COLLOQUIUM MATHEMATICUM

XLII DÉDIÉ À LA MÉMOIRE D'EDWARD MARCZEWSKI 1979

SETS OF UNIQUENESS ON NONCOMMUTATIVE LOCALLY COMPACT GROUPS. II

BY

MAREK BOŻEJKO (WROCŁAW)

This note is a continuation of [1]. We are now interested in the class $U(G)$ of compact sets of uniqueness on a locally compact group.

We refer to Eymard's thesis [4] for the basic definitions and properties of $A(G)$, $B_\varrho(G)$, $B(G)$, $C_\varrho^*(G)$, $C^*(G)$, and $VN(G)$.

Let E be a closed subset of a locally compact group G. We recall that

$$J_E^0 = \{u \in A(G)\colon u = 0 \text{ on some neighborhood of } E, \text{ support of } u \text{ is compact}\},$$

and J_E denotes the closure of J_E^0 in $A(G)$.

Definition. A closed subset E of G is called a *set of uniqueness* ($E \in U(G)$) if $T \in C_\varrho^*(G)$ and $\operatorname{supp} T \subseteq E$ imply $T = 0$.

PROPOSITION 1. *For a closed set E in a locally compact group G the following statements are equivalent*:

(a) $E \in U(G)$.

(b) *The ideal J_E is $\sigma\big(B_\varrho(G), C_\varrho^*(G)\big)$-dense in $B_\varrho(G)$.*

(c) *The ideal J_E is $\sigma\big(B(G), C^*(G)\big)$-dense in $A(G)$.*

Proof follows from the following simple facts:

(1) The algebra $A(G)$ is $\sigma\big(B_\varrho(G), C_\varrho^*(G)\big)$-dense in $B_\varrho(G)$.

(2) $\{T \in C_\varrho^*(G)\colon \operatorname{supp} T \subseteq E\} = \{T \in C_\varrho^*(G)\colon (T, u) = 0 \text{ for all } u \in J_E\}$.

(3) On $B_\varrho(G)$ the topologies $\sigma\big(B_\varrho(G), C_\varrho^*(G)\big)$ and $\sigma\big(B(G), C^*(G)\big)$ coincide.

Since, for $T \in VN(G)$, $\|T\| = \sup\{|(T, u)|\colon u \in A(G),\ \|u\| = 1\}$, we obtain (1).

From the definition of the support of $T \in VN(G)$ one can verify (2).

To show (3) note that (see [4])

$$(*) \qquad C_\varrho^*(G) = C^*(G)/N_\varrho',$$

where $N_\varrho' = \{T \in C^*(G)\colon (T, u) = 0 \text{ for all } u \in B_\varrho(G)\}$.

Suppose that $u_a \in B_\varrho(G)$ and $(u_a, T) \to 0$ for every $T \in C^*(G)$. By $(*)$ we have $T = T_1 + S$, where $T_1 \in C^*_\varrho(G)$ and $S \in N'_\varrho$, whence $(u_a, T) = (u_a, T_1)$ and $u_a \to 0$ with respect to the topology $\sigma(B_\varrho(G), C^*_\varrho(G))$.

The converse implication follows in the same way.

THEOREM 1. *If $E_1, E_2 \in U(G)$, then $E_1 \cup E_2 \in U(G)$.*

Proof. Let $T \in C^*_\varrho(G)$ and $\operatorname{supp} T \subseteq E_1 \cup E_2$.

Consider two cases:

(α) $\operatorname{supp} T \subset E_1 \cap E_2$. Then $T = 0$, since the subset of the set of uniqueness is also a set of uniqueness.

