COLLOQUIUM MATHEMATICUM

XXVI DÉDIÉ À M. WŁADYSŁAW ŚLEBODZIŃSKI 1972

CANONICAL FORM ON A GROUPOID RELATED TO REGULAR PROLONGATIONS OF LIE GROUPS

BY

A. SZYBIAK (RZESZÓW)

A theory of immersed manifolds gave rise to a development of the theory of fibre bundles of higher order and of related geometric objects. The theory of prolongations of G-structures has been undertaken by Cartan and now it is provided with strict foundations (cf. [3]-[5] and [14]). Now there are two directions of further development of this theory: one based on the jet calculus of Ehresmann (cf. [1], [2] and [13]) and another in which a certain algorithm on external forms is used essentially (cf. [9] and [10]). This algorithm was algebraized in [17]. However, relations between the two directions are not evident. Nevertheless, in spite of the fact that operating with co-frames (i.e., with external forms), as it is presented in [9], seems to be unsatisfactory nowadays, many local problems of the local differential geometry were solved by this method. Thus there is a good reason to investigate relations between the two mentionned directions in the theory of prolongations (cf. [4]-[8], [11], [12], [15] and [16]).

In this paper we show that linear forms which satisfy the "structure equations" of G. F. Laptev can be obtained by some intrinsic construction by means of jet calculus. Although this construction will be presented on a trivial bundle and a trivial groupoid only, transferring results to a bundle a base of which is a differentiable manifold is rather easy.

All considerations are in the cathegory C^∞.

1. Let G be a δ-dimensional Lie group and R^n a Cartesian space of dimension n. Consider a trivial bundle $R^n \times G \to R^n$ and define a pseudogroup $\mathscr{P}(G, n)$ as follows:

Elements of $\mathscr{P}(G, n)$ are quadruples of the form (A, f, a, B), where A and B are open sets in R^n, f is a diffeomorfism of A into B, and a is a differentiable cross-section over A in our trivial bundle. A rule of composition in this groupoid will be defined as follows:

$$(A, f, a, B)\cdot(B, g, b, C) := \big(A, g\circ f, a\cdot(b\circ f), C\big).$$

Thus $a\cdot(b\circ f)$ is a cross-section over A, $A\ni x\mapsto a(x)\cdot\big(b\circ f(x)\big)\in G$. Dots will denote throughout multiplication in a group, a pseudogroup or a groupoid being just in question. We introduce a sign $^{\urcorner}$ for denoting a reciprocal of any mapping, because we reserve $^{-1}$ for an inverse in algebraic structures. Thus we have $(A,f,a,B)^{-1}=\big(B,f^{\urcorner},(a\circ f^{\urcorner})^{-1},A\big)$ in the groupoid $\mathscr{P}(G,n)$.

$\mathscr{P}(G,n)$ acts on the open sets in $R^n\times G$ as follows: if $(x,g)\in R^n\times G$ and $(A,f,a,B)\in\mathscr{P}(G,n)$, then $(A,f,a,B)\cdot(x,g):=\big(f(x),a(x)\cdot g\big)$.

Fix a positive integer r. If we take jets of order r of the mappings which appear in the elements of $\mathscr{P}(G,n)$, then we obtain a groupoid the elements of which are quadruples of the form (x,j_rf,j_ra,y), where x and y are a source and a target of the jet j_rf, respectively, and x and $a(x)$ are those for j_ra.

Algebraical structure of this groupoid is induced by the structure of $\mathscr{P}(G,n)$. We denote this groupoid by $\mathscr{G}_r^n$. Evidently, $\mathscr{G}_r^n$ is a Lie groupoid. We denote by α, β and γ the following three mappings:

$$\alpha\colon (x,j_rf,j_ra,y)\mapsto x,$$

$$\beta\colon (x,j_rf,j_ra,y)\mapsto y\ \ (=f(x)),$$

$$\gamma\colon (x,j_rf,j_ra,y)\mapsto a(x).$$

The following three propositions are immediate consequences of the general theory of differentiable groupoids [13].

PROPOSITION 1. *The groupoid structure restricted to a set* $\{u\in\mathscr{G}_r^n\,|\,\alpha(u)=\beta(u)=c$, *c is fixed in* $R^n\}$ *is a Lie group. If c varies over* R^n, *then we obtain a family of Lie groups, all being isomorfic one to another.*

We choose the group corresponding to $c=0$ as a representative. Denote it by G_r^n.

