COLLOQUIUM MATHEMATICUM

VOL. XXXIV 1975 FASC. 1

REMARKS ON OPERATOR-STABLE MEASURES

BY

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1. In this paper we consider Borel probability measures on the Euclidean space R^N. With the topology of weak convergence and multiplication defined by the convolution, the set of all such measures becomes a topological semigroup. We denote the convolution of two measures λ and μ by $\lambda * \mu$, and the probability measure concentrated at the point $a \in R^N$ by δ_a. The *characteristic function* $\hat{\lambda}$ of a probability measure λ on R^N is defined by the formula

$$\hat{\lambda}(x) = \int_{R^N} \exp\{i(x, y)\} \lambda(dy),$$

where (x, y) denotes the inner product in R^N. We call a probability measure on R^N *full* if its support is not contained in any $(N-1)$-dimensional hyperplane of R^N.

Let $\{X_n\}$ be a sequence of independent and identically distributed R^N-valued random variable . Consider affine modifications of partial sums

$$A_n \sum_{k=1}^{n} X_k + a_n, \tag{1}$$

where $\{A_n\}$ is a sequence of non-singular linear operators on R^N, and $\{a_n\}$ a sequence of vectors from R^N. The limit distribution of (1), if it exists, is called *operator-stable*. The problem of describing the class of all full operator-stable probability measures on Euclidean spaces has been completely solved by Sharpe in [5]. In particular, he proved the following statements (*)-(***).

(*) *Every full operator-stable probability measure on R^N is infinitely divisible.*

Hence the characteristic function φ of such a measure is of the form

(2)

$$\varphi(x) = \exp\left\{i(x, a) - \frac{1}{2}(Sx, x) + \int_{R^N \setminus \{0\}} \left(e^{i(x,y)} - 1 - \frac{i(x, y)}{1 + \|y\|^2}\right) \frac{1 + \|y\|^2}{\|y\|^2} \gamma(dy)\right\},$$

where a is a vector from R^N, S is a symmetric non-negative operator in R^N, and the spectral measure γ is a finite Borel measure on $R^N \setminus \{0\}$. Moreover, the triple (a, S, γ) is uniquely determined by φ. Hence, for every positive real number t, it is possible to define the t-th power λ^t of λ in the sense of convolution. Namely, if λ is represented by the triple (a, S, γ), the λ^t will denote the infinitely divisible measure with the triple $(at, tS, t\gamma)$. The convolution semigroup $\{\lambda^t\colon t > 0\}$ is weakly continuous.

(**) *A full measure λ is operator-stable if and only if there is a non-singular linear operator B in R^N such that*

$$(3) \quad \lambda^t = t^B \lambda * \delta_{a_t} \quad \text{for } t > 0, \quad \text{where } t^B = \exp\{\log t B\} \text{ and } a_t \in R^N.$$

Moreover, the spectrum of B is in the half-plane $\mathrm{Re}\, z \geqslant \frac{1}{2}$ *and the eigenvalues lying on the line* $\mathrm{Re}\, z = \frac{1}{2}$ *are simple.*

In the sequel every operator-stable measure satisfying (3) will be called *B-stable.*

(***) *Any full operator-stable measure λ on R^N can be decomposed into a product $\lambda = \lambda_1 * \lambda_2$ of measures λ_1 and λ_2 concentrated in subspaces P_1 and P_2, respectively, $R^N = P_1 \oplus P_2$, where λ_1 is a full Gaussian measure on P_1, and λ_2 is a full operator-stable measure on P_2 having no Gaussian component. Moreover, the eigenvalues of the operator B lying in P_2 have real parts greater than $\frac{1}{2}$, and real parts of eigenvalues of the operator B in P_1 are equal to $\frac{1}{2}$.*

Hence, in particular, it follows that a full operator-stable measure has no Gaussian component if and only if it is B-stable for some operator B whose eigenvalues have real parts greater than $\frac{1}{2}$.

M. Sharpe pointed out that the set of all spectral measures corresponding to B-stable probability measures is a convex cone which can be investigated by the extreme point method. The aim of the present paper is to give, using this method, a representation of the characteristic function of full operator-stable probability measures.

