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ON σ-COMPLETE PRIME IDEALS IN BOOLEAN ALGEBRAS

BY

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Luxemburg [2] posed the following problem:

Does there exist a complete non-atomic Boolean algebra which has a σ-complete prime ideal?

He answered this question neqatively in [3] under the hypothesis that there is no measurable cardinal. We prove that if a measurable cardinal exists, the answer is positive.

We identify an ordinal with the set of smaller ordinals. Cardinals are initial ordinals.

Definition 1. Let $\varkappa$ be a cardinal. Measure μ, defined on $\mathscr{P}(\varkappa)$, is called a *$\varkappa$-complete* measure in $\mathscr{P}(\varkappa)$ if it satisfies the following conditions:

(a) for every $X \subseteq \varkappa$, either $\mu(X) = 0$ or $\mu(X) = 1$;

(b) $\mu(\varkappa) = 1$ and $\mu(\{a\}) = 0$ for every $a \in \varkappa$;

(c) for every sequence $\{A_\xi \mid \xi \in \lambda\}$, if $\lambda < \varkappa$ and $\mu(A_\xi) = 0$ for each $\xi \in \lambda$, then $\mu(\bigcup \{A_\xi \mid \xi \in \lambda\}) = 0$.

$\varkappa$ is a *measurable cardinal* if there is a $\varkappa$-complete measure in $\mathscr{P}(\varkappa)$.

LEMMA 1. *Suppose that $(\mathfrak{P}, \precsim)$ is a partial ordering satisfying the following condition*:

($*$) *If $p, q \in \mathfrak{P}$ and p* non $\precsim$ *q, then there is an $r \precsim p$ such that no $s \in \mathfrak{P}$ satisfies simultaneously $s \precsim q$ and $s \precsim r$.*

Then there is a complete Boolean algebra $(\mathfrak{B}, \precsim)$ such that $\mathfrak{P}$ is a dense subset of $\mathfrak{B}$ and the Boolean inclusion $\precsim$ in $\mathfrak{B}$ is an extension of $\precsim$ in $\mathfrak{P}$.

Lemma 1 is a slightly amplified version of a statement in [6], p. 38. There the word "complete" does not occur in the conclusion. Lemma 1 is obtained by considering the minimal completion of an algebra, the existence of which is asserted in [6]. We also remark that a set $\mathfrak{S} \subseteq \mathfrak{B}$, is said to be *dense* in $\mathfrak{B}$, if for every $b \in \mathfrak{B}$, $b \neq \mathbf{0}$, there is an $s \in \mathfrak{S}$ such that $s \neq \mathbf{0}$ and $s \precsim b$.

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Definition 2. For any set A, $[A]^n$ denotes the set of subsets of A of cardinality n; $[A]^{<\omega_0} \overset{\text{def}}{=} \bigcup\{[A]^n : n \in \omega_0\}$.

For any two finite sets of ordinals r, s, $r \sqsubset s$ means that r is an initial segment of s with respect to the natural ordering of ordinals. Also, $r \sqsubset r$.

LEMMA 2 (Rowbottom [5]). *Let $\varkappa$ be a measurable cardinal. Then there is a $\varkappa$-complete measure μ in $\mathscr{P}(\varkappa)$ satisfying the following conditions:*

(a) *Suppose that $\mu(A_\xi) = 1$ for each $\xi \in \varkappa$. Then*

$$\mu(\{\eta \mid (\forall \xi < \eta)(\eta \in A_\xi)\}) = 1.$$

(b) *Suppose that $\mu(U) = 1$, $\lambda < \varkappa$ and $f: [U]^{<\omega_0} \to \lambda$. Then there is a set $V \subseteq U$ such that $\mu(V) = 1$, and for every $n \in \omega_0$, f is constant on $[V]^n$.*

Such a measure is said to be *normal*.

Proof. See [4], lemma 1.4, remarks 1.5, 1.6 and theorem 1.9.

Definition 3. Let $\varkappa$ be a measurable cardinal and μ a fixed normal measure in $\mathscr{P}(\varkappa)$. From now on, $\mathfrak{P}$ will be the set of ordered pairs $\langle s, A\rangle$, where $s \in [\varkappa]^{<\omega_0}$, $\mu(A) = 1$, and $\max s < \min A$. If $\langle s, A\rangle \in \mathfrak{P}$ and $\langle r, B\rangle \in \mathfrak{P}$, we put $\langle r, B\rangle \precsim \langle s, A\rangle$ iff $s \sqsubset r$, $B \subseteq A$ and $r - s \subseteq A$.