Definition 1. We call G_r^n the *r-th prolongation* of G with respect to n.

PROPOSITION 2. *A manifold formed of the set* $\{u\in\mathscr{G}_r^n\,|\,\alpha(u)=c$, *c is fixed in* $R^n\}$ *can be provided with the structure of a principal fibre bundle. A base of this bundle is* R^n, *its structure group is* G_r^n, *and canonical mapping is* β.

In what follows we always assume $c=0$. The bundle for $c=0$ will be denoted by $\mathscr{B}_r(R^n,G)$.

PROPOSITION 3. *Groupoid* $\mathscr{G}_r^n$ *acts transitively on* $\mathscr{B}_r(R^n,G)$ *by the following rule:*

$$(0,j_rh,j_rg,x)\cdot(x,j_rf,j_ra,y)=\big(0,j_r(f\circ h),j_r\big(g\cdot(a\circ h)\big),y\big).$$

Now let M be any differentiable manifold. Fix $a\in M$ and consider a set Φ of all C^∞ mappings of R^n such that 0 is send to a. Then the tangent space of the first order at a, $(TM)_a$, can be defined as a linear structure on the set of all jets of the first order of elements of Φ, targets of these jets being at the point a.

Notation. The mapping which assigns to any element $\varphi \in \Phi$ the vector $j_1\varphi$ will be denoted by $\boldsymbol{d}$.

Consider bundle $T\mathscr{B}_{r-1}(R^n, G)$ which is the tangent bundle over $\mathscr{B}_{r-1}(R^n, G)$.

PROPOSITION 4. *Groupoid $\mathscr{G}_r^n$ acts on $T\mathscr{B}_{r-1}(R^n, G)$ in the following way:*

If $\mathfrak{X} \in T\mathscr{B}_{r-1}(R^n, G)$, then there exists a mapping u, $R \ni t \mapsto u_t \in \mathscr{B}_{r-1}(R^n, G)$, such that $\mathfrak{X} = \boldsymbol{d}u$.

Denote βu_0 by x_0. Let $w = (x_0, j_r l, j_r k, y)$ be an element of $\mathscr{G}_r^n$ and let $\overline{w} = (x_0, j_{r-1} l, j_{r-1} k, y)$ be its natural map on $\mathscr{G}_{r-1}^n$. Then we can write u explicitly as follows:

$$u_t = \big(x_t, j_{r-1}s_t, j_{r-1}e_t, s_t(x_t)\big).$$

If t is sufficiently close to 0, then there is a mapping

$$t \mapsto \big(0, j_{r-1}(l \circ s_t), j_{r-1}\big(e_t \cdot (k \circ s_t)\big), l \circ s_t\big).$$

Its first order jet at 0 (i.e., having 0 as its source) is a vector which belongs to $T\mathscr{B}_{r-1}(R^n, G)$ and depends on $\mathfrak{X}$ and w only.

Thus we have obtained

PROPOSITION 5. *For any element $w \in \mathscr{G}_r^n$, $w = (x, j_r l, j_r k, y)$, there is a linear mapping w_* which sends each vector tangent to $\mathscr{B}_r(R^n, G)$ at the point $(0, j_r s, j_r e, x)$ to a vector tangent to $\mathscr{B}_{r-1}(R^n, G)$ at the point $(0, j_r\, id., j_r\, id., 0)$.*

Denote by $\imath$ the $(r-1)$-jets of identical mapping of R^n onto itself, the target of $\imath$ being 0. If o denotes the unit element of G, then we denote by $\hat{\imath}$ the $(r-1)$-jet of the mapping $R^n \to o$. Let T_0 be the space tangent to $\mathscr{B}_{r-1}(R^n, G)$ at $(0, \imath, \hat{\imath}, 0)$. According to propositions (6) and (7), a linear mapping

$$(u^{-1})_* : \big(T\mathscr{B}_{r-1}(R^n, G)\big)_{\overline{u}} \to T_0$$

is associated with each element $u \in \mathscr{B}_{r-1}(R^n, G)$, where by u^{-1} we have denoted an inverse of u considered in the groupoid $\mathscr{G}_{r-1}^n$, and $\overline{u}$ is obtained from u by the natural projection of r-jets to $(r-1)$-jets.

