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ON GENERALIZED CURVATURE TENSORS ON SOME RIEMANNIAN MANIFOLDS

BY

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1. Introduction. An n-dimensional ($n > 3$) Riemannian manifold (not necessarily of definite metric form) is said to be *conformally symmetric* [2] if its Weyl's conformal curvature tensor

$$(1) \qquad C^h{}_{ijk} = R^h{}_{ijk} - \frac{1}{n-2}(g_{ij}R^h{}_k - g_{ik}R^h{}_j + \delta^h_k R_{ij} - \delta^h_j R_{ik}) + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik})$$

satisfies the condition

$$(2) \qquad C^h{}_{ijk,l} = 0,$$

where the comma denotes covariant differentiation with respect to the metric.

It follows easily from (1) and (2) that every conformally flat ($n > 3$) as well as every locally symmetric Riemannian manifold ($n > 3$) is necessarily conformally symmetric. The converse statement fails in general ([8], Theorem 1).

Investigating conformally symmetric manifolds, Głodek discovered ([4], Theorem 2) that a connected conformally symmetric manifold with a positive-definite metric is conformally flat or its scalar curvature is constant. Using Głodek's result, the present author was able to prove the following

THEOREM A ([8], Theorem 2). *Let M be a connected conformally symmetric manifold with a positive-definite metric*, $\dim M > 4$ (1). *If M is not conformally flat, then M is locally symmetric.*

In the present paper we shall prove a slight generalization of this theorem (Theorem 1). The proof is based on the following remarkable result of Tanno:

(1) Recently, it has been proved that Theorem A is valid also for $\dim M = 4$ (A. Derdziński and W. Roter, *On conformally symmetric manifolds with metrics of indices* 0 *and* 1, to appear).

THEOREM B ([12], Theorem 6). *A connected conformally symmetric manifold (not necessarily of definite metric form) is conformally flat or its scalar curvature is constant.*

The remainder of this paper deals with a generalization of Tanno's Theorem B as well as with some conditions for a Riemannian manifold to be locally symmetric.

All considered manifolds are connected and of class C^∞.

2. Generalization of Theorem A. The following lemmas are essential tools for this section:

LEMMA 1 ([8], (12)). *Let M be a conformally symmetric manifold of contant scalar curvature. Then the metric tensor of M satisfies the relation*

$$\begin{aligned}(3)\quad & g_{hl}R_{rm,p}C^r{}_{ijk} - g_{hm}R_{rl,p}C^r{}_{ijk} + R_{hl,p}C_{mijk} - R_{hm,p}C_{lijk} - \\ & - g_{il}R_{rm,p}C^r{}_{hjk} + g_{im}R_{rl,p}C^r{}_{hjk} + R_{il,p}C_{hmjk} - R_{im,p}C_{hljk} + \\ & + g_{jl}R_{rm,p}C^r{}_{khi} - g_{jm}R_{rl,p}C^r{}_{khi} + R_{jl,p}C_{himk} - R_{jm,p}C_{hilk} + \\ & + g_{kl}R_{rm,p}C^r{}_{jih} - g_{km}R_{rl,p}C^r{}_{jih} + R_{kl,p}C_{hijm} - R_{km,p}C_{hijl} = 0.\end{aligned}$$

LEMMA 2 ([8], Lemma 4). *Let M be a conformally symmetric manifold of constant scalar curvature. If M has dimension $n > 4$, then its Weyl's conformal curvature tensor satisfies the condition*

$$(4)\qquad R_{rm,p}C^r{}_{ijk} = 0.$$

Remark 1. It was assumed in [8] that all manifolds under consideration have positive-definite metric forms. However, as one can easily verify (see [8], proofs of Lemmas 3 and 4), both Lemmas 1 and 2 remain true without this assumption.

