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A SUFFICIENT CONDITION FOR GRAPHS WITH 1-FACTORS

BY

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The famous theorem of Petersen [2] states that every cubic bridgeless graph contains a 1-factor. Using different terminology, we may rephrase this result as follows:

Every 3-regular, 2-edge connected graph contains a 1-factor.

It is also the case that every cubic graph with at most two bridges contains a 1-factor or, equivalently, every 3-regular, 1-edge connected graph containing at most two edge cut sets of cardinality 1 contains a 1-factor. With this strengthening of Petersen's theorem at hand, we propose to present a further generalization in this article.

A *1-factor* of a graph G is a 1-regular spanning subgraph of G. A well-known characterization of graphs containing 1-factors is due to Tutte [3]:

A graph G has a 1-factor if and only if, for every proper subset S of the vertex set of G, the number $k_0(G-S)$ of odd components of $G-S$ does not exceed the cardinality of S.

An *edge cut set* of a connected graph G is a set of edges whose deletion from G results in a disconnected graph. A graph G is *n-edge connected* if there exists no edge cut set of cardinality less than n.

We are now prepared to present our main result.

THEOREM 1. *Let G be an $(r-2)$-edge connected graph $(r \geqslant 3)$ of even order such that*

(i) $\deg v \equiv r \pmod 2$ *for every vertex v of G;*

(ii) $\sum(\deg v - r) = 2x < 2r$, *where the summation is taken over all vertices v of G whose degrees are at least r.*

If the maximum number of pairwise disjoint edge cut sets of cardinality $r-2$ in G is less than $r-x$, then G contains a 1-factor.

Proof. Suppose, on the contrary, that G does not contain a 1-factor. Then, by Tutte's theorem, there exists a proper subset S of the vertex set of G such that $k_0(G-S) > |S|$. Denote the odd components of $G-S$ by $G_1, G_2, \ldots, G_n$, so labeled that the numbers a_i of edges in G joining G_i and S are in non-decreasing order. Since G is $(r-2)$-edge connected, $a_i \geqslant r-2$ for each i $(1 \leqslant i \leqslant n)$.

Assume that the maximum number of pairwise disjoint edge cut sets of cardinality $r-2$ in G is j. By hypothesis, $j < r-x$. For each i, we note that $a_i \neq r-1$; for suppose that $a_t = r-1$, say. Let the vertex set of G_t be $\{u_1, u_2, \ldots, u_m\}$. Then

$$(1) \qquad \sum_{i=1}^{m} \deg_{G_t} u_i = \sum_{i=1}^{m} \deg_G u_i - (r-1) \equiv (m-1)r+1 \pmod 2,$$

which is impossible since the right-hand side of (1) is odd and the left-hand side is even. Therefore, $a_i \geqslant r-2$ for all i $(1 \leqslant i \leqslant n)$ and $a_i \geqslant r$ for $i > j$.

Thus, the number of edges joining $G-S$ and S is at least

$$\sum_{i=1}^{n} a_i \geqslant j(r-2)+(n-j)r = nr-2j.$$

On the other hand, by hypothesis (ii), the number of edges joining $G-S$ and S cannot exceed $kr+2x$, where $k = |S|$. Hence,

$$nr-2j \leqslant kr+2x,$$

which implies that $(n-k)r \leqslant 2j+2x$ and that $n-k \leqslant (2j+2x)/r$. Since $j < r-x$, we have $n-k < 2$; however, $n > k$ so that $n = k+1$. This shows that the cardinalities of the sets S and $\{G_1, G_2, \ldots, G_n\}$ are of opposite parity, which contradicts the fact that G has even order.

Later we shall discuss the sharpness of the result given in Theorem 1. At present, however, we consider a consequence of this theorem.

The degrees of the vertices of the graph G of Theorem 1 may have any of the values $r-2, r, r+2, r+4$, etc., provided that $\sum(\deg v - r) < 2r$, where the summation is taken over all vertices v whose degrees are at least r. If only vertices of degree r are allowed, then we have the following corollary, which gives an alternative proof of a result by Chartrand and Nebeský [1].