Definition 2. The just described field of linear forms on $\mathscr{B}_r(R^n, G)$ is called a *canonical form*.

We denote the canonical form at $u \in \mathscr{B}_r(R^n, G)$ by Ω_u and its value at vector $\mathscr{X}$ by $\langle \Omega, \mathscr{X} \rangle$.

Remark. Analogous form can be defined on the groupoid $\mathscr{G}_r^n$ but we prefer to consider it on the bundle in order to have a fixed set if its values.

We have to present some computations. Recall that $\mathfrak{X}$ corresponds to a curve which is parametrized as follows:

$$t \mapsto \big(0, j_{r-1}s_t, j_{r-1}e_t, s_t(0)\big).$$

We perform a multiplication in pseudogroup $\mathscr{P}(G, n)$,

$$\big(\theta, s_t, e_t, s(\theta)\big)\cdot\big(s_0(\theta), s_0^{\urcorner}, (e_0\circ s_0)^{-1}, \theta\big) = \big(\theta, s_0^{\urcorner}\circ s_t, e_t\cdot(e_0\circ s_0^{\urcorner}\circ s_t)^{-1}, \theta'\big),$$

which makes sense if θ is a convenient neighbourhood of 0 in R^n and t is sufficiently close to the number 0. If we pass to jets which have their sources at 0, then we obtain a curve

(1) $$t \mapsto \Big(0, j_{r-1}s_0^{\urcorner}\circ s_t, j_{r-1}\big(e_t\cdot(e_0\circ s^{\urcorner}\circ s_t)\big), s_0^{\urcorner}\circ s_t(o)\Big).$$

Applying to this curve the operator $\boldsymbol{d}$, we obtain the vector $\langle\Omega, \mathfrak{X}\rangle$.

It follows from (9) that the form Ω can be split into two forms, say ω and φ. Namely, $\langle\omega, \mathfrak{X}\rangle$ is equal to $\boldsymbol{d}(j_{r-1}s_0^{\urcorner}\circ s)$ and $\langle\varphi, \mathfrak{X}\rangle$ is $\boldsymbol{d}j_{r-1}\big(e\cdot(e_0\circ s^{\urcorner}\circ s)\big)$. The form ω is well known. Its intrinsic construction has been given by Kobayashi [6] (cf. also [11], [12], [15] and [16]). Note that Ω reduces to ω if G is a trivial one-element group. Vector $\boldsymbol{d}s_0^{\urcorner}\circ s$ is a projection of $\mathfrak{X}$ to R^n.

If we present jets by formal polynomials, we can split both ω and φ into components

(2) $$\omega \sim \left[\omega^i + \sum_{s=1}^{r-1}\frac{1}{s!}\,\omega^i_{j_1,\ldots,j_s}t^{j_1}\ldots t^{j_s}\right]_{i=1,\ldots,n},$$

(3) $$\varphi \sim \left[\varphi^a + \sum_{s=1}^{r-1}\frac{1}{s!}\,\varphi^a_{j_1,\ldots,j_s}t^{j_1}\ldots t^{j_s}\right]_{a=1,\ldots,\delta},$$

where $\delta = \dim G$.

In order to write following computations in a possibly compact manner, we introduce the following notations. Let

$$[X^A_{i_1\ldots i_s}]_{A=1,\ldots,N;\ 0\leqslant i_1,\ldots,i_s\leqslant n}$$

be a system of local coordinates of N holonomical jets. Define inductively the following operation on these systems:

$$(X^1\ldots X^N)_{\{i\}} = X^1_iX^2\ldots X^N + X^1X^2_i\ldots X^N + \ldots + X^1X^2\ldots X^N_i,$$

$$(X^1\ldots X^N)_{\{i_1\ldots i_si_{s+1}\}} = \big((X^1\ldots X^N)_{\{i_1\ldots i_s\}}\big)_{\{i_{s+1}\}}.$$

