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ON UNIVERSAL ALGEBRAS HAVING BASES OF DIFFERENT CARDINALITIES

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The aim of this note is to prove some remarks for abstract algebras with two bases of different cardinalities. Our terminology and notation are standard (see [2] and [3]).

In [3] Marczewski has proposed the following conjecture:

If an algebra $\mathfrak{A}$ has two bases of different cardinalities, then for each natural number n there exists an essentially n-ary algebraic operation over $\mathfrak{A}$. (For partial results see [1] and [4].)

The results of this paper are not quite related to Marczewski's conjecture, they are concerned with algebras of a variety $\mathscr{A}_{m,n}$ defined in the sequel.

Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$ be bases of an algebra $\mathfrak{A} = (A, F)$. We assume that $m < n$. It is easy to see that there exist unique algebraic operations f_i and g_j over $\mathfrak{A}$ such that

$$b_j = f_j(a_1, a_2, \ldots, a_n) \quad \text{and} \quad a_i = g_i(b_1, b_2, \ldots, b_m)$$
$$\text{for } j = 1, 2, \ldots, m \text{ and } i = 1, 2, \ldots, n.$$

Then the algebra $\mathfrak{A}_{m,n} = (A, f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_n)$ belongs to the variety $\mathscr{A}_{m,n}$ $(m < n)$ defined by the following identities:

(*) $f_j(g_1, g_2, \ldots, g_n) = e_j^{(m)}$,

(**) $g_i(f_1, f_2, \ldots, f_m) = e_i^{(n)}$.

The following notation will be used in the sequel: for any algebra $\mathfrak{A}$ let $\mathscr{S}(\mathfrak{A})$ denote the set of all n's such that in $\mathfrak{A}$ there exists an essentially n-ary algebraic operation.

If $F^* \subset A(F)$, then the algebra (A, F^*) is said to be a *reduct* of a given algebra $\mathfrak{A} = (A, F)$. The most interesting reducts of an algebra $\mathfrak{A}$ are: the trivial reduct $(A, \emptyset)$, the idempotent reduct $I(\mathfrak{A}) = (A, I(F))$, where $I(F)$ consists of all idempotent algebraic operations (i.e., $f \in I(F)$ iff, for $x \in A$, $f(x, x, \ldots, x) = x$), and $S(\mathfrak{A}) = (A, S(F))$, where $S(F)$ denotes the set of all completely symmetric algebraic operations over $\mathfrak{A}$. The algebras $I(\mathfrak{A})$ and $S(\mathfrak{A})$ were investigated by Urbanik in [6] and [7].

PROPOSITION 1. *For any non-one-element algebra $\mathfrak{A} \in \mathscr{A}_{1,n}$ and any positive integer s there exists an essentially s-ary algebraic operation $d(x_1, x_2, \ldots, x_s)$ over $\mathfrak{A}$ such that (A, d) is an s-dimensional diagonal algebra.*

COROLLARY 1. *If $\mathfrak{A}$ has bases consisting of 1 and $m \geqslant 2$ elements, then for all s it contains an essentially s-ary diagonal operation, and hence*

$$\mathscr{S}(\boldsymbol{I}(\mathfrak{A})) = \mathscr{S}(\mathfrak{A}) = \{1, 2, \ldots\}.$$

PROPOSITION 2. *Let $\mathfrak{A}$ be an algebra and $\mathfrak{B}$ a reduct of $\mathfrak{A}$ such that $\mathfrak{B} \in \mathscr{A}_{m,n}$ for some m and n. Then $\mathscr{S}(\boldsymbol{I}(\mathfrak{A}))$ is infinite.*

COROLLARY 2. *$\mathscr{S}(\mathfrak{A})$ is infinite for all non-one-element algebras $\mathfrak{A} \in \mathscr{A}_{m,n}$.*

PROBLEM. Is it true that $\mathscr{S}(\boldsymbol{I}(\mathfrak{A})) = \{1, 2, \ldots\}$ for all non-one-element algebras $\mathfrak{A} \in \mathscr{A}_{m,n}$? **(P 1083)**

If the answer is affirmative, then Marczewski's conjecture has a positive solution.

