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CONCERNING NON-COMMUTATIVE BANACH ALGEBRAS OF TYPE ES

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In paper [5], we have introduced a class of commutative Banach algebras which we have called *ES-algebras* (after the term "extension from subalgebras"). A commutative Banach algebra A is said to belong to the class ES (or is of type ES, written as $A \in \mathrm{ES}$) if for every closed subalgebra $A_0 \subset A$ every multiplicative linear functional defined on A_0 can be extended to such a functional defined on the whole of A. Theorem 1 of [5] states that a commutative complex Banach algebra is of type ES if and only if each of its elements has a totally disconnected spectrum.

In this paper, we extend the concept of ES-algebras onto non-commutative complex Banach algebras, we study some properties of ES-algebras, and, as an illustration, we prove that for every compact group G the algebra $L_p(G)$, $1 \leqslant p < \infty$, is an ES-algebra. The following theorem is a starting point for our considerations:

THEOREM 1. *Let A be a complex Banach algebra with unit e. The following conditions are equivalent*:

(a_1) *Every commutative* (*closed*) *subalgebra of A is an ES-algebra.*

(b_1) *For every $x \in A$ its spectrum $\sigma(x)$ is totally disconnected.*

(c_1) *For every* (*closed*) *subalgebra $A_0 \subset A$, containing the unit e, we have*

$$G(A_0) = G(A) \cap A_0, \tag{1}$$

where $G(A)$ and $G(A_0)$ denote the groups of invertible elements in A and in A_0, respectively.

Proof. (a_1) $\Rightarrow$ (b_1). Since the spectrum of an element $x \in A$ is the same in A as in a commutative subalgebra $A_0 \subset A$ which contains x and the set $\{(x+\lambda e)^{-1}: x+\lambda e \in G(A)\}$, and since $A_0 \in \mathrm{ES}$, it follows, by theorem 1 of [5], that $\sigma(x)$ is totally disconnected.

(b_1) $\Rightarrow$ (a_1). Let A_0 be a commutative subalgebra of A and let A_1 be a maximal commutative subalgebra of A containing A_0. Since the spectrum of each element $x \in A_1$ is the same in A_1 as in A, we have by theorem 1 of [5], $A_1 \in \mathrm{ES}$. Consequently, $A_0 \in \mathrm{ES}$.

non (c_1) $\Rightarrow$ non (a_1). Suppose that (1) does not hold. Then there is a subalgebra $A_0 \subset A$, and an element $x_0 \in A_0$ invertible in A, but singular in A_0. If we denote by A_1 the subalgebra with a unity generated by x_0, then $A_1 \subset A_0$, and x_0 is singular in A_1. Let A_2 be a maximal commutative subalgebra of A containing x_0. We have $A_1 \subset A_2$ and x_0 is invertible in A_2. Since x_0 is singular in A_1, there exists on A_1 a multiplicative linear functional f such that $f(x_0) = 0$. Clearly the functional f cannot be extended to a multiplicative linear functional on A_2, and so $A_2 \notin \mathrm{ES}$.

non (b_1) $\Rightarrow$ non (c_1). If (b_1) does not hold, then for some $x_0 \in A$ the spectrum $\sigma(x_0)$ contains a continuum. By lemma 1 of [5] there exists an element $z \in G(A)$ which is non-invertible in the subalgebra generated by z, and so (c_1) does not hold.

Theorem 1 gives a motivation to the following definition:

Definition 1. Let A be a complex Banach algebra with a unity. It is called an *ES-algebra* if any one of equivalent conditions (a_1)-(c_1) is satisfied in it.

In order to extend this definition onto algebras without unity we recall the concept of quasi-regularity (cf. [3]). If A is a Banach algebra with unity e, then $(e-x)(e-y) = e$ is equivalent with $xy-x-y = 0$, and the last equation does not involve the unity. Writing $x \circ y = xy-x-y$, we see that $x \circ y = y \circ x = 0$ implies $(e-x)^{-1} = e-y$. If there is no unity in A, then $x \circ y = y \circ x = 0$ implies that $e-x$ is invertible in A_1 obtained from A by adjoining the unity e. If for an $x \in A$ there exists such a $y \in A$ that $x \circ y = y \circ x = 0$, then y is said to be a *quasi-inverse* of x, and x is said to be *quasi-invertible* or *quasi-regular*. Since the mapping $x \to e-x$ sends quasi-invertible elements onto invertible elements, and at the same time it sends circle product $x \circ y$ onto ordinary product xy, it follows that the set $Q(A)$ of all quasi-regular elements in A forms a group under the circle multiplication $x \circ y$, and it is an open set in A (no matter whether there is a unity in A or not).