Dots as subscripts of X's mean that we delete those summands which do not contain any lower index at the X signed by a dot. In particular, we have for $N = 2$

$$(X^1X^2)_{\{i_1\dots i_s\}} = X^1_{i_1\dots i_s}X^2 + sX^1_{(i_1\dots i_{s-1}}X^2_{i_s)} + \dots +$$
$$+ \frac{s!}{p!(s-p)!}(X^1_{(i_1\dots i_p}X^2_{i_{p+1}+\dots i_s)} + \dots + X^1X^2_{i_1\dots i_s},$$
$$(X^{\cdot 1}X^2)_{\{i_1\dots i_s\}} = (X^1X^2)_{\{i_1\dots i_s\}} - X^1X^2_{i_1\dots i_s},$$
$$(X^{\cdot 1}X^{\cdot 2})_{\{i_1\dots i_s\}} = (X^1X^2)_{\{i_1\dots i_s\}} - X^1_{i_1\dots i_s}X^2 - X^1X^2_{i_1\dots i_s}.$$

Our symbols differ slightly from those used in [10], [11] and [16].

Using above-mentioned notation we may write briefly differential identities satisfied by components $\omega^i, \omega^i_j, \dots, \omega^i_{j_1\dots j_{r-1}}$ of ω defined by (2). We have (cf. [9]-[12], [15] and [16]):

$$d\omega^i = -\omega^i_h \wedge \omega^h, \quad \dots, \quad d\omega^i_{j_1\dots j_{r-1}} = -(\omega^i_h \wedge \omega^h)_{\{j_1\dots j_{r-1}\}}.$$

In formula (1) a vector $\boldsymbol{d}(j_{r-1}e\circ s_0^{\neg}\circ s)$ appears. Recall that the mapping $t \mapsto s_0^{\neg}\circ s$ yields a vector $\langle\omega, \mathfrak{X}\rangle$. The mapping $\mathfrak{X} \mapsto \boldsymbol{d}(j_{r-1}e\circ s_0^{\neg}\circ s)$ is linear. Denote the corresponding linear form by ψ. We have to find its coordinate expression. In order to do it, write a polynomial expression for $j_{r-1}e$:

$$j_{r-1}e \sim \Big[a^a + \sum_{s=1}^{r-1}\frac{1}{s!}a^a_{i_1\dots i_s}t^{i_1}\dots t^{i_s}\Big]_{a=1,\dots,\delta}. \tag{4}$$

PROPOSITION 6. *Mapping $\mathfrak{X} \mapsto \boldsymbol{d}(j_{r-1}e\circ f_0^{\neg}\circ f)$ is a linear form which obeys the representation*

$$\psi \sim \Big[\sum_{s=1}^{r-1}\frac{1}{s!}(a^a_j\omega^j)_{\{i_1\dots i_s\}}t^{i_1}\dots t^{i_s}\Big]_{a=1,\dots,\delta}$$

or, in an equivalent form,

$$\langle\psi, \mathfrak{X}\rangle \sim \Big[\sum_{s=1}^{r-1}\frac{1}{s!}(a^a_j\langle\omega^j, \mathfrak{X}\rangle)_{\{i_1\dots i_s\}}t^{i_1}\dots t^{i_s}\Big]_{a=1,\dots,\delta}.$$

Our proposition will follow from the following

LEMMA 1. *Let $[p^\varkappa]$ be an N-tuple of polynomials in n variables, $N \geqslant 1$, $n \geqslant 1$. Let Q be an open neighbourhood of 0 in R^1. Let $Q \ni t \mapsto [r^k_t]$ be a C^∞ family of n-tuples of polynomials in n variables. Put*

$$p^\varkappa(z^1, \dots, z^n) = P^\varkappa + \sum\frac{1}{s!}P^\varkappa_{i_1\dots i_s}z^{i_1}\dots z^{i_s},$$

$$r_t^k(z^1, \ldots, z^n) = \mathscr{R}^k(t) + \sum \frac{1}{s!} \mathscr{R}^k_{i_1 \ldots i_s}(t) z^{i_1} \ldots z^{i_n},$$

$$dr^k(z^1, \ldots, z^n) = \boldsymbol{r}^k + \sum \frac{1}{s!} \boldsymbol{r}^k_{i_1 \ldots i_s} z^{i_1} \ldots z^{i_s},$$

and consider a superposition

$$p^\varkappa \circ r_t(z^1, \ldots, z^n) = p^\varkappa\big(r_t^1(z^1, \ldots, z^n), \ldots, r_t^n(z^1, \ldots, z^n)\big).$$