THEOREM. *If $\mathfrak{A}$ is an algebra having bases of different cardinalities, then $\mathfrak{A}$ contains infinite independent sets.*

Proof of Proposition 1. Let $\mathfrak{A}_{1,n} \in \mathscr{A}_{1,n}$; then

$$\mathfrak{A}_{1,n} = (A, f, g_1, g_2, \ldots, g_n).$$

Consider now

$$d(x_1, x_2, \ldots, x_n) = f\big(g_1(x_1), g_2(x_2), \ldots, g_n(x_n)\big).$$

Of course, conditions $(*)$ and $(**)$ imply that d is idempotent, depends on each its variable and its diagonality results from axiom (II) in [5]. Now, it is easy to verify that every operation $d'(x_1, x_2, \ldots, x_m) = d(x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ $(1 \leqslant i_p \leqslant m$ for $p = 1, 2, \ldots, n)$ is a diagonal algebraic operation and that $\mathscr{S}\big((A, d)\big) = \{1, 2, \ldots, n\}$. For the last fact see [6]. Now, if $\mathfrak{A} \in \mathscr{A}_{m,n}$, then by the Ryll-Nardzewski theorem [2] we have

$$\mathfrak{A}^{(n)} = (A^{(n)}, F) \cong \mathfrak{A}^{(m)} = (A^{(m)}, F)$$

and, therefore, $\mathfrak{A}^{(n)}$ has bases whose cardinalities form an arithmetic progression. If $\mathfrak{A} \in \mathscr{A}_{1,n}$, then the cardinalities of the bases of $\mathfrak{A}^{(1)} = (A^{(1)}, F)$ are of the form $1 + (n-1)k$, where $k = 1, 2, \ldots$ and n is the smallest cardinality of a basis of $\mathfrak{A}$ greater than 1.

Let $s \leqslant 1 + (n-1)k = r$ for some k and let $\{f_1, f_2, \ldots, f_r\}$ be a basis of $\mathfrak{A}^{(1)}$. Then there exists an algebraic operation f over $\mathfrak{A}$ such that

$$\tilde{f}(f_1, f_2, \ldots, f_r) = e_1^{(1)}$$

and

$$\tilde{f}_i\big(\tilde{f}(e_1^{(r)}, e_2^{(r)}, \ldots, e_r^{(r)})\big) = e_i^{(r)} \quad \text{for } i = 1, 2, \ldots, r.$$

The last formula can be rewritten as follows:

$$f\big(f_1(x), f_2(x), \ldots, f_r(x)\big) = x,$$

$$f_i\big(f(x_1, x_2, \ldots, x_r)\big) = x_i \quad \text{for all } x_1, x_2, \ldots, x_r, x \in A.$$

So far we have shown that the algebra $\mathfrak{A}$ contains a reduct $\mathfrak{B} \in \mathscr{A}_{1,r}$. But $A(\mathfrak{A}) \supset A(\mathfrak{B})$ and we can repeat the previous argument for $\mathfrak{B}$ to prove the existence of an essentially r-ary diagonal operation in the algebra $\mathfrak{A}$. This completes the proof.

Corollary 1 follows immediately.

Proof of Proposition 2. Let $\mathfrak{A} = (A, \boldsymbol{F})$, $\boldsymbol{F}_1 \subset \boldsymbol{A}(\boldsymbol{F})$ and $\mathfrak{B} = (A, \boldsymbol{F}_1)$. Assuming $\mathfrak{B} \in \mathscr{A}_{m,n}$ we set

$$\mathfrak{B}^{(m)} \cong \big(\boldsymbol{A}^{(m)}(\boldsymbol{F}_1), \boldsymbol{F}_1\big) \cong \big(\boldsymbol{A}^{(n)}(\boldsymbol{F}_1), \boldsymbol{F}_1\big)$$

and

$$\mathscr{S}\big(\boldsymbol{I}(\mathfrak{B}^{(n)})\big) \subset \mathscr{S}\big(\boldsymbol{I}(\mathfrak{A}^{(m)})\big) \subset \mathscr{S}\big(\boldsymbol{I}(\mathfrak{A})\big).$$

As $\mathfrak{B}^{(m)}$ has bases of different cardinalities, our assertion follows from Theorem 1 of [2].