COROLLARY 1a. *If G is an r-regular, $(r-2)$-edge connected graph $(r \geqslant 3)$ of even order containing less than r distinct edge cut sets of cardinality $r-2$, then G contains a 1-factor.*

For $r = 3$ the preceding result yields the following statement:

COROLLARY 1b. *If G is a cubic graph containing at most two bridges, then G has a 1-factor.*

If G is an $(r-1)$-edge connected graph $(r \geqslant 3)$, then G is also $(r-2)$-edge connected and has no edge cut sets of cardinality $r-2$. Therefore, Corollary 1a has the following consequence:

COROLLARY 1c. *Every r-regular, $(r-1)$-edge connected graph $(r \geqslant 3)$ of even order contains a 1-factor.*

Specializing Corollary 1c to $r = 3$ returns us to Petersen's theorem.

COROLLARY 1d. *Every cubic bridgeless graph contains a 1-factor.*

For each $r \geqslant 3$, Corollary 1a is the best possible in the following sense. Let r be given. For r even, write

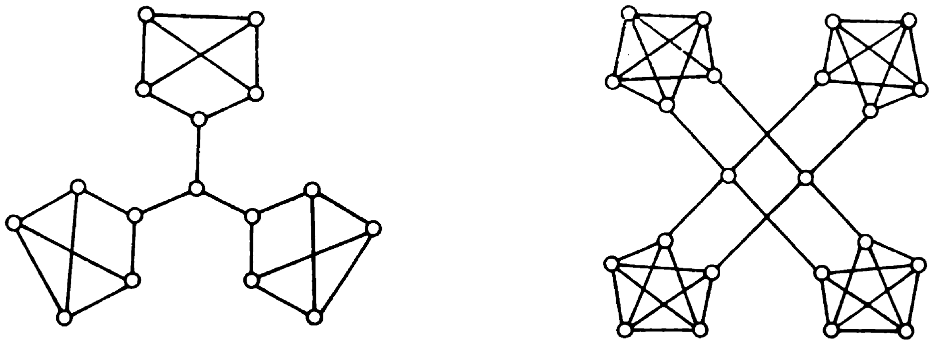
$$H_r = [(r-2)/2]K_2 \cup (3K_1),$$

i.e., H_r consists of $(r-2)/2$ copies of K_2 and 3 copies of K_1. For r odd, write

$$H_r = P_r \cup K_2,$$

where P_r denotes the path of order r. For r even or odd, put $G_r = \bar{H}_r$, the complement of H_r. Construct a graph G by taking r copies of G_r, a set S of $r-2$ vertices, and $r(r-2)$ additional edges, namely $r-2$ edges joining the $r-2$ vertices of degree $r-1$ in G_r to the vertices of S, in a one-to-one manner, for each of the r copies of G_r in G.

The resulting graph G is r-regular, $(r-2)$-edge connected and contains r edge cut sets of cardinality $r-2$. However, $k_0(G-S) = r$ while $|S| = r-2$, so that, by Tutte's theorem, G does not contain a 1-factor. Thus, if we increase the number of edge cut sets of cardinality $r-2$ in the statement of Corollary 1a, then we cannot be assured that G contains a 1-factor. For $r = 3$ and $r = 4$, the graphs G so constructed are shown in Fig. 1 (a) and (b), respectively.



(a) (b)

Fig. 1

If r is even, the graph G constructed to illustrate the sharpness of Corollary 1a has order r^2+2r-2; while if r is odd, then G has order r^2+3r-2. These examples are, in a sense, the best possible. In order to verify this, we present the following result:

THEOREM 2. *Let G be an $(r-2)$-edge connected graph $(r \geqslant 3)$ of even order p such that* $\deg v \geqslant r$ *for every vertex v of G, and let* $2x = \sum(\deg v - r)$, *where the summation is taken over all vertices v of G. If*

$$p < (r-x)\left(2\left\{\frac{r}{2}\right\}+1\right)+\max\{1, r-2-x\}, \tag{2}$$

then G has a 1-factor.

