''$, $i = i', i''$, *and* $h = h', h''$, *are weakly conjugate iff* $s'' = s'$ *and there exists a natural number* a *such that*

$$i'' \equiv ai' \text{ or } a(s-i') \pmod{s} \quad \text{and} \quad h'' \equiv ah' \text{ or } a(j-h') \pmod{j}.$$

If $a = 1$, *then the actions are conjugate. We have*

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(s)+1}{2}\right]\left[\frac{\varphi(j)+1}{2}\right], \quad \xi_w\big((\mathrm{a})\big) = \left[\frac{\varphi\big(\text{g.c.d.}(s,j)\big)+1}{2}\right].$$

Each class of actions (up to weak conjugation) contains $\xi_c\big((\mathrm{a})\big)/\xi_w\big((\mathrm{a})\big)$ *actions, up to conjugation. Moreover, we have* $\mathrm{Iz}(M) = \mathrm{Fix}(T^s) = S^1$ *and* $M^* = N$.

(b) *Actions of type* $(1;\ n, s, i, g_{(2,1)}C_0)_{C_0}$, *where* $n = 2s$, *and* s, i *are odd. Two such actions, for* $i = i', i''$, *are conjugate iff* $i'' = i'$; *hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Iz}(M) = \mathrm{Fix}(T^s) = S^1 \mathring{\cup} S^1, \quad M^* = N.$$

(c) *Actions of type* $(1;\ n, s, i, g_{(n,h)})_C$, *where* $0 < h \leqslant n/s = j$, j *is even,* i *is odd, and* $(j, h) = 1$. *Two such actions, for* $s = s', s''$, $i = i', i''$, *and* $h = h', h''$, *are weakly conjugate iff* $s'' = s'$ *and there exists a natural number* a *such that*

$$i'' \equiv ai' \text{ or } a(2s-i') \pmod{2s} \quad \text{and} \quad h'' \equiv ah' \text{ or } a(j-h') \pmod{j}.$$

If $a = 1$, *then the actions are conjugate. We have*

$$\xi_c\big((\mathrm{c})\big) = \left[\frac{\varphi(2s)+1}{2}\right]\left[\frac{\varphi(j)+1}{2}\right], \quad \xi_w\big((\mathrm{c})\big) = \left[\frac{\varphi\big(\text{g.c.d.}(2s,j)\big)+1}{2}\right].$$

Each class of actions (up to weak conjugation) contains $\xi_c\big((\mathrm{c})\big)/\xi_w\big((\mathrm{c})\big)$ *actions, up to conjugation. Moreover,*

$$\mathrm{Iz}(M) = \begin{cases} \emptyset & \text{if } j = 2, \\ \mathrm{Fix}(T^{2s}) = S^1 & \text{if } j \neq 2, \end{cases} \quad M^* = S^1 \times P^2.$$

(d) *Actions of type* $(1;\ n, s, i, g_{(n/2,h)})_C$, *where* $0 < h \leqslant n/s = j$, $j/2$ *and* i *are odd, and* $(n/2, h) = 1$. *Two such actions, for* $s = s' s''$, $i = i', i''$, *and* $h = h', h''$, *are weakly conjugate iff* $s'' = s'$ *and there exists a natural number* a *such that*

$$i'' \equiv ai' \text{ or } a(2s-i') \pmod{2s} \quad \text{and} \quad h'' \equiv ah' \text{ or } a(j-h') \pmod{j}.$$

If $a = 1$, then the actions are conjugate. We have

$$\xi_c\big((\mathrm{d})\big) = \left[\frac{\varphi(2s)+1}{2}\right]\left[\frac{\varphi(j)+1}{2}\right], \quad \xi_w\big((\mathrm{d})\big) = \left[\frac{\varphi\big(\text{g.c.d.}(2s, j)\big)+1}{2}\right].$$

Each class of actions (up to weak conjugation) contains $\xi_c\big((\mathrm{d})\big)/\xi_w\big((\mathrm{d})\big)$ actions, up to conjugation. Moreover,