LEMMA 3. *$(\mathfrak{P}, \precsim)$ is a partial ordering which satisfies $(*)$ of Lemma 1.*

Proof. The first part is left to the reader.

Suppose that $\langle s, A\rangle, \langle r, B\rangle \in \mathfrak{P}$ and $\langle r, B\rangle$ non $\precsim$ $\langle s, A\rangle$, i.e. the premise of $(*)$.

First of all, let us assume that for some $\langle t, C\rangle \in \mathfrak{P}$, $\langle t, C\rangle \precsim \langle s, A\rangle$ and $\langle t, C\rangle \precsim \langle r, B\rangle$. Then $r \sqsubset t$ and $s \sqsubset t$. Hence $s \sqsubset r$ or $r \sqsubset s$. In any case, $r - s \subseteq t - s \subseteq A$. The last inclusion follows by $\langle t, C\rangle \precsim \langle s, A\rangle$ and definition 3.

Thus, if $r - s \nsubseteq A$, or $s \sqsubset r$ and $r \sqsubset s$ both fail, then $\langle t, C\rangle \precsim \langle s, A\rangle$ and $\langle t, C\rangle \precsim \langle r, B\rangle$ cannot hold simultaneously. So if p, q of Lemma 1 are $\langle r, B\rangle$, $\langle s, A\rangle$ respectively, we can take for r of Lemma 1 the p itself.

Therefore we can assume $r - s \subseteq A$, and either $s \sqsubset r$ or $r \sqsubset s$. Let us consider first the case $s \sqsubset r$. Then $B \nsubseteq A$, because otherwise $\langle r, B\rangle \precsim \langle s, A\rangle$, contrary to our hypothesis. In fact, $B \nsubseteq A \cup s$, since $s \subseteq r$ and $B \cap r = 0$. Choose a $\beta \in B - (A \cup s)$. Let $B' = B - \{\xi \mid \xi \leqslant \beta\}$ and $r' = r \cup \{\beta\}$. It is easy to see that $\langle r', B'\rangle \precsim \langle r, B\rangle$. It is claimed that for no $\langle t, C\rangle \in \mathfrak{P}$ do we have simultaneously $\langle t, C\rangle \precsim \langle r', B'\rangle$ and $\langle t, C\rangle \precsim \langle s, A\rangle$. Assume to the contrary. Hence $r' \sqsubset t$, which results in $\beta \in t$. This, together with the choice of β, gives $\beta \in t - s$. From $\langle t, C\rangle \precsim \langle s, A\rangle$ we get $t - s \subseteq A$. This implies $\beta \in A$, which is a contradiction.

Finally, suppose that $r \sqsubset s$ and not $s \sqsubset r$. Because $\mu(B) = 1$, we can choose a $\beta \in B$ such that $\beta > \max s$. Put $r' = r \cup \{\beta\}$ and $B' = B - \{\xi \mid \xi \leqslant \beta\}$. Hence

$$\min B' > \beta > \max s > \max r$$

and $\mu(B') = 1$. It follows that $\langle r', B'\rangle \in \mathfrak{P}$ and $\langle r', B'\rangle \preceq \langle r, B\rangle$. Now $\langle t, C\rangle \preceq \langle r', B'\rangle$ and $\langle t, C\rangle \preceq \langle s, A\rangle$ cannot hold simultaneously. Otherwise either $s \sqsubset r'$ or $r' \sqsubset s$. Since $\beta \in r'$ and $\beta > \max s$, the former has to hold. But $s \sqsubset r'$, together with $r' = r \cup \{\beta\}$ and $\beta > \max s$, give $s \sqsubset r$. This is a contradiction. Hence Lemma 3 is proved.

LEMMA 4. *There is a complete Boolean algebra* $(\mathfrak{B}, \preceq)$ *such that* $\mathfrak{P}$ *is a dense subset of* $\mathfrak{B}$ *and* $\preceq$ *in* $\mathfrak{B}$ *is an extension of* $\preceq$ *in* $\mathfrak{P}$. *In what follows,* $(\mathfrak{B}, \preceq)$ *will be such an algebra.*

Proof. Lemma 3 and Lemma 1.

Definition 4. Let

$$\mathfrak{J} = \{b \mid b \in \mathfrak{B} \;\&\; (\exists U)(\mu(U) = 1 \;\&\; \langle 0, U\rangle \preceq -b_i)\}.$$

LEMMA 5. *For every* $b \in \mathfrak{B}$ *there is an* A *such that* $\mu(A) = 1$ *and either* $\langle 0, A\rangle \preceq b$ *or* $\langle 0, A\rangle \preceq -b$.