If we have $r_0^i(z^1, \ldots, z^n) = z^i$ identically with respect to i and z, then we have

$$(5) \qquad (dp^\varkappa \circ r)(z^1, \ldots, z^n) = \sum \frac{1}{s!} (P_j^\varkappa r^j)_{\{i_1 \ldots i_s\}} z^{i_1} \ldots z^{i_s}.$$

Proof of lemma follows by a direct computation in which we use the following identity:

$$(p^\varkappa \circ r_t)(z^1, \ldots, z^n)$$
$$= P^\varkappa + \sum \frac{1}{p!} \sum \frac{1}{s!} P^\varkappa_{h_1 \ldots h_p}\big(\mathscr{R}^{h_1}(t) \ldots \mathscr{R}^{h_p}(t)\big)_{\{i_1 \ldots i_s\}} z^{i_1} \ldots z^{i_p}.$$

After a differentiation we obtain, under assumptions of the lemma, formula (5).

Proof of proposition 6. In lemma 1 take the polynomial expressions instead of $p^\varkappa$ and take the polynomial representation of $j_{r-1}(s_0^{\urcorner} \circ s)$ instead of r. Assumptions of the lemma are evidently satisfied. Thus the numbers $\boldsymbol{r}^k, \boldsymbol{r}^k_i, \ldots, \boldsymbol{r}^k_{i_1 \ldots i_{r-1}}$ correspond to $\langle \omega^k, \mathfrak{X} \rangle$, $\langle \omega^k_i, \mathfrak{X} \rangle, \ldots,$ $\langle \omega^k_{i_1 \ldots i_{r-1}}, \mathfrak{X} \rangle$ and the proposition follows.

We have to express in local coordinates a rule of multiplication inside G. Let g be any element of G. If $(\mathcal{O}, m)$ is a local map covering the unity o of G, i.e., $\mathcal{O}$ is a domain in G and $m\colon \mathcal{O} \to R^\delta$, then using the right translations we obtain two other maps, say $(\mathcal{O}_g, m_g)$ and $({}_g\mathcal{O}, {}_gm)$, such that the first of them covers g and the second covers g^{-1}. If we assume that $m(o) = 0$; then we have $m_g(g) = 0$ and ${}_gm(g^{-1}) = 0$. Thus there exist C^∞ functions $L^1, \ldots, L^\delta$ such that if $a \in \mathcal{O}_g$, $b \in {}_g\mathcal{O}$ and $a \cdot b \in \mathcal{O}$, then

$$m^a(a \cdot b) = L^a\big(m_g^1(a), \ldots, m_g^\delta(a); m^1(a), \ldots, {}_gm^\delta(a)\big),$$

where $a = 1, \ldots, \delta$.

Denote by $L^a_{\mu;}$ (respectively, by $L^a_{;\mu}$) the derivative of L^a with respect to the μ-th (respectively, to the $(\delta+\mu)$-th) argument. Using traditional notation, we should write

$$L^a_{\mu;}(x^1, \ldots, x^\delta; y^1, \ldots, y^\delta) = \frac{\partial L^a}{\partial x^\mu}(x^1, \ldots, x^\delta; y^1, \ldots, y^\delta),$$

$$L^a_{;\mu}(x^1, \ldots, x^\delta; y^1, \ldots, y^\delta) = \frac{\partial L^a}{\partial y^\mu}(x^1, \ldots, x^\delta; y^1, \ldots, y^\delta).$$

Assume that $m^1(o) = \ldots = m^\delta(o) = 0$ and put

$$q^a_\mu(b) = L^a_\mu(0, \ldots, 0; b^1, \ldots, b^\delta).$$

Matrix q is inversible and it is equal to the unit matrix if $b = 0$. Set

$$q^a_{\mu\nu}(b) = \partial_\nu L^a_{\mu;}(b), \qquad \text{where } b = (b^1, \ldots, b^\delta),$$
$$q^a_{\mu\nu}(0) = p^a_{\mu\nu}.$$