2. In what follows B will denote an operator in R^N with eigenvalues lying in the half-plane $\mathrm{Re}\, z > \frac{1}{2}$. Let S^m be the m-dimensional unit sphere, L the open half-line $(0, \infty)$, and $\bar{L} = L \cup \{0\} \cup \{\infty\}$ the compactified half-line. Put $H^N = S^{N-1} \times \bar{L}$. Obviously, the space H^N is compact. Further, we define a congruence relation in H^N by $\langle x, t\rangle \sim \langle y, u\rangle$, where $x, y \in S^{N-1}$ and $t, u \in L$, if there exists a positive real number s such that $s^B x = y$ and $s = tu^{-1}$.

PROPOSITION 1. *The quotient space $M_B = H^N/\sim$ is compact.*

Proof. Suppose that $\langle x_n, t_n\rangle \sim \langle y_n, u_n\rangle$ $(n = 1, 2, \ldots)$ and that the sequences $\{\langle x_n, t_n\rangle\}$ and $\{\langle y_n, u_n\rangle\}$ converge to $\langle x, t\rangle$ and $\langle y, u\rangle$, respectively. Then, for some positive real numbers s_n, we have $s_n^B x_n = y_n$

($n = 1, 2, \ldots$). Since the eigenvalues of B have positive real parts, the last equation and the compactness of S^{N-1} imply that the sequence $\{s_n\}$ is bounded. If s is its limit point, then $0 < s < \infty$ and $s^B x = y$ and $s = tu^{-1}$. Thus $\langle x, t\rangle \sim \langle y, u\rangle$. Consequently, the relation $\sim$ is closed. Hence and from [1], p. 97, the space M_B is compact.

An element of M_B, i.e. an equivalence class containing $\langle x, t\rangle$ from H^N, will be denoted by $[x, t]$. We define a one-parameter group T_s ($s \in L$) of transformations of M_B by assuming

$$T_s[x, t] = [x, st]. \tag{4}$$

Further, for every element $[x, t] \in M_B$, we put

$$|[x, t]| = \|t^B x\| \text{ if } t \in L, \qquad |[x, \infty]| = \infty \quad \text{and} \quad |[x, 0]| = 0. \tag{5}$$

Since

$$\lim_{t\to 0} t^B = 0 \quad \text{and} \quad \lim_{t\to\infty} \|t^B z\| = \infty \text{ for every } z \in R^N\setminus\{0\},$$

each element $z \in R^N\setminus\{0\}$ can be represented in the form $z = t^B x$, where $x \in S^{N-1}$ and $t \in L$. In general, this representation is not unique. But $z = u^B y$, where $y \in S^{N-1}$ and $u \in L$, if and only if $\langle x, t\rangle \sim \langle y, u\rangle$. Thus the mapping

$$\pi_B(t^B x) = [x, t], \qquad x \in S^{N-1},\ t \in L, \tag{6}$$

is an embedding of $R^N\setminus\{0\}$ into M_B. Obviously, for all $y \in R^N\setminus\{0\}$ and $t \in L$,

$$\|y\| = |\pi_B(y)|, \tag{7}$$

$$\pi_B(t^B y) = T_t \pi_B(y). \tag{8}$$

We say that a subset $\mathscr{E}$ of M_B is *bounded from below* if $\inf\{|a| : a \in \mathscr{E}\} > 0$. Let λ be a finite Borel measure on M_B. For any Borel subset $\mathscr{E}$ of M_B bounded from below we put

$$I_\lambda(\mathscr{E}) = \int_{\mathscr{E}} (1 + |u|^{-2})\lambda(du), \tag{9}$$

where the integrand is assumed to be 1 if $|u| = \infty$.

Let $\mathscr{M}_B$ be the set of all finite Borel measures λ on M_B satisfying the condition

$$tI_\lambda(\mathscr{E}) = T_t I_\lambda(\mathscr{E}) \tag{10}$$

for all $t \geqslant 0$ and for all Borel subsets $\mathscr{E}$ bounded from below. It is clear that the set $\mathscr{M}_B$ is convex and closed. Let $\mathscr{K}_B$ be the subset of $\mathscr{M}_B$ consisting of probability measures. Obviously, the set $\mathscr{K}_B$ is convex and compact.