Proof. First we note that $2x$ is an even integer, since $\sum \deg v = pr+ +2x$, and $\sum \deg v$ and pr are both even. Hence x is an integer. Further, inequality (2) implies that $x < r-1$.

Assume, on the contrary, that G has no 1-factor. As before, there exists a proper subset S of $V(G)$ for which

$$n = k_0(G-S) > |S| = k,$$

where the odd components of $G-S$ are $G_1, G_2, \ldots, G_n$. It follows that $n \neq k+1$, since, otherwise, the cardinalities of the sets S and $\{G_1, G_2, \ldots \ldots, G_n\}$ are of opposite parity, which is impossible because p is even. Therefore, $n \geqslant k+2$.

Let a_i $(1 \leqslant i \leqslant n)$ denote the number of edges joining G_i and S. We claim that if $|V(G_i)| = r-t$ for $0 \leqslant t \leqslant r-1$, then $a_i \geqslant (r-t)(t+1)$. This follows because every vertex v_i of G_i is adjacent with at most $r-t-1$ vertices of G_i and, therefore, adjacent with at least $t+1$ vertices of S, since $\deg v \geqslant r$ for all vertices v of G. Therefore, $a_i \geqslant (r-t)(t+1)$, as claimed. In particular, since $(r-t)(t+1) \geqslant r$ for $0 \leqslant t \leqslant r-1$, we conclude that if $|V(G_i)| \leqslant r$, then $a_i \geqslant r$.

Let m denote the number of odd components G_i of $G-S$ for which $|V(G_i)| \geqslant r+1$. (Note that, since each G_i has odd order, if r is odd, then $|V(G_i)| \geqslant r+1$ implies $|V(G_i)| \geqslant r+2$.) Therefore,

$$\begin{aligned} \sum_{i=1}^{n} a_i &= \sum_{|V(G_i)| \leqslant r} a_i + \sum_{|V(G_i)| \geqslant r+1} a_i \\ &\geqslant \sum_{|V(G_i)| \leqslant r} r + \sum_{|V(G_i)| \geqslant r+1} (r-2) = nr-2m. \end{aligned} \tag{3}$$

Further, it follows that

$$\sum_{i=1}^{n} a_i \leqslant \sum_{v \in S} \deg v \leqslant kr+2x. \tag{4}$$

Thus, from (3) and (4) we obtain

$$nr-2m \leqslant kr+2x$$

or

$$m \geqslant r-x, \tag{5}$$

since $n \geqslant k+2$. Now, by (4),

$$kr+2x \geqslant \sum_{i=1}^{n} a_i \geqslant n(r-2) \geqslant (k+2)(r-2),$$

so that

$$k \geqslant r-2-x. \tag{6}$$

Hence, for r even, by (5) and (6) we have

$$p \geqslant m(r+1)+k \geqslant (r-x)(r+1)+\max\{1, r-2-x\},$$

which contradicts hypothesis (2); while, for r odd, we have

$$p \geqslant m(r+2)+k \geqslant (r-x)(r+2)+\max\{1, r-2-x\},$$

which again contradicts (2).

If the graph G of Theorem 2 is r-regular, then we have the following result:

COROLLARY 2a. *If G is an r-regular, $(r-2)$-edge connected graph $(r \geqslant 3)$ of even order p such that*

$$p < \begin{cases} r^2+2r-2 & \text{if } r \text{ is even}, \\ r^2+3r-2 & \text{if } r \text{ is odd}, \end{cases} \tag{7}$$

then G has a 1-factor.

Proof. Applying Theorem 2 for G r-regular, we see that $x = 0$ and $\max\{1, r-2-x\} = r-2$. Thus, by (7), we have (2).