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T^{n/2}) = S^1 \times S^1 & \text{if } j = 2, \\ \mathrm{Fix}(T^{n/2}) \cup \mathrm{Fix}(T^{2s}) = S^1 \times S^1 \overset{\circ}{\cup} S^1 & \text{if } j > 2, \end{cases} \qquad M^* = S^1 \times D^2.$$

(e) *Actions of type* $(1;\, n, s, i, C_0)_C$, *where $n = 2s$, and i is odd. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$; hence*

$$\xi_c\big((\mathrm{e})\big) = \left[\frac{\varphi(n)+1}{2}\right],$$

$$\mathrm{Iz}(M) = \mathrm{Fix}(T^s) = \begin{cases} S^1 \times S^1 & \text{if } s \text{ is even}, \\ B & \text{if } s \text{ is odd}, \end{cases} \qquad M^* = Bs.$$

(f) *Actions of type* $(2;\, n, g_{(n,h)})_C$, *where n is even and $(n, h) = 1$. Two such actions, for $h = h', h''$, are conjugate iff $h'' = h'$ or $n - h'$; hence*

$$\xi_c\big((\mathrm{f})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = 2 \text{ points},$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = 2 \text{ points} & \text{if } n = 2, \\ \mathrm{Fix}(T^2) = S^1 & \text{if } n > 2, \end{cases}$$

$$M^* = \langle 0, 1\rangle \times S^2/\sim, \quad (0, x) \sim \big(0, g_{(2,1)}(x)\big) \quad \text{and} \quad (1, x) \sim (1, -x).$$

(g) *Actions of type* $(2;\, n, g_{(n/2,h)})_C$, *where $n/2$ and h are odd, and $(n/2, h) = 1$. Two such actions, for $h = h', h''$, are conjugate iff $h'' = h'$; hence*

$$\xi_c\big((\mathrm{g})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} S^2 \overset{\circ}{\cup} S^1 & \text{if } n = 2, \\ 2 \text{ points} & \text{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = S^2 \overset{\circ}{\cup} S^1 & \text{if } n = 2, \\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = S^1 \cup (S^2 \overset{\circ}{\cup} S^1) & \text{if } n > 2, \end{cases} \qquad M^* = D^3.$$

(h) *Actions of type* $(2;\ n, g_{(n,h)}C)_{g_{(2,1)}C}$, *where* $n/2$ *is odd and* $(n/2, h) = 1$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$ *or* $n-h'$; *hence*

$$\xi_c\big((\mathrm{h})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} S^2 & \text{if } n = 2, \\ 2 \text{ points} & \text{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = S^2 & \text{if } n = 2, \\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = S^1 \cup S^2 & \text{if } n > 2, \end{cases} \qquad M^* = P^3 \# D^3.$$

(i) *Actions of type* $(2;\ n, g_{(n,h)})_{g_{(2,1)}C}$, *where* n *is even and* $(n, h) = 1$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$ *or* $n-h'$; *hence*

$$\xi_c\big((\mathrm{i})\big) = \left[\frac{\varphi(n)+1}{2}\right], \quad \mathrm{Fix}(T) = \begin{cases} 2 \text{ points} \cup S^1 & \text{if } n = 2, \\ 2 \text{ points} & \text{if } n > 2, \end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = 2 \text{ points} \cup S^1 & \text{if } n = 2, \\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = S^1 \overset{\circ}{\cup} S^1 & \text{if } n > 2 \text{ and } n/2 \text{ is odd}, \\ \mathrm{Fix}(T^2) = S^1 & \text{if } n \text{ is a multiple of } 4, \end{cases}$$

$$M^* = \begin{cases} \langle 0, 1\rangle \times S^2/\sim,\ (0, x) \sim \big(0, g_{(2,1)}(x)\big) \text{ and } (1, x) \sim \big(1, C(x)\big) & \text{if } n/2 \text{ is odd}, \\ \langle 0, 1\rangle \times S^2/\sim,\ (0, x) \sim \big(0, g_{(2,1)}(x)\big) \text{ and } (1, x) \sim \big(1, A(x)\big) & \text{if } n/2 \text{ is even}. \end{cases}$$