Proof. Put $b_0 = -b$, $b_1 = b$. For $i = 0, 1$ we put

$$S_i = \{s \mid s \in [\varkappa]^{<\omega_0} \;\&\; (\exists U)(\langle s, U\rangle \in \mathfrak{P} \;\&\; \langle s, U\rangle \preceq b)\};$$
$$S_2 = [\varkappa]^{<\omega_0} - (S_0 \cup S_1).$$

First of all, $S_0 \cap S_1 = 0$. If not, then for some s, $s \in S_0 \cap S_1$. Hence there are U_0, U_1 such that $\mu(U_0) = \mu(U_1) = 1$, $\langle s, U_0\rangle \preceq b_0$ and $\langle s, U_1\rangle \preceq b_1$. Let $U = U_0 \cap U_1$. Thus $\mu(U) = 1$ and, consequently, $\langle s, U\rangle \in \mathfrak{P}$. Clearly, $\langle s, U\rangle \preceq \langle s, U_i\rangle$ for $i = 0, 1$. By transitivity, $\langle s, U\rangle \preceq b_i$ for $i = 0, 1$. This is a contradiction, because $b_0 \wedge b_1 = \mathbf{0}$ and $\langle s, U\rangle \neq \mathbf{0}$.

By Lemma 2, there is a set A with the properties

(0) $\mu(A) = 1$;

(1) for every $n \in \omega_0$ there is an $i = 0, 1$, or 2 satisfying $[A]^n \subseteq S_i$.

It is claimed that either $\langle 0, A\rangle \preceq b_0$ or $\langle 0, A\rangle \preceq b_1$. Suppose not. Then there are b_0', b_1' such that $b_i' \wedge b_i = \mathbf{0}$ and $b_i' \preceq \langle 0, A\rangle$ for $i = 0, 1$. Because $\mathfrak{P}$ is dense in $\mathfrak{B}$, there are $\langle s_i, A_i\rangle \preceq b_i'$ for $i = 0, 1$. Thus $\langle s_i, A_i\rangle \wedge b_i = \mathbf{0}$, i.e. $\langle s_i, A_i\rangle \preceq b_{1-i}$ for $i = 0, 1$. Hence $s_i \in S_{1-i}$ for $i = 0,1$.

We can assume that $|s_0| = |s_1| = k$ for some $k \in \omega_0$. Because for example, if $|s_0| < |s_1|$, we can find $\langle s_0', A_0'\rangle \preceq \langle s_0, A_0\rangle$ such that $|s_0'| = |s_1|$. Then we would consider $\langle s_0', A_0'\rangle, \langle s_1, A_1\rangle$ instead of $\langle s_0, A_0\rangle, \langle s_1, A_1\rangle$. To obtain such an $\langle s_0', A_0'\rangle$, let $a_1, a_2, \ldots, a_{|s_1|-|s_0|}$ be the first $|s_1| - |s_0|$ elements of A_0. Put $s_0' = s_0 \cup \{a_1, a_2, \ldots, a_{|s_1|-|s_0|}\}$ and $A_0' = A_0 - \{a_1, a_2, \ldots, a_{|s_1|-|s_0|}\}$.

Hence assuming $|s_0| = |s_1| = k$, we have $[A]^k \cap S_0 \neq 0$ and $[A]^k \cap S_1 \neq 0$. This, together with the disjointness of S_i for $i = 0, 1, 2$, preclude $[A]^k \subseteq S_i$ for any fixed $i = 0, 1$, or 2. That contradicts the choice of A. Hence either $\langle 0, A\rangle \preceq b_0$ or $\langle 0, A\rangle \preceq b_1$, which proves the lemma.

LEMMA 6. *For every* $b \in \mathfrak{B}$ *either* $b \in \mathfrak{J}$ *or* $-b \in \mathfrak{J}$. $\mathbf{1} \notin \mathfrak{J}$.

Proof. The first part is an immediate consequence of Lemma 5 and definition 4. As for $\mathbf{1} \notin \mathfrak{J}$, assume to the contrary. Hence for some $A \subseteq \varkappa$, $\mu(A) = 1$ and $\langle 0, A\rangle \preceq -\mathbf{1} = \mathbf{0}$. This is impossible.