PROPOSITION 2. *The extreme points of $\mathcal{K}_B$ are measures concentrated on one of the sets* $\{[x, \infty]\}$, *or* $\{[x, 0]\}$, *or* $\{[x, t]:\ t \in L\}$, *where* $x \in S^{N-1}$.

Proof. If a Borel subset $\mathcal{E}$ of M_B is T_t-invariant for all $t \in L$ and $\lambda \in \mathcal{M}_B$, then the restriction $\lambda | \mathcal{E}$ belongs to $\mathcal{M}_B$ because of the equation

$$tI_{\lambda|\mathcal{E}}(\mathcal{A}) - T_t I_{\lambda|\mathcal{E}}(\mathcal{A}) = tI_\lambda(\mathcal{E} \cap \mathcal{A}) - T_t I_\lambda(\mathcal{E} \cap \mathcal{A}).$$

Hence the extreme points of the set $\mathcal{K}_B$ are measures concentrated on orbits of elements of M_B, which completes the proof.

We proceed to describing the extreme points of $\mathcal{K}_B$ supported by the set $F_x = \{[x, t]:\ t \in L\}$.

PROPOSITION 3. *For every measure λ from $\mathcal{K}_B$ concentrated on F_x there exists a constant $c_\lambda(x)$ such that, for every bounded continuous function on F_x, we have*

$$\int_{F_x} f(z)\lambda(dz) = c_\lambda(x) \int_0^\infty f([x, t]) \frac{\|[x, t]\|^2}{(1 + \|[x, t]\|^2)t^2}\, dt, \tag{11}$$

where

$$c_\lambda^{-1}(x) = \int_0^\infty \frac{\|[x, t]\|^2}{(1 + \|[x, t]\|^2)t^2}\, dt.$$

Proof. Put

$$J_\lambda(u) = I_\lambda\{[x, t]:\ t \geqslant u\}, \qquad u \in L. \tag{12}$$

It is easy to verify that $\lambda \in \mathcal{K}_B$ if and only if equality (10) holds for all $t \geqslant 0$ and for all subsets $\mathcal{E}$ of the form $\{[x, t]:\ a \leqslant t < b\}$, where $a < b$ and $a, b \in L$. Taking into account the formulae

$$I_\lambda\{[x, t]:\ a \leqslant t < b\} = J_\lambda(a) - J_\lambda(b),$$

$$T_s I_\lambda\{[x, t]:\ a \leqslant t < b\} = I_\lambda\left\{[x, t]:\ \frac{a}{s} \leqslant t < \frac{b}{s}\right\} = sI_\lambda\{[x, t]:\ a \leqslant t < b\},$$

we infer that $\lambda \in \mathcal{K}_B$ if and only if, for every triple $a, b, s \in L$ satisfying the conditions $a < b$ and $s > 0$, the equality

$$J_\lambda\left(\frac{a}{s}\right) - J_\lambda\left(\frac{b}{s}\right) = s\big(J_\lambda(a) - J_\lambda(b)\big) \tag{13}$$

holds. On the other hand, it follows from (9), (12) and (13) that

$$J_\lambda\left(\frac{a}{s}\right) = sJ_\lambda(a) \quad \text{for every } s > 0 \text{ and } a > 0. \tag{14}$$

Hence, putting $a = s$, we have

(15) $$J_\lambda(s) = \frac{1}{s} J_\lambda(1),$$

and further

(16) $$J_\lambda(a) - J_\lambda(b) = \left(\frac{1}{a} - \frac{1}{b}\right) J_\lambda(1).$$

Put $J_\lambda(1) = c_\lambda(x)$. By (10), (12) and (16) we have the formula

(17) $$\int\limits_{F_x} f(z)\lambda(dz) = c_\lambda(x) \int\limits_0^\infty f([x, t]) \frac{|[x, t]|^2}{(1+|[x, t]|^2)t^2} dt.$$

Moreover,

$$c_\lambda^{-1}(x) = \int\limits_0^\infty \frac{|[x, t]|^2}{(1+|[x, t]|^2)t^2} dt,$$

which completes the proof.