Thus we have the following expressions for structure constants $C^a_{\mu\nu}$ of the Lie algebra of G:

$$C^a_{\mu\nu} = p^a_{\mu\nu} - p^a_{\nu\mu}.$$

Turn to formula (3). We find explicit expressions for components $\varphi^a, \varphi^a_i, \ldots, \varphi^a_{i_1\ldots i_{r-1}}$ of the form Ω. It follows from formulas (1), (4), and from proposition 6 that the following identities must be satisfied:

$$a^a + da^a + (a^a_i + da^a_i)t^i + \ldots + \frac{1}{s!}(a^a_{i_1\ldots i_s} + da^a_{i_1\ldots i_s})t^{i_1}\ldots t^{i_s} \tag{6}$$

$$= L^a(\text{polynomials representing } \varphi;\ \text{polynomials representing } \psi)$$
$$(a = 1, \ldots, \delta;\ s = 1, \ldots, r-1).$$

We write

$$q^a_{\beta\gamma} = \partial_\gamma q^a_\beta, \qquad \ldots, \qquad q^a_{\beta\gamma_1\ldots\gamma_p} = \partial_{\gamma_p}\ldots\partial_{\gamma_1} q^a_\beta.$$

Then we expand the right-hand member of (6) into a power series at the point $(0, \ldots, 0; a^1, \ldots, a^\delta)$. A comparison of coefficients at t^{i_k} yields

$$\begin{aligned}
da^a &= a^a_j\omega^j + q^a_\beta(a)\varphi^\beta,\\
da^a_i &= a^a_{ji}\omega^j + a^a_j\omega^j_i + q^a_{\beta\gamma}(a)a^\gamma_i\varphi^\beta + q^a_\beta(a)\varphi^\beta_i,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
da^a_{i_1\ldots i_s} &= (a^a_j\omega^j)_{\{i_1\ldots i_s\}} +\\
&\quad + \sum \frac{1}{p!} q^a_{\beta\gamma_1\ldots\gamma_p}(a)(a^{\gamma_1}_{\cdot}\ldots a^{\gamma_p}_{\cdot}\varphi^\beta_{\cdot})_{\{i_1\ldots i_s\}} + q^a_\beta(a)\varphi^\beta_{i_1\ldots i_s}
\end{aligned} \tag{7}$$
$$\text{for } s = 1, \ldots, r-1.$$

Hence we obtain directly the following recurrent formulas for the components of φ:

$$\begin{aligned}
\varphi^a &= l^a_\mu(a)(da^\mu - a^\mu_j\omega^j),\\
\varphi^a_i &= l^a_\mu(a)\big(da^\mu_i - (a^\mu_j\omega^j)_{\{i\}} - q^\mu_{\beta\gamma}(a)a^\gamma_i\varphi^\beta\big),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
\varphi^a_{i_1\ldots i_s} &= l^a_\mu(a)\big(da^\mu_{i_1\ldots i_s} - (a^\mu_j\omega^j)_{\{i_1\ldots i_s\}} -\\
&\quad - \sum_\lambda \frac{1}{\lambda!} q^\mu_{\beta\gamma_1\ldots\gamma_\lambda}(a)(a^{\gamma_1}_{\cdot}\ldots a^{\gamma_\lambda}_{\cdot}\varphi^\beta_{\cdot})_{\{i_1\ldots i_s\}}\big).
\end{aligned}$$

$[l^a_\mu]_{a,\mu=1,\ldots,\delta}$ denotes here the matrix inverse to the matrix $[q^\mu_\beta]$.

PROPOSITION 7. *If we set $\omega^i = 0$ for $i = 1, \ldots, n$, then φ reduces to the fundamental left invariant form on G_r^n.*

This proposition follows directly from the construction of G_r^n and of Ω (see proposition 1). Observe that $\omega^i = 0$ corresponds to the case $a(u) = \beta(u) = 0$. Thus we can restrict ourselves to a standard fibre of the bundle $\mathscr{B}_r(R^n, G)$, i.e., to G_r^n.

2. Our purpose is to show that the components of the form Ω satisfy certain differential identities which in fact are structure equations of Cartan and Laptev. Then we conclude that the process of prolongation based on the theory of jets gives us prolonged groups together with their fundamental forms and these forms are identical with those of Laptev [10].