LEMMA 7. *$\mathfrak{J}$ is $\varkappa$-complete.*

Proof. Suppose that $b_\xi \in \mathfrak{J}$, for each $\xi < \lambda < \varkappa$. Then there are sets $A_\xi \subseteq \varkappa$ such that $\mu(A_\xi) = 1$ and $\langle 0, A_\xi\rangle \preceq -b_\xi$. Let $A = \cap\{A_\xi \mid \xi \in \lambda\}$. By the $\varkappa$-completeness of μ, $\mu(A) = 1$. Obviously, $\langle 0, A\rangle \preceq \langle 0, A_\xi\rangle$ for each $\xi \in \lambda$. Hence,

$$\langle 0, A\rangle \preceq \inf\{\langle 0, A_\xi\rangle \mid \xi \in \lambda\} \preceq \inf\{-b_\xi \mid \xi \in \lambda\} = -\sup\{b_\xi \mid \xi \in \lambda\}.$$

Hence $\sup\{b_\xi \mid \xi \in \lambda\} \in \mathfrak{J}$.

THEOREM 1. *$(\mathfrak{B}, \preceq)$ is a non-atomic complete Boolean algebra and $\mathfrak{J}$ is a $\varkappa$-complete prime ideal in $\mathfrak{B}$.*

Proof. Let $b \in \mathfrak{B}$, $b \neq \mathbf{0}$. Then, due to the density of $\mathfrak{P}$ in $\mathfrak{B}$, there is an $\langle r, A\rangle \in \mathfrak{P}$ such that $\langle r, A\rangle \preceq b$. Let $r' = r \cup \{\min A\}$ and $A' = A - \{\min A\}$. Then $\langle r', A'\rangle \preceq \langle r, A\rangle \preceq b$ and $\langle r', A'\rangle \neq \langle r, A\rangle$. Thus $\langle r', A'\rangle \preceq b$ and $\langle r', A'\rangle \neq b$, which proves that $\mathfrak{B}$ is non-atomic. The rest follows from Lemmas 6, 7.

The following result is a slight strengthening of Theorem 1:

THEOREM 2. *Every proper principal ideal in $(\mathfrak{B}, \preceq)$ can be extended to a $\varkappa$-complete prime ideal.*

Proof. For each $\langle r, A\rangle \in \mathfrak{P}$, we let

$$\mathfrak{B}(\langle r, A\rangle) = \{b \mid b \in \mathfrak{B} \ \& \ b \preceq \langle r, A\rangle\}.$$

Clearly, $\big(\mathfrak{B}(\langle r, A\rangle), \preceq\big)$ is a complete Boolean algebra. We define

$$\mathfrak{J}(\langle r, A\rangle) = \big\{b \mid b \in \mathfrak{B}(\langle r, A\rangle) \ \& \ (\exists U)\big(\mu(U) = 1 \ \& \ \langle r, U\rangle \preceq \langle r, A\rangle - b\big)\big\}.$$

Similarly as in Theorem 1, it can be shown that $\mathfrak{J}(\langle r, A\rangle)$ is a $\varkappa$-complete prime ideal in $\big(\mathfrak{B}(\langle r, A\rangle), \preceq\big)$.

Let $b \in \mathfrak{B}$, $b \neq \mathbf{1}$ and

$$\mathfrak{J}_b = \{c \mid c \in \mathfrak{B} \ \& \ c \preceq b\}.$$

The proper principal ideals of $\mathfrak{B}$ are exactly such $\mathfrak{J}_b$'s. Because $-b \neq \mathbf{0}$ an $\langle r, A\rangle \in \mathfrak{P}$ can be chosen so that $\langle r, A\rangle \preceq -b$. We put

$$\mathfrak{J} = \big\{c \mid c \in \mathfrak{B} \ \& \ (\exists d, e)\big(d \in \mathfrak{J}_b \ \& \ e \in \mathfrak{J}(\langle r, A\rangle) \ \& \ c = d \vee e\big)\big\}.$$

It is easy to see that $\mathfrak{J}_b \subseteq \mathfrak{J}$ and $\mathfrak{J}$ is a $\varkappa$-complete prime ideal.

We would like to prove one more result about $\mathfrak{B}$.

THEOREM 3. *$(\mathfrak{B}, \preceq)$ has no dense subset of cardinality $\varkappa$.*

Proof. It is obviously sufficient to prove that $(\mathfrak{P}, \preceq)$ has no dense subset of cardinality $\varkappa$. In order to do this, suppose that the sequence

$\mathfrak{A} = \{\langle s_\xi, A_\xi\rangle | \xi \in \varkappa\}$ is a sequence of elements of $\mathfrak{P}$. Let

$$A = \{\eta | (\forall \xi < \eta)(\eta \in A_\xi)\}.$$

It follows immediately that $A - \xi \subseteq A_\xi - \xi$. By the normality of μ and Lemma 2(a), $\mu(A) = 1$. Let $B \subseteq A$ be such that $\mu(B) = 1$ and $|A - B| = \varkappa$. Hence $|A_\xi - B| = \varkappa$ and, consequently, $A_\xi - B \neq 0$ for each $\xi \in \varkappa$. So for no $\xi \in \varkappa$, do we have $\langle s_\xi, A_\xi\rangle \leqslant \langle 0, B\rangle$. This means that $\mathfrak{A}$ is not dense in $\mathfrak{P}$. Because this is true for any $\mathfrak{A}$ as defined above, the theorem is proved.