In the sequel we use the following lemma:

LEMMA. *For every $x \in R^N$, the integral*

(18) $$\int\limits_0^\infty \frac{\|t^B x\|^2}{(1+\|t^B x\|^2)t^2} dt$$

is finite. Moreover, there exist constants M and b such that

$$\int\limits_0^\infty \frac{\|t^B x\|^2}{(1+\|t^B x\|^2)t^2} dt \leqslant 1 + M \|x\|^{1/b}.$$

Proof. We assumed that all eigenvalues, say $a_1, \ldots, a_r$, of B lie in the half-plane $\operatorname{Re} z > \frac{1}{2}$. Consequently, for $\frac{1}{2} < b < \operatorname{Re} a_j$ $(j = 1, 2, \ldots, r)$, we have

$$\lim_{t \to 0} t^{-b} t^B = 0.$$

Hence

$$\sup_{0 \leqslant t \leqslant 1} \|t^{-b} t^B\| = a < \infty.$$

Thus $\|t^B x\| \leqslant a t^b \|x\|$ for $x \in R^N$ and $0 \leqslant t \leqslant 1$. Since the function $u^2/(1+u^2)$ is monotone and non-decreasing on the right half-line, we have the inequality

$$\int\limits_0^\infty \frac{\|t^B x\|^2}{(1+\|t^B x\|^2)t^2} dt \leqslant \int\limits_0^1 \frac{a^2 t^{2b} \|x\|^2}{(1+a^2 \|x\|^2 t^{2b})t^2} dt + \int\limits_1^\infty \frac{1}{t^2} dt$$

$$\leqslant 1 + \int\limits_0^\infty \frac{a^2 t^{2b} \|x\|^2}{(1+a^2 \|x\|^2 t^{2b})t^2} dt = 1 + (a\|x\|)^{1/b} \int\limits_0^\infty \frac{u^{(2b-1)-1}}{1+u^{2b}} du.$$

Since

$$\int_0^\infty \frac{u^{r-1}}{1+u^k}\,du = \frac{\pi}{k\sin(r\pi/k)} \quad \text{for } k \geqslant r > 0 \quad \text{and} \quad 2b-1>0,$$

putting

$$M = a^{1/b}\frac{\pi}{2b\sin(\pi/2b)},$$

we have

$$\int_0^\infty \frac{\|t^B x\|^2}{(1+\|t^B x\|^2)t^2}\,dt \leqslant 1 + M\|x\|^{1/b},$$

which completes the proof.

COROLLARY 1. *There exists a constant M such that*

$$\int_0^\infty \frac{|[x,t]|^2}{(1+|[x,t]|^2)t^2}\,dt < M \quad \textit{for every } x \in S^{N-1}.$$

PROPOSITION 4. *For every $x \in S^{N-1}$, the orbit F_x supports a measure from $\mathscr{K}_B$.*

Proof. Given $x \in S^{N-1}$, we put

$$c^{-1}(x) = \int_0^\infty \frac{|[x,t]|^2}{(1+|[x,t]|^2)t^2}\,dt.$$

Corollary 1 yields $c(x) > 0$. We define a probability measure for every bounded and continuous function f on F_x by the formula

$$\int_{F_x} f(z)\lambda(dz) = c(x)\int_0^\infty f([x,t])\frac{|[x,t]|^2}{(1+|[x,t]|^2)t^2}\,dt.$$

It is easy to verify that this measure satisfies condition (10) and, consequently, belongs to $\mathscr{K}_B$.

From Propositions 3 and 4 we get the following corollary:

COROLLARY 2. *Each orbit F_x supports exactly one measure from $\mathscr{K}_B$.*

Given $x \in S^{N-1}$ and $a \in L$, we denote by $m_{[x,a]}$ the unique probability measure from $\mathscr{K}_B$ concentrated on the orbit of the element $[x,a]$. By (11), for all continuous functions f on M_B we have the formula

$$\int_{M_B} f(z)m_{[x,a]}(dz) = c_{[x,a]}\int_0^\infty f(T_t[x,a])\frac{|T_t[x,a]|^2}{(1+|T_t[x,a]|^2)t^2}\,dt, \tag{19}$$

where

$$c_{[x,a]}^{-1} = \int_0^\infty \frac{|T_t[x, a]|^2}{(1+|T_t[x, a]|^2)t^2}\, dt. \tag{20}$$

Further, we put

$$m_{[x,a]} = \delta_{[x,a]} \quad \text{if either } a = 0 \text{ or } a = \infty. \tag{21}$$