Keeping in mind the introduced notation we have

LEMMA 2. *The following identities hold*:

$$q^a_{\beta\gamma}(a)q^\gamma_\delta(a) - q^a_{\gamma\delta}(a)q^\gamma_\beta(a) = q^a_\varkappa(a)C^\varkappa_{\beta\gamma}.$$

Proof. If a, b, c are in a suitable domain in R^n, then $L^a(c; L(b; a)) = L^a(L(c; b); a)$, where $L = (L^1, \ldots, L^\delta)$. Differentiating both members with respect to the β-th coordinate of c and substituting $c = 0$ we obtain

$$L^a_{\beta;}(0; L(b; a)) = L^a_{\lambda;}(b; a)L^\lambda_{;\beta}(0; b).$$

The same procedure with b yields

$$L^a_{\beta;\delta}(0; a)L^\delta_{\gamma;}(0, a) = L^a_{\lambda;}(0, a)L^\lambda_{\beta\gamma;}(0, 0) + L^a_{\lambda;\gamma}(0, a)L^\lambda_{\beta;}(0, 0). \tag{8}$$

Making use of common identities $L^\lambda_{\beta;}(0; 0) = \delta^\lambda_\beta$, and of the above-made assumptions $L^\lambda_{\beta;\gamma}(0; a) = q^\lambda_{\beta\gamma}(0)$, $q^a_{\beta\gamma}(0) = p^a_{\beta\gamma}$, $C^a_{\beta\gamma} = p^a_{\beta\gamma} - p^a_{\gamma\beta}$, we can write (8) as follows:

$$q^a_{\beta\gamma}(a)q^\gamma_\delta(a) = q^a_\lambda(a) + L^\lambda_{\beta\gamma}(0; 0)L^a_{\lambda;}(a).$$

In view of the identity $L^a_{\beta\gamma}(0; a) = L^a_{\gamma\beta}(0; a)$, we obtain the lemma by alternating lower indices.

THEOREM 1. *Components $\varphi^a, \varphi^a_i, \ldots, \varphi^a_{i_1 \ldots i_s}$ of the form Ω satisfy the following differential identities*:

$$\begin{aligned}
d\varphi^\beta &= \frac{1}{2} C^\beta_{\gamma\delta}\varphi^\gamma \wedge \varphi^\delta + \omega^j \wedge \varphi^\beta_j, \\
d\varphi^\beta_i &= \frac{1}{2} C^\beta_{\gamma\delta}\varphi^\gamma_i \wedge \varphi^\delta + \frac{1}{2} C^\beta_{\gamma\delta}\varphi^\gamma \wedge \varphi^\delta_i + \omega^j_i \wedge \varphi^\beta_j + \omega^j \wedge \varphi^\beta_{ji}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
d\varphi^\beta_{i_1 \ldots i_s} &= \frac{1}{2} C^\beta_{\gamma\delta}(\varphi^\gamma \wedge \varphi^\delta)_{\{i_1 \ldots i_s\}} + (\omega^j \wedge \varphi^\beta_j)_{\{i_1 \ldots i_s\}}.
\end{aligned} \tag{9}$$

Proof. We prove our theorem by induction. We have (see (7))

$$da^a = a^a_j \omega^j + q^a_\beta(a)\omega^\beta,$$

$$(10) \qquad da^a_i = a^a_{ji}\omega^j + a^a_j \omega^j_i + q^a_{\beta\gamma}(a) a^\gamma_i \varphi^\beta + q^a_\beta(a)\varphi^\beta_i.$$

If we differentiate externally the first of these equalities, then on the left-hand side appears 0. We replace da^a_i which appears in the right-hand member by its expression (10) and we replace $d\omega^j$ by $\omega^k \wedge \omega^j_k$. We also use equalities

$$dq^a_\beta(a) = q^a_{\beta\gamma}(a)(a^\gamma_j \omega^j + q^\gamma_\delta(a)\varphi^\delta).$$

Then we obtain

$$\begin{aligned} 0 = a^a_{kj}\omega^k \wedge \omega^j + a^a_k \omega^k_j \wedge \omega^j + a^a_j \omega^k \wedge \omega^j_k + \\ + q^a_{\beta\gamma}(a) a^\gamma_j \varphi^\beta \wedge \omega^j + q^a_{\beta\gamma}(a) a^\gamma_j \omega^j \wedge \varphi^\beta + \\ + q^a_\beta(a)\varphi^\beta_j \wedge \omega^j + q^a_{\beta\gamma}(a) q^\gamma_\delta(a)\varphi^\delta \wedge \varphi^\beta + q^a_\beta(a) d\varphi^\beta. \end{aligned}$$