Remarks. The hypothesis of the normality of μ simplifies the construction, but it is not necessary. An analogous construction can be carried out for an arbitrary $\varkappa$-complete μ. Instead of Lemma 2, we would use Theorem 1.36, [4].

It is proved in [1] that if $\varkappa$ satisfies a certain condition stronger than measurability, the following holds: In every $\varkappa$-distributive complete Boolean algebra every $\varkappa$-complete proper ideal can be extended to a $\varkappa$-complete prime ideal. See [1], Theorem 4.16 iv), p. 290, and Theorem 5.10, p. 304.

In [4], $(\mathfrak{P}, \precsim)$ is investigated in the context of forcing of P. Cohen. The Boolean algebra $(\mathfrak{B}, \precsim)$, which is determined by $(\mathfrak{P}, \precsim)$ up to isomorphism, is also studied in [4]. There, to obtain $(\mathfrak{B}, \precsim)$, we used a method of [7]. However, the description of $(\mathfrak{B}, \precsim)$, as stated in [4], is too general for the purposes of the present paper. The following theorem about $(\mathfrak{B}, \precsim)$ was proved in [4]. Here we would like to prove it using the results presented above.

THEOREM. (a) $(\mathfrak{B}, \precsim)$ *is $\varkappa$-distributive, i.e. the $(\lambda, 2)$ distributive law holds for each $\lambda < \varkappa$.*

(b) *The $(\aleph_0, \varkappa)$ distributive law fails in $(\mathfrak{B}, \precsim)$.*

Proof. (a) This follows from Theorem 2, and Theorems 0.5, 0.6, p. 236, [1].

(b) For each $n \in \omega_0$, $n \neq 0$, let $\{b_n^\xi | \xi \in \varkappa\}$ be an enumeration of $\{\langle r, \varkappa\rangle | r \in [\varkappa]^n\}$. It is claimed that

(0) $\sup\{b_n^\xi | \xi \in \varkappa\} = \mathbf{1}$ for each $n \in \omega_0$, $n \neq 0$;

(1) $\inf\{b_n^{\xi(n)} | n \in \omega_0\} = \mathbf{0}$ for every sequence $\{\xi(n) | n \in \omega_0\}$ of ordinals $< \varkappa$.

(0) Suppose that for some $n \in \omega_0$, $n \neq 0$, $\sup\{b_n^\xi | \xi \in \varkappa\} = b \neq \mathbf{1}$. Hence $-b \neq \mathbf{0}$ and therefore there is an $\langle s, B\rangle \in \mathfrak{P}$ such that $\langle s, B\rangle \precsim -b$. Similarly as in the proof of Lemma 5, we can assume that $|s| > n$. Let $a_1 < a_2 < \ldots < a_n < \ldots < a_m$ be the elements of s in increasing order. Let $r = \{a_1, a_2, \ldots, a_n\}$ and $A = B \cup \{a_{n+1}, \ldots, a_m\}$. Obviously $\langle s, B\rangle \precsim \langle r, A\rangle \precsim \langle r, \varkappa\rangle$. Now for some $\xi \in \varkappa$, $\langle r, \varkappa\rangle = b_n^\xi$. Hence $\langle s, B\rangle$

$\precsim \sup\{b_n^\xi \mid \xi \in \varkappa\} = b$. This gives $\langle s, B\rangle \precsim b \wedge (-b) = \mathbf{0}$, which is a contradiction. This proves (0).

(1) Let $\langle r, U\rangle \in \mathfrak{P}$, $n \in \omega_0$ and $n > |r|$. Then for some $s \in [\varkappa]^n$, $b_n^{\xi(n)} = \langle s, \varkappa\rangle$. It follows from $|s| > |r|$ that $\langle r, U\rangle$ non $\precsim \langle s, \varkappa\rangle$. Hence $\langle r, U\rangle$ non $\precsim \inf\{b_n^{\xi(n)} \mid n \in \omega_0\}$. Because $\langle r, U\rangle$ was an arbitrary element of $\mathfrak{P}$ and $\mathfrak{P}$ is dense in $\mathfrak{B}$, we must have $\inf\{b_n^{\xi(n)} \mid n \in \omega_0\} = \mathbf{0}$. This proves (1) and therefore (b).

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