We note that the mapping $z \to m_z$ from M_B onto the set of extreme points of $\mathscr{K}_B$ is one-to-one. Indeed, if $m_{[x,a]} = m_{[y,b]}$ for $x, y \in S^{N-1}$ and $a, b \in L$, then the orbits of points $[x, a]$ and $[y, b]$ coincide. Hence $\langle x, a\rangle \sim \langle y, b\rangle$, and so we have $[x, a] = [y, b]$. Moreover, the mapping $z \to m_z$ is continuous at every point $[x, a]$ with $a \in L$. Further, it is easy to see that $m_{[x_n,a]}$ tends to $m_{[x,a]}$ whenever $x_n \to x$ in S^{N-1} and either $a = 0$ or $a = \infty$.

Suppose that $x_n \to x$ in S^{N-1}, $a_n \in L$ and $a_n \to \infty$, i.e. $[x_n, a_n] \to [x, \infty]$. Then, by (4) and (5),

$$\lim_{n\to\infty} |T_t[x_n, a_n]| = \infty$$

uniformly in t in every finite interval. Hence and from (20) it follows that

$$\lim_{n\to\infty} c_{[x_n,a_n]} = 0. \tag{22}$$

Given $\varepsilon > 0$ and a continuous function f on M_B, we can choose a number t_0 and an integer n_0 such that, for all $t > t_0$ and $n \geqslant n_0$, we have

$$|f([x_n, t]) - f([x, \infty])| < \varepsilon. \tag{23}$$

Consequently,

$$|f(T_t[x_n, a_n]) - f([x, \infty])| < \varepsilon \tag{24}$$

whenever $n \geqslant n_0$ and $t > a_n^{-1}t_0$. Since, by (4),

$$\int_0^{a_n^{-1}t_0} \frac{|T_t[x_n, a_n]|^2}{(1+|T_t[x_n, a_n]|^2)t^2}\, dt = a_n^{-1} \int_0^{t_0} \frac{|[x_n, u]|^2}{(1+|[x_n, u]|^2)u^2}\, du \tag{25}$$

and $\|x_n\| = 1$, we infer, by virtue of the Lemma, that integrals (25) are commonly bounded. Thus, by (23),

(26)

$$\lim_{n\to\infty} c_{[x_n,a_n]} \int_0^{a_n^{-1}t_0} \big(f(T_t[x_n, a_n]) - f([x, \infty])\big) \frac{|T_t[x_n, a_n]|^2}{(1+|T_t[x_n, a_n]|^2)t^2}\, dt = 0.$$

Further, by (24),

$$c_{[x_n,a_n]} \int_{a_n^{-1}t_0}^{\infty} \big(f(T_t[x_n, a_n]) - f([x, \infty])\big) \frac{|T_t[x_n, a_n]|^2}{(1+|T_t[x_n, a_n]|^2)t^2}\, dt \leqslant \varepsilon$$

whenever $n \geqslant n_0$. The arbitrariness of ε and (26) show that $m_{[x_n,a_n]} \to m_{[x,\infty]}$. Thus the mapping $z \to m_z$ is also continuous at the point z of the form $[x, \infty]$.

Suppose that $x_n \to x$ in S^{N-1}, $b_n \in L$ and $b_n \to 0$, i.e. $[x_n, b_n] \to [x, 0]$. Then, by (4),

$$\lim_{n\to\infty} T_t[x_n, b_n] = [x, 0]$$

uniformly in t, which, by (19), implies the relation $m_{[x_n,b_n]} \to m_{[x,0]}$. Thus the mapping $z \to m_z$ is continuous at the points z of the form $[x, 0]$. This completes the proof of the continuity of the mapping $z \to m_z$. Hence, by a well known theorem ([3], p. 11), we conclude that this mapping is a homeomorphism between M_B and the set of extreme points of $\mathscr{K}_B$. Thus we have the following

PROPOSITION 5. *The set of measures m_z ($z \in M_B$) defined by formulae* (19) *and* (21) *coincides with the set of extreme points of $\mathscr{K}_B$. Moreover, the mapping $z \to m_z$ is a homeomorphism between M_B and the set of extreme points of $\mathscr{K}_B$.*

If the extreme points of $\mathscr{K}_B$ are found, we can apply the Theorem of Krein-Milman-Choquet ([2], see also [4], Chapter 3).