The first and the second rows vanish. Now we apply lemma 2 to the second term in the third row. This yields

$$q^a_\beta(a)\left(d\varphi^\beta - \frac{1}{2} C^\beta_{\gamma\delta}\varphi^\gamma \wedge \varphi^\delta - \omega^j \wedge \varphi^\beta_j\right) = 0.$$

Thus, in view of the non-singularity of the matrix q, we obtain the first identity of theorem 1. Second step of the induction is analogous. Assuming that theorem 1 holds for $s = 0, 1, \ldots, p$, $p \leqslant r-2$, we differentiate the p-th of equalities (9) and make use of these equalities up to the order $p+1$ and of the assumption of the induction. Thus we obtain some identity which is to be reduced to $(p+1)$-th one of (9). We apply again lemma 2 and identities which are obtained from lemma 2 by a direct differentiation term by term with respect to all variables. Calculation does not contain difficult steps, but since it is rather inconvenient for presenting it in details, we leave it.

Following Laptev's theory, the existence of prolonged groups is a consequence of the following facts:

1° Starting from the structure equations

$$d\varphi^a = \frac{1}{2} C^a_{\beta\gamma}\varphi^\beta \wedge \varphi^\gamma + \omega^j \wedge \varphi^a_j$$

one can prove by means of generalized lemma of Cartan that there exist forms $\varphi^a_i, \varphi^a_{i_1 i_2}, \ldots, \varphi^a_{i_1 \ldots i_{r-1}}$ which satisfy identities (9).

2° If $\omega^1 = \ldots = \omega^n = 0$, then structure equations (9) become Maurer-Cartan equations of some group (more precisely, of some local Lie group). These groups form a sequence of prolongations of the initial group G.

[5] — *Su prolongamenti normali e su distribuzioni strutturali. Applicazioni a spazii fibrati [On normal prolongations and on structural distributions. Applications to fibre bundles]*, Atti del Convegno Internazionale di Geometria Differenziale, Bologna 1947.

[6] S. Kobayashi, *Canonical forms on frame bundles of higher order contact*, Proceedings of Symposia (Differential Geometry) American Mathematical Society, (1961), p. 186-193.

[7] I. Kolář, *Geometric objects of submanifolds of a space with fundamental Lie pseudogroups*, Commentationes Mathematicae Universitatis Carolinae 11. 2 (1970), p. 587-589.

[8] — *Canonical forms on the prolongations of principale fibre bundles*, Revue Roumaine de Mathématiques Pures et Appliquées 16. 7 (1971), p. 1091-1106.

[9] Г. Ф. Лаптев, *Дифференциальная геометрия погружённых многообразий*, Труды московского математического общества 2 (1953), p. 275-372.

[10] — *Структурные уравнения главного расслоенного многобразия*, Труды геометрического семинара 2 (1969), p. 161-179.

[11] Л. Е. Евтушик, *Дифференциальные связности и инфинитезимальные преобразования продолженной псевдогруппы*, ibidem 2 (1969), p. 119-150.

[12] O. Nanu, *Sur les prolongements d'une variété différentiable*, Comptes Rendus de l'Académie des Sciences du Paris 265 (1967), p. 852-855.

[13] Ngô van Que, *Du prolongement des espaces fibrés et des structures infinitésimales*, Annales de l'Institut Fourier de Grenoble 17 (1967), p. 157-223.

[14] S. Sternberg, *Lectures on differential geometry*, Prentice Hall 1964.

[15] A. Szybiak, *On prolongation of fibre structures and connections of higher order*, Bulletin de l'Académie Polonaise des Sciences, Série des sciences mathématiques, physiques et astronomiques, 13 (1965), p. 661-664.

[16] — *Generalized tangent bundles*, ibidem 17 (1969), p. 289-297.

[17] А. М. Васильев, *Дифференциальная алгебра как аппарат дифференциальной геометрии*, Труды геометрического семинара 1 (1966), p. 33-61.

Reçu par la Rédaction le 11. 11. 1970