Proof. Since the proof of Theorem 2.1 is similar to that of Theorem 6.5 in [9], we give only its outline. It follows from Theorem 1.7 and Corollary 4.3 in [8] that each Z_n-action on $S^1 \hat{\times} S^2$ is obtained by using a multiple or a connected sum for $F_i = S^2$. We have the following possibilities (we use the terminology of 3.1 and 3.2 from [8]):

I. Case of multiple. We deduce from the formulas in Section 3.5 of [8] that

$$(S^1 \hat{\times} S^2, Z_n) = ([0, 1] \times S^2, g)^s_{(\{0\}\times S^2, \{1\}\times S^2, f, r)} \quad \text{and} \quad j_0 = 1$$

(see Definition 3.2 in [8]).

Since g, extended to S^3, has at least 2 fixed points, we infer, using Proposition 1.3, that g is equal either to $\mathrm{Id} \times g_{(n_1,h)}$ or to $\mathrm{Id} \times g_{(n_1,h)}C$, or to $\mathrm{Id} \times g_{(n_1/2,h)}C$. Now we study these possibilities:

Possibilities for g	Possibilities for f, up to equivariant isotopy (see Lemma 1.6)	Actions described in particular subcases of Theorem 2.1
1. $g = \mathrm{Id} \times g_{(n_1,h)}$	$f = \mathrm{Id}$ $f = C$ $f = C_0, n_1 = 2$ $f = CC_0, n_1 = 2$	I.1(a) (by Lemma 2.2 in [9]) I.2(a) or II(a) I.2(b) or II(b) I.1(b)
2. $g = \mathrm{Id} \times g_{(n_1,h)} C$	$f = \mathrm{Id}$ or C $f = C_0$ or $C_0 C$	I.2(a) or II(c) no new actions
3. $g = \mathrm{Id} \times g_{(n_1/2,h)} C$	$f = \mathrm{Id}$ or C $f = C_0$ or $C_0 C, n_1 = 2$ (so $g = \mathrm{Id} \times C$)	I.2(c) or II(d) I.2(b) or II(e)

II. Case of connected sum. We deduce from the formulas in Section 3.5 of [8] that

$$(S^1 \hat{\times} S^2, Z_n) = (\langle 0, 1\rangle \times S^2, T_1) \hat{\#}_{(\{0\}\times S^2, \{0\}\times S^2, f)} (\langle 0, 1\rangle \times S^2, T_2)$$

and $s_1 = s_2 = 1$, $j_0 = 2$ (see Definition 3.1 in [8]). It follows from Proposition 1.3 that T_1 (and also T_2) is equal either to $A \times g_{(n,h)} C$ or to $A \times g_{(n,h)}$, or to $A \times g_{(n/2,h)} C$, or to $A \times g_{(n/2,h)}$. We may assume that $f = \mathrm{Id}$ up to equivariant isotopy. We have the following possibilities:

Possibilities for T_1 and T_2	Actions described in particular subcases of Theorem 2.1
1. $T_1 = T_2 = A \times g_{(n,h)} C$	I.1(c)
2. $T_1 = T_2 = A \times g_{(n,h)}$	I.2(d)
3. $T_1 = T_2 = A \times g_{(n/2,h)} C$	I.1(d)
4. $T_1 = T_2 = A \times g_{(n/2,h)}$	I.2(e)
5. $T_1 = A \times g_{(n,h)}$, $T_2 = A \times g_{(n,h)} C$	II(f)
6. $T_1 = A \times g_{(n/2,h)}$, $T_2 = A \times g_{(n/2,h)} C$	II(g)
7. $T_1 = A \times g_{(n,h)} C$, $T_2 = A \times g_{(n/2,(h\mp n/2)/2)} C$ ($n/2$ is odd)	I.1(e)
8. $T_1 = A \times g_{(n,h)}$, $T_2 = A \times g_{(n/2,(h\mp n/2)/2)}$ ($n/2$ is odd)	I.2(f)
9. $T_1 = A \times g_{(n,h)} C$, $T_2 = A \times g_{(n/2,(h\mp n/2)/2)}$ ($n/2$ is odd)	II(h)
10. $T_1 = A \times g_{(n,h)}$, $T_2 = A \times g_{(n/2,(h\mp n/2)/2)} C$ ($n/2$ is odd)	II(i)
11. $T_1 = A \times g_{(n,h)} C$, $T_2 = A \times g_{(n,h\mp n/2)} C$	I.1(e)
12. $T_1 = A \times g_{(n,h)}$, $T_2 = A \times g_{(n,h\mp n/2)}$	I.2(f)
13. $T_1 = A \times g_{(n,h)}$, $T_2 = A \times g_{(n,h\mp n/2)} C$	II(i)