Since the spectrum of an element x in an algebra A without unity is defined as the spectrum of x in A_1, where A_1 is obtained from A by adjoining a unity e, we may by above remarks, reformulate theorem 1 as follows:

THEOREM 2. *In a complex Banach algebra A the following conditions are equivalent*:

(a_2) *Every commutative* (*closed*) *subalgebra of A is an ES-algebra.*

(b_2) *For every $x \in A$ the spectrum $\sigma(x)$ is totally disconnected.*

(c_2) *For every closed subalgebra $A_0 \subset A$ we have*

$$Q(A_0) = Q(A) \cap A_0, \tag{2}$$

where $Q(A)$ and $Q(A_0)$ denote the groups of quasi-regular elements in A and in A_0, respectively.

We can, in turn, give a general definition of a complex Banach of type ES.

Definition 2. A complex Banach algebra is called an *ES-algebra* if any one of equivalent conditions (a_2)-(c_2) is satisfied in it.

We prove now some properties of ES-algebras. First we prove that a homomorphic image of an ES-algebra is again an ES-algebra.

THEOREM 3. *Let A and $\tilde{A}$ be two complex Banach algebras and let $A \in ES$. If there exists a (continuous) homomorphism T of A onto $\tilde{A}$, then $\tilde{A} \in ES$.*

Proof. Let A_1 and $\tilde{A}_1$ denote the algebras obtained from A and $\tilde{A}$ by adjoining unity e_1 and $\tilde{e}_1$ (we can do it even if A and $\tilde{A}$ already possess unities e and $\tilde{e}$; in this case e and $\tilde{e}$ become idempotents in A_1 and $\tilde{A}_1$). We can now extend T to a homomorphism of A_1 onto $\tilde{A}_1$ by setting

$$T(x+\lambda e_1) = Tx+\lambda \tilde{e}_1.$$

By theorem 5 of [2] we have

$$\sigma_{\tilde{A}_1}(Tx) \subset \sigma_{A_1}(x) \tag{3}$$

for every x in A_1 (we denote here by $\sigma_B(x)$ the spectrum of x in B in the case when more then one algebra is involved). On the other hand, it is easy to see that

$$\sigma_{A_1}(x) = \sigma_A(x) \cup \{0\}$$

for every $x \in A \subset A_1$. So, by formula (3) and (b_2), $\sigma_{\tilde{A}}(x)$ is totally disconnected for every $x \in \tilde{A}$, which means that $\tilde{A} \in$ ES.

Remark. If $A \in$ES and T is a homomorphism of A onto a normed algebra R, then the completion $\bar{P}$ of R need not to be an ES-algebra. If we take the algebra $\Lambda_\alpha(E)$ of all α-Lipschitz functions defined on the Cantor set E, we obtain an ES-algebra (cf. [5]). On the other hand, if T is the identity mapping of $\Lambda_\alpha(E)$ into $C(E)$, then $\overline{T\Lambda_\alpha(E)} = C(E)$ is not an ES-algebra.

As a corollary we obtain

THEOREM 4. *Let A be a complex Banach algebra, and $A \in ES$. If I is a two-sided closed ideal in A, then $A/I \in ES$.*

THEOREM 5. *Let A be the cartesian product of a finite number of complex Banach algebras $A_1, \ldots, A_n$. Then $A \epsilon ES$, provided that $A_i \epsilon ES$, $i = 1, 2, \ldots, n$.*

Proof. If $x \epsilon A$, then $x = (x_1, \ldots, x_n)$, $x_i \epsilon A_i$, $i = 1, 2, \ldots, n$, and

$$\sigma(x) = \bigcup_{i=1}^{n} \sigma_{A_i}(x_i).$$

So, by (b_2), $\sigma(x)$ is totally disconnected as a finite union of totally disconnected compact sets.

We turn now to an example. Let G be a compact group and let $L_p(G)$ be taken with respect to the normalized Haar measure on G. It is known (cf e.g. [4]) that $L_p(G)$, $1 \leqslant p < \infty$, is a Banach algebra under the convolution

$$x * y = \int x(\tau^{-1}t) y(\tau) d\tau,$$

and we have $|x*y|_p \leqslant |x|_p |y|_p$, where $|x|_p = \left[\int |x|^p dt\right]^{1/p}$. We also have $L_p(G) \subset L_1(G)$, and

$$|x|_1 \leqslant |x|_p \tag{4}$$

for every $x \epsilon L_p(G)$, $p \geqslant 1$. We shall need in the sequel the following

LEMMA. *Let A_0 be a subalgebra of a complex Banach algebra A. Suppose that for every $x \epsilon A_0$ the spectrum $\sigma_A(x)$ is totally disconnected. Then $A_0 \epsilon ES$.*

Proof. Since $\sigma_A(x)$ is nowhere dense and fails to separate the complex plane, we have, by [1], Chapter IX, section 1, corollary 10, $\sigma_A(x) = \sigma_{A_0}(x)$ for every $x \epsilon A_0$, and so $A_0 \epsilon$ES.