Since each element of $\mathscr{M}_B$ is of the form cv, where $c \geqslant 0$ and $v \in \mathscr{K}_B$, we get the following

PROPOSITION 6. *A measure μ belongs to M_B if and only if there exists a finite Borel measure λ on M_B such that, for each continuous function f on M_B, we have*

$$\int_{M_B} f(u)\mu(du) = \int_{M_B}\int_{M_B} f(u) m_z(du)\lambda(dz).$$

3. We proceed now to a characterization of full operator-stable measures without Gaussian component. We know that every such measure is B-stable for some operator B whose eigenvalues have the real parts greater than $\frac{1}{2}$. For such an operator B we have the following proposition, where π_B denotes the embedding of $R^N \setminus \{0\}$ into M_B defined by formula (6):

PROPOSITION 7. *A finite Borel measure γ on R^N is the spectral measure of B-stable measure if and only if the induced measure $\pi_B\gamma$ belongs to $\mathscr{M}_B$.*

Proof. From (2) and (3) for every $t > 0$ we get the equations

$$\exp\left\{i(y, ta) + \int\limits_{R^N\setminus\{0\}} \left(\exp[i(x, y)] - 1 - \frac{i(x, y)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}(t\gamma)(dx)\right\}$$
$$= \exp\left\{i(y, t^B a) + \int\limits_{R^N\setminus\{0\}} \left(\exp[i(t^B x, y)] - 1 - \frac{i(t^B x, y)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\gamma(dx)\right\}$$
$$= \exp\left\{i(y, t^B a) + \int\limits_{R^N\setminus\{0\}} \left(\exp[i(t^B x, y)] - 1 - \frac{i(t^B x, y)}{1+\|t^B x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\gamma(dx) + \right.$$
$$\left. + \int\limits_{R^N\setminus\{0\}} \left(\frac{i(t^B x, y)}{1+\|t^B x\|^2} - \frac{i(t^B x, y)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\gamma(dx)\right\}.$$

Hence, putting

$$(b_t, y) = \int\limits_{R^N\setminus\{0\}} \left(\frac{i(t^B x, y)}{1+\|t^B x\|^2} - \frac{i(t^B x, y)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\gamma(dx) + (y, t^B a),$$

we have the equality

$$\exp\left\{i(y, ta) + \int\limits_{R^N\setminus\{0\}} \left(\exp[i(x, y)] - 1 - \frac{i(x, y)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}(t\gamma)(dx)\right\}$$
$$= \exp\left\{i(y, b_t) + \int\limits_{R^N\setminus\{0\}} \left(\exp[i(t^B x, y)] - 1 - \frac{i(t^B x, y)}{1+\|t^B x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2}\gamma(dx)\right\}.$$

Therefore, the measure in question is B-stable if and only if, for every Borel subset $\mathcal{E}$ of $R^N\setminus\{0\}$, we have

$$t\int\limits_{\mathcal{E}} \frac{1+\|x\|^2}{\|x\|^2}\gamma(dx) = \int\limits_{\mathcal{E}} \frac{1+\|t^{-B}x\|^2}{\|t^{-B}x\|^2}\gamma(dx).$$

Taking into account (5)-(8) for each Borel subset $\mathcal{E}$ of M_B bounded from below, we have the formula

$$t\int\limits_{\mathcal{E}} (1+|u|^{-2})(\pi_B\gamma)(du) = \int\limits_{T_t^{-1}\mathcal{E}} (1+|u|^{-2})(\pi_B\gamma)(du).$$

Hence

$$tI_{\pi_B\gamma}(\mathcal{E}) = T_t I_{\pi_B\gamma}(\mathcal{E}),$$

which completes the proof.

THEOREM 1. *A function φ on R^N is the characteristic function of a full operator-stable measure without Gaussian component if and only if*

(27)
$$\varphi(y) = \exp\left\{i(a, y) + \int\limits_{R^N\setminus\{0\}}\int\limits_0^\infty \left(\exp[i(t^B x, y)] - 1 - \frac{i(t^B x, y)}{1+\|t^B x\|^2}\right)\frac{1}{t^2}\,dt\nu(du)\right\},$$

where a is a vector from R^N, ν is a finite Borel measure on $R^N\setminus\{0\}$, and all eigenvalues of B have the real parts greater than $\frac{1}{2}$.