The proof of Corollary 2 is trivial.

Proof of the Theorem. Assume that the algebra $\mathfrak{A}$ has a basis $B_1 = \{b_1^1, b_2^1, \ldots, b_m^1\}$ of m elements and also a basis $B_2 = \{b_1^2, b_2^2, \ldots, b_{m+r}^2\}$ of $m+r$ elements. Now we construct a new basis B_3 by letting

$$b_i^3 = g_i(b_1^2, b_2^2, \ldots, b_m^2) \quad \text{for } i = 1, 2, \ldots, m+r$$

and

$$b_i^3 = b_{m+k}^2 \quad \text{for } i = m+r+k \text{ and } k = 1, 2, \ldots, r.$$

So

$$\begin{aligned} B_3 &= \{g_1(b_1^2, b_2^2, \ldots, b_m^2), g_2(b_1^2, b_2^2, \ldots, b_m^2), \ldots, \\ &\qquad g_{m+r}(b_1^2, b_2^2, \ldots, b_m^2), b_{m+1}^2, b_{m+2}^2, \ldots, b_{m+r}^2\} \\ &= \{b_1^3, b_2^3, \ldots, b_{m+2r}^3\}. \end{aligned}$$

It is an easy matter to show that $|B_3| = m+2r$. Let us check that $B_3 \in \operatorname{Ind}(\mathfrak{A})$. Assume that for $F_1, F_2 \in \mathfrak{A}^{(m+2r)}$ the following holds:

$$F_1(b_1^3, b_2^3, \ldots, b_{m+2r}^3) = F_2(b_1^3, b_2^3, \ldots, b_{m+2r}^3).$$

This means that

$$\begin{aligned} &F_1\big(g_1(b_1^2, b_2^2, \ldots, b_n^2), \ldots, g_{m+r}(b_1^2, b_2^2, \ldots, b_m^2), b_{m+1}^2, \ldots, b_{m+r}^2\big) \\ &\quad = F_2\big(g_1(b_1^2, b_2^2, \ldots, b_m^2), \ldots, g_{m+r}(b_1^2, b_2^2, \ldots, b_m^2), b_{m+1}^2, \ldots, b_{m+r}^2\big). \end{aligned}$$

The last equality is satisfied on the independent set B_2, therefore on A. Thus by using formula (**) we obtain

$$F_1(x_1, x_2, \ldots, x_{m+2r}) = F_2(x_1, x_2, \ldots, x_{m+2r})$$
$$\text{for all } x_1, x_2, \ldots, x_{m+2r} \in A.$$

This implies $B_3 \in \mathrm{Ind}(\mathfrak{A})$. Since $f_j(b_1^3, b_2^3, \ldots, b_{m+r}^3) = b_j^2$ for $j = 1, 2, \ldots, m$, we infer that the subalgebra generated by B_3 contains B_2. Hence B_3 generates $\mathfrak{A}$. Let us write

$$J_1 = \{b_{m+1}^2, b_{m+2}^2, \ldots, b_{m+r}^2\},$$
$$J_2 = J_1 \cup \{g_{m+1}(b_1^2, b_2^2, \ldots, b_m^2), \ldots, g_{m+r}(b_1^2, b_2^2, \ldots, b_m^2)\}.$$

Then $|J_1| = r$ and $|J_2| = 2r$ with $J_1 \subset J_2 \in \mathrm{Ind}(\mathfrak{A})$. Analogously, starting with the bases B_2 and B_3 we construct a new base B_4 with $|B_4| = m+3r$ and, in general,

$$B_s = \{b_1^s, b_2^s, \ldots, b_{m+(s-1)r}^s\} \quad \text{with } |B_s| = m+(s-1)r.$$

Also, for a given s we construct J_s as follows:

$$J_s = J_{s-1} \cup \{b_{m+r}^{s+1}, b_{m+r+1}^{s+1}, \ldots, b_{m+2r}^{s+1}\}.$$

Since (J_s) is an increasing sequence of independent sets, its union is infinite and independent, which completes the proof of the Theorem.

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