THEOREM 6. *Let G be a compact group, and $1 \leqslant p < \infty$. Then the algebra $L_p(G)$ is an ES-algebra.*

Proof. First of all we imbed $L_p(G)$ in an algebra of operators. We may consider elements of $L_p(G)$ as endomorphisms of $L_p(G)$, but, since usually there is no unity in $L_p(G)$, the operator norm is usually non-equivalent with the original norm in $L_p(G)$. So first we adjoin, if necessary, a unity e to the algebra $L_p(G)$ and obtain in this way an algebra A_1. The elements of A_1 are of the form $x + \lambda e$, where $x \epsilon L_p(G)$, λ is a complex scalar, and A_1 is complete in the norm $\|x + \lambda e\| = |x|_p + |\lambda|$. The algebra A_1 can be now imbedded in the algebra A of all endomorphisms of A_1 and the operator norm is there equivalent with the original norm in A_1. It follows that

$$L_p(G) \subset A_1 \subset A,$$

and the operator norm on $L_p(G)$ is equivalent with $|\cdot|_p$.

Let z be a continuous function on G, and assume that $z \in L_p(G)$ (and this holds for every $p \geqslant 1$). On these conditions we shall show that the operator generated by z on A_1, given by

(5) $$x + \lambda e \to z * x + \lambda z$$

is a compact endomorphism of A_1. To this end consider the sets

$$E_z' = \{z * x \in L_p(G) : x \in L_p(G),\ |x|_p \leqslant 1\},$$

and

$$E_z'' = \{\lambda z \in L_p(G) :\ |\lambda| \leqslant 1\}.$$

Clearly, E_z'' is a compact set in $L_p(G)$, and so in A_1. We shall show that E_z' is precompact. First we show that the functions in E_z' are equicontinuous. In fact, since z is uniformly continuous, there exists, for each $\varepsilon > 0$, a neighbourhood U of the unit element in G such that $u^{-1} v \in U$ implies

(6) $$|z(u) - z(v)| < \varepsilon.$$

We have now

(7) $$|z * x(u) - z * x(v)| \leqslant \int |z(t^{-1}u) - z(t^{-1}v)|\,|x(t)|\,dt.$$

Let $|x|_p \leqslant 1$. Since $(t^{-1}u)^{-1}(t^{-1}v) = u^{-1}v \in U$, we have, by (4), (6) and (7),

$$|z * x(u) - z * x(v)| \leqslant \varepsilon |x|_1 \leqslant \varepsilon |x|_p \leqslant \varepsilon,$$

and so the family E_z' is equicontinuous. We shall now show that E_z' is a uniformly bounded family. By the Hölder inequality we have

$$|z * x(t)| \leqslant \int |z(\tau^{-1}t)|\,|x(\tau)|\,d\tau \leqslant |z|_p |x|_p \leqslant |z|_q,$$

where $1/p + 1/q = 1$. It follows that E_z' is precompact in $C(G)$ and a fortiori in $L_p(G)$. Since the mapping (5) sends the unit ball in A_1 into the set

$$E_z = \{x + y : x \in E_z',\ y \in E_z''\},$$

and since E is precompact in A_1, it follows that the operator of left multiplication by z is a compact operator in A. Since continuous functions form a dense subset in $L_p(G)$, and since the set of all compact operators in A is closed in the operator norm, it follows that $L_p(G)$ consists of compact operators. Since the spectrum of a compact operator is at most denumerable, it follows that $\sigma_A(x)$ is totally disconnected for every $x \in L_p(G)$. Applying now the lemma to $L_p(G) = A_0 \subset A_1$ we obtain the desired conclusion.

Remark. Theorem 6 is clearly false in the case of $p = \infty$.

Theorems 1-5 remain true if we replace Banach algebras by complete locally bounded algebras (p-normed algebras, for the definition see [6]). We do not know whether the results of [5] and of this paper are true for multiplicatively convex B_0-algebras (cf. [6]). So we pose the following question:

PROBLEM. Is it true that a commutative complex m-convex B_0-algebra is an ES-algebra if and only if the spectrum of none of its elements contains a continuum? (**P 654**)

Here we mean by an ES-algebra an algebra A with the property that every multiplicative-linear continuous functional defined on any subalgebra of A can be extended to such a functional defined on the whole of A.

Added in proof. A positive answer to P 654 has been given in [7].

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