Proof. Suppose that μ is full operator-stable without Gaussian component. By Proposition 7, the induced measure $\pi_B\gamma$ on M_B belongs to M_B. By Proposition 6, there exists a finite Borel measure ω on M_B such that, for every continuous function f on M_B, we have

$$\int_{M_B} f(u)\pi_B\gamma(du) = \int_{M_B}\int_{M_B} f(u)m_z(du)\,\omega(dz), \tag{28}$$

where m_z ($z\in M_B$) denotes the extreme point of $\mathscr{K}_B$. It is clear that the measure $\pi_B\gamma$ is concentrated on the set $U_N = \pi_B(R^N\setminus\{0\})$. Consequently, by (28), the measure ω is also concentrated on U_N. Since, for $z\in U_N$, the measures m_z are concentrated on U_N, formula (28) can be rewritten in the form

$$\int_{U_N} f(u)\pi_B\gamma(du) = \int_{U_N}\int_{U_N} f(u)m_z(du)\,\omega(dz) \tag{29}$$

for any continuous and bounded function on U_N. Let

$$\lambda = \pi_B^{-1}\omega \quad\text{and}\quad \nu(\mathscr{E}) = \int_{\mathscr{E}} c(x)\lambda(dx),$$

where $\mathscr{E}$ are Borel subsets of $R^N\setminus\{0\}$ and

$$c^{-1}(x) = \int_0^\infty \frac{\|t^Bx\|^2}{(1+\|t^Bx\|^2)t^2}\,dt.$$

By the Lemma, ν is a finite measure on $R^N\setminus\{0\}$. Further, for every continuous and bounded function g on $R^N\setminus\{0\}$ we get, by a simple calculation with (5)-(8) and (29), the formula

$$\int_{R^N\setminus\{0\}} g(x)\gamma(dx) = \int_{R^N\setminus\{0\}}\int_0^\infty g(t^Bx)\frac{\|t^Bx\|^2}{(1+\|t^Bx\|^2)t^2}\,dt\,\nu(dx). \tag{30}$$

Setting

$$g(x) = \left(e^{i(x,y)}-1-\frac{i(x,y)}{1+\|x\|^2}\right)\frac{1+\|x\|^2}{\|x\|^2} \qquad (y\in R^N)$$

into the last formula and taking into account (2), we get representation (27). The necessity of the conditions is thus proved.

Sufficiency. Suppose that the function φ is given by formula (27). First, note that φ is a limit of products of a Poissonian characteristic functions of the form

$$\exp\{i(a,y)\}\exp\left\{c\left(e^{i(y,b)}-1-\frac{i(y,b)}{1+\|b\|^2}\right)\right\},$$

where $c\geqslant 0$ and $a,b\in R^N\setminus\{0\}$. Thus φ is the characteristic function of an infinitely divisible measure, say μ. It is easy to verify that $\mu^t = t^B\mu * \delta_{a_t}$ for $t>0$ and for some $a_t\in R^N$, which completes the proof.

As a consequence of Theorem 1 and Sharpe's Decomposition Theorem (***) we get the following representation theorem:

THEOREM 2. *A function φ on R^N is the characteristic function of a full operator-stable measure if and only if*

$$\varphi(y) = \exp\Big\{i(a, y) - \frac{1}{2}(Sy, y) + \int\limits_{R^N\setminus\{0\}}\int\limits_0^\infty \Big(\exp[i(t^B x, y)] - 1 - \frac{i(t^B x, y)}{1+\|t^B x\|^2}\Big)\frac{1}{t^2}\,dt\nu(dx)\Big\},$$

where $a \in R^N$, all eigenvalues of B lie on the half-plane $\mathrm{Re}\, z \geqslant \frac{1}{2}$, *$S$ is a non-negative symmetric operator vanishing on the subspace H of R^N spanned by eigenvalues of B lying on the half-plane* $\mathrm{Re}\, z > \frac{1}{2}$, *and ν is a finite Borel measure on R^N concentrated on H.*

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Reçu par la Rédaction le 15. 7. 1974