Thus, Corollary 2a shows that the examples presented to illustrate that Corollary 1a is the best possible are themselves of smallest possible order. Conversely, these examples show that Corollary 2a is the best possible.

We now return to Theorem 1 and show that this result is sharp in the following sense. Namely, we show that for each $r \geqslant 3$ there exists an $(r-2)$-edge connected graph G of even order, satisfying hypotheses (i) and (ii), having exactly $r-x$ distinct edge cut sets of cardinality $r-2$, and not containing a 1-factor. Furthermore, the degrees exceeding r can be specified as can the number of vertices of degree $r-2$, and G can be constructed to have these added properties.

Let G_r denote the graph defined in the example following Corollary 1a. To construct the desired graph G we consider two cases.

Suppose first that the number of vertices of G whose degrees are to exceed r is at most $r-2$. To construct G, we begin with a set S of $r-2$ vertices. The graph G also consists of pairwise disjoint graphs $G_1', G_2', \ldots, G_r'$ such that $G_i' \cong G_r$ for $i = 1, 2, \ldots, r-x$ and $G_i' \cong G_r - e$ for any $i > r-x$, where e is an edge joining two vertices of degree r in G_r. By the definition of G_r, such an edge e exists. Thus, for $i = 1, 2, \ldots, r-x$, G_i' contains $r-2$ vertices of degree $r-1$, while, for any $i > r-x$, G_i' contains

r vertices of degree $r-1$. All other vertices of $G_1', G_2', \ldots, G_r'$ have degree r. Now, for $i = 1, 2, \ldots, r$, we join $r-2$ vertices of degree $r-1$ to the vertices of S in a one-to-one manner. At this point, each vertex of S has degree r. If degrees exceeding r have been specified, such vertices can be produced in S by joining the appropriate vertices of S to the $2x$ remaining vertices of $G_{r-x+1}', \ldots, G_r'$ having degree $r-1$. The graph G so constructed has the desired properties. The sets of edges joining G_i' and S for $i = 1, 2, \ldots$ $\ldots, r-x$ are the only edge cut sets of cardinality $r-2$. The graph G does not have a 1-factor, since

$$k_0(G-S) = r > r-2 = |S|.$$

If l vertices of degree $r-2$ have been specified, where, necessarily, $l \leqslant r-x$, then $G_1', G_2', \ldots, G_l'$ may be contracted to single vertices, and then the resulting graph has all the desired properties.

Next, suppose that the number of vertices whose degrees exceed r is $r-1$. Then, necessarily, each of these vertices has degree $r+2$, and $x = r-1$. In this case, the graph G is constructed by beginning with a set S of $r-1$ vertices. The graph G also consists of graphs $G_1', G_2', \ldots, G_{r+1}'$ with $G_1' \cong G_r$ and, for $2 \leqslant i \leqslant r+1$, $G_i' \cong G_r - e$, where e is an edge joining two vertices of degree r. Thus, G_1' contains $r-2$ vertices of degree $r-1$ with all others of degree r, while each of $G_2', G_3', \ldots, G_{r+1}'$ has r vertices of degree $r-1$ and all others of degree r. We now join the $r-2$ vertices of degree $r-1$ in G_1' to $r-2$ vertices of S in a one-to-one manner. For $i = 2, 3, \ldots, r+1$, we join the r vertices of degree $r-1$ in G_i' to the vertices of S, and this is done in such a way that each vertex of S has degree $r+2$. This is possible since the total number of edges between the graphs $G_1', G_2', \ldots, G_{r+1}'$ and S is r^2+r-2. This graph G has the required properties. The set of edges joining G_1' and S is the only edge cut set of cardinality $r-2$. However, G does not have a 1-factor, since

$$k_0(G-S) = r+1 > r-1 = |S|.$$

If a vertex of degree $r-2$ is specified, then G_1' may be contracted to a single vertex to produce a graph with the desired properties.

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