To complete the proof of Theorem 2.1 it remains to verify the following:

I. Actions described in distinct subcases I.1(a), ..., II(i) of Theorem 2.1 are not weakly conjugate.

II. The actions in each subcase of Theorem 2.1 are well classified. The proof differs in details from that of Theorem 6.5 in [9] and will be omitted.

3. Actions of Z_n on $S^1 \times P^2$.

3.1. THEOREM. *Each effective action of Z_n on $M = S^1 \times P^2$ (generated by T) takes one of the following forms (up to conjugation). Each of the cases* (b)-(e) *describes exactly one class of weakly conjugate actions.*

(a) *Actions of type $(1;\ n, s, i, g_{(n,h)})_{\mathrm{Id}}$, where $0 < h \leqslant j = n/s$ and $(j, h) = 1$. Two such actions, for $s = s', s''$, $i = i', i''$, and $h = h', h''$, are weakly conjugate iff $s'' = s'$ and there exists a natural number a such that*

$$i'' \equiv ai' \text{ or } a(s-i') \pmod s \quad \text{and} \quad h'' \equiv ah' \text{ or } a(j-h') \pmod j.$$

If $a = 1$, then the actions are conjugate. For given s we have

$$\xi_c\big((\mathrm{a})\big) = \left[\frac{\varphi(s)+1}{2}\right]\left[\frac{\varphi(j)+1}{2}\right] \quad \text{and} \quad \xi_w\big((\mathrm{a})\big) = \left[\frac{\varphi\big(\mathrm{g.c.d.}(s, j)\big)+1}{2}\right].$$

Each class of actions (up to weak conjugation) contains $\xi_c\big((\mathrm{a})\big)/\xi_w\big((\mathrm{a})\big)$ actions, up to conjugation. Moreover,

$$\mathrm{Iz}(M) = \begin{cases} \emptyset & \text{if } j = 1, \\ \mathrm{Fix}(T^s) = S^1 & \text{if } j > 1 \text{ and } j \text{ is odd}, \\ \mathrm{Fix}(T^s) \cup \mathrm{Fix}(T^{n/2}) = S^1 \mathring{\cup} S^1 \times S^1 & \text{if } j \text{ is even}, \end{cases}$$

$$M^* = \begin{cases} S^1 \times P^2 & \text{if } j \text{ is odd}, \\ S^1 \times D^2 & \text{if } j \text{ is even}. \end{cases}$$

(b) *Actions of type $(1;\ n, s, i, C_0)_{g(2,1)}$, where $n = 2s$ and i is odd. Two such actions, for $i = i', i''$, are conjugate iff $i'' = i'$ or $s - i'$; hence*

$$\xi_c\big((\mathrm{b})\big) = \left[\frac{\varphi(s)+1}{2}\right],$$

$$\mathrm{Iz}(M) = \mathrm{Fix}(T^s) = \begin{cases} S^1 \mathring{\cup} S^1 \times S^1 & \text{if } s \text{ is even}, \\ S^1 \mathring{\cup} B & \text{if } s \text{ is odd}, \end{cases} \qquad M^* = Bs.$$