(β) There exists an $a \in \operatorname{supp} T \setminus (E_1 \cap E_2)$. Without loss of generality we can assume that $a \notin E_1$. Since the algebra $A(G)$ is regular, there exists $u \in A(G)$ such that $u(a) \neq 0$ and $u = 0$ on a neighborhood of E_1. Consider $uT \in C^*_\varrho(G)$. Since $E_2 \in U(G)$ and $\operatorname{supp}(uT) \subseteq \operatorname{supp}(u) \cap \operatorname{supp}(T) \subseteq E_2$, we have $uT = 0$. Hence, by Proposition 4.4, (ii), of Eymard [4], $u(a) = 0$ and $T = 0$.

Now we present another proof of the theorem that every residual closed set belongs to $U(G)$. The original proof in [1] was rather long and incomplete.

THEOREM 2. *If G is a nondiscrete locally compact group and E is closed and residual (i.e., including no nonempty perfect sets), then E is a set of uniqueness.*

Proof. Since, for nondiscrete groups G, $C^*_\varrho(G)$ has no unit, we infer that $\varrho(x) \notin C^*_\varrho(G)$ for every $x \in G$.

Let $T \in C^*_\varrho(G)$ and $\operatorname{supp} T \subseteq E$. Since $\operatorname{supp} T$ is a closed subset of E, it contains an isolated point $a \in \operatorname{supp} T$. Again, by the regularity of the algebra $A(G)$ there exists a $u \in A(G)$ such that $u(a) \neq 0$ and $u = 0$ in a neighborhood of $\operatorname{supp} T \setminus \{a\}$. Hence $uT \in C^*_\varrho(G)$ and $\operatorname{supp}(uT) = \{a\}$. Therefore, by Theorem 4.9 of Eymard [4], $uT = \lambda\varrho(a)$ for some complex λ. But $\varrho(a) \notin C^*_\varrho(G)$ implies $uT = 0$, whence $T = 0$ since $a \in \operatorname{supp}(u) \cap \operatorname{supp} T$.

One can see that no set E of positive Haar measure is a set of uniqueness. There is a conjecture that, for some class of nonabelian groups, every set of Haar measure zero is a set of uniqueness.

Now we give the negative answer to that problem in the case of separable infinite compact groups.

THEOREM 3. *If G is a separable infinite compact group, then there exists a closed set E of Haar measure zero which is not a set of uniqueness.*

Proof. Figà-Talamanca [5] showed that $A(G) \subsetneqq B_\varrho(G) \cap c_0(G)$ for the large family of groups, and the very same ideas as those in the proof in [5] yield that if G is a separable infinite compact group, then

$$L_1(G)^+ \subsetneqq M(G)^+ \cap C^*_\varrho(G).$$

Choose $\mu \in M(G)^+ \cap C_\varrho^*(G)$ and $\mu \notin L_1(G)$. Then using the Lebesgue decomposition we have $\mu = f + \mu_s$, where $f \in L_1(G)$ and μ_s is singular with respect to the Haar measure on G.

Since μ is a positive measure, μ_s is absolutely continuous with respect to μ, so the theorem of [3] implies $\mu_s \in C_\varrho^*(G)$.

Put $E = \operatorname{supp}(\mu_s)$. Then the Haar measure of E is equal to zero, but E is not a set of uniqueness since $\mu_s \in C_\varrho^*(G)$.

We finish our note with a short list of problems.

1. When is $L_1(G) = M(G) \cap C_\varrho^*(G)$? (**P 1075**)

2. Let $E_n \in U(G)$ and $\bigcup_n E_n$ be closed. Does $\bigcup_n E_n \in U(G)$? (**P 1076**)

3. In a nonabelian group find an uncountable set of uniqueness. (**P 1077**)

4. Let E be compact and residual, $T \in VN(G)$ and $\operatorname{supp} T \subseteq E$. Does then $T \in C_\varrho^*(G_d)$? (See Loomis [6] in the abelian case.) (**P 1078**)

5. Let H be a closed subgroup of G of the Haar measure zero. Does $H \in U(G)$? (**P 1079**)

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WROCŁAW

Reçu par la Rédaction le 1. 6. 1978