(c) *Actions of type* $(2;\ n, g_{(n,h)})_{\mathrm{Id}}$, *where* $(n,h) = 1$ *and* n *is even. Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$ *or* $n-h'$; *hence*

$$\xi_c\big((\mathrm{c})\big) = \left[\frac{\varphi(n)+1}{2}\right], \qquad \mathrm{Fix}(T) = \begin{cases} S^1 \overset{\circ}{\cup} S^1 \cup 2 \textit{ points} & \textit{if } n = 2,\\ 2 \textit{ points} & \textit{if } n > 2,\end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = S^1 \overset{\circ}{\cup} S^1 \cup 2 \textit{ points} & \textit{if } n = 2,\\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = S^1 \overset{\circ}{\cup} S^1 \overset{\circ}{\cup} S^1 & \textit{if } n > 2 \textit{ and } n/2 \textit{ is odd},\\ \mathrm{Fix}(T^{n/2}) = S^1 \overset{\circ}{\cup} S^1 \times S^1 & \textit{if } n \textit{ is a multiple of } 4,\end{cases}$$

$$M^* = \begin{cases} D^1 \times P^2/\sim,\ (i,z) \sim \big(i, g_{(2,1)}(z)\big)\ (i = 1,2), & \textit{if } n/2 \textit{ is odd},\\ D^1 \times D^2/\sim,\ (i,z) \sim (i, -z)\ (i = 1,2), & \textit{if } n \textit{ is a multiple of } 4.\end{cases}$$

(d) *Actions of type* $(2;\ n, g_{(n/2,h)})_{\mathrm{Id}}$, *where* $n/2$ *is odd and* $(n/2, h) = 1$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$ *or* $n/2-h'$; *hence*

$$\xi_c\big((\mathrm{d})\big) = \left[\frac{\varphi(n)+1}{2}\right], \qquad \mathrm{Fix}(T) = \begin{cases} P^2 \overset{\circ}{\cup} P^2 & \textit{if } n = 2,\\ 2 \textit{ points} & \textit{if } n > 2,\end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = P^2 \overset{\circ}{\cup} P^2 & \textit{if } n = 2,\\ \mathrm{Fix}(T^{n/2}) \cup \mathrm{Fix}(T^2) = (P^2 \overset{\circ}{\cup} P^2) \cup S^1 & \textit{if } n > 2,\end{cases} \qquad M^* = D^1 \times P^2.$$

(e) *Actions of type* $(2;\ n, g_{(n,h)})_{g_{(2,1)}}$, *where* n *is even and* $(n,h) = 1$. *Two such actions, for* $h = h', h''$, *are conjugate iff* $h'' = h'$, $n-h'$, $h' \mp n/2$ *or* $n-h' \mp n/2$; *hence*

$$\xi_c\big((\mathrm{e})\big) = \left[\frac{\varphi(n/2)+1}{2}\right], \qquad \mathrm{Fix}(T) = \begin{cases} S^1 \overset{\circ}{\cup} P^2 \cup 1 \textit{ point} & \textit{if } n = 2,\\ 2 \textit{ points} & \textit{if } n > 2,\end{cases}$$

$$\mathrm{Iz}(M) = \begin{cases} \mathrm{Fix}(T) = S^1 \overset{\circ}{\cup} P^2 \cup 1 \textit{ point} & \textit{if } n = 2,\\ \mathrm{Fix}(T^2) \cup \mathrm{Fix}(T^{n/2}) = (S^1 \overset{\circ}{\cup} S^1) \cup P^2 & \textit{if } n > 2 \textit{ and } n/2 \textit{ is odd},\\ \mathrm{Fix}(T^{n/2}) = S^1 \overset{\circ}{\cup} S^1 \times S^1 & \textit{if } n \textit{ is a multiple of } 4,\end{cases}$$

$$M^* = \begin{cases} [0,1] \times P^2/\sim,\ (1,z) \sim \big(1, g_{(2,1)}(z)\big), & \textit{if } n/2 \textit{ is odd},\\ [0,1] \times D^2/\sim,\ (i,z) \sim (i, -z)\ (i = 1,2), & \textit{if } n \textit{ is a multiple of } 4.\end{cases}$$

The proof of Theorem 3.1 is similar to that of Theorem 2.1 (apply Corollary 4.4 from [8], Propositions 3.5, 3.6 from [8], Corollary 1.5, Lemma 1.6, and Theorem 1.7) and we omit it.

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