THEOREM 1. *Let M be a conformally symmetric manifold of dimension $n > 4$. Then its Weyl's conformal curvature tensor C is null (i.e., $\langle C, C\rangle = 0$) on M or M is locally symmetric.*

Proof. Suppose that the Weyl conformal curvature tensor is not null on M. Since M cannot be conformally flat, Theorem B yields $R = \text{const}$ and Lemmas 1 and 2 work.

Hence, in view of (4), equation (3) takes the form

$$\begin{aligned}(5)\quad & R_{hl,p}C_{mijk} - R_{hm,p}C_{lijk} + R_{il,p}C_{hmjk} - R_{im,p}C_{hljk} + \\ & + R_{jl,p}C_{himk} - R_{jm,p}C_{hilk} + R_{kl,p}C_{hijm} - R_{km,p}C_{hijl} = 0.\end{aligned}$$

Transvecting now (5) with C^{mijk}, using the well-known formulas $C_{hijk} = C_{jkhi} = -C_{ihjk} = -C_{hikj}$, and applying (4) again, we get

$$R_{hl,p}C^{mijk}C_{mijk} = 0.$$

It follows easily from (2) that $C^{mijk}C_{mijk} = \text{const}$. Now, since the Weyl conformal curvature tensor is not null on M by assumption, we obtain $C^{mijk}C_{mijk} = \text{const} \neq 0$. Hence $R_{ij,k} = 0$ which, together with (1) and (2), implies $R_{hijk,l} = 0$.

Thus the theorem is proved.

Remark 2. If the metric form of M is positive-definite, Theorem A is an immediate consequence of Theorem 1.

3. Generalization of Theorem B. In the sequel we need the following lemmas:

LEMMA 3 ([9], Lemma 1). *If e_j and T_{ij} are numbers satisfying*

$$e_i T_{mj} + e_j T_{mi} = 0, \tag{6}$$

then either all the e_j are zero or all the T_{ij} are zero.

LEMMA 4. *If e_j and D_{hijk} are numbers satisfying*

$$e_i D_{lhkj} + e_h D_{lijk} + e_j D_{lkhi} + e_k D_{ljih} = 0, \tag{7}$$

$$D_{hijk} = D_{jkhi} = -D_{ihjk} = -D_{hikj}, \tag{8}$$

then either each e_j is zero or each D_{hijk} is zero.

Proof. Suppose that one of the e's, say e_q, is not zero. Then (7) with $i = l = j = q$ gives $2e_q D_{qhkq} = 0$ since $D_{qhkq} = D_{qkhq}$, and, therefore, $D_{qhkq} = 0$ for all h and k. Putting $i = l = q$ into (7), we obtain $e_q D_{qhkj} = 0$ and, therefore, $D_{qhkj} = 0$ for all h, k and j. Putting now $i = q$ into (7) and making use of $D_{qhkj} = 0$, we get $e_q D_{lhkj} = 0$, which leads immediately to our assertion. The lemma is proved.

Remark 3. Lemma 4 is implicitly contained in the proof of Tanno's Theorem B. We have included its proof for completeness.

Definition ([6] and [10]). Let M be a Riemannian manifold with not necessarily definite metric form, $\dim M \geqslant 3$. A $(1,3)$-tensor B of class C^∞ (with components $B^h{}_{ijk}$) will be called a *generalized curvature tensor* on M if

$$B^h{}_{ijk} + B^h{}_{jki} + B^h{}_{kij} = 0 \quad \text{(the first Bianchi identity)}, \tag{9}$$

$$B^h{}_{ijk} = -B^h{}_{ikj}, \qquad B_{hijk} = B_{jkhi}, \tag{10}$$

where $B_{hijk} = g_{rh} B^r{}_{ijk}$.

The tensor B is said to be *proper* if it satisfies the second Bianchi identity

$$B^h{}_{ijk,l} + B^h{}_{ikl,j} + B^h{}_{ilj,k} = 0.$$

For every generalized curvature tensor B there is a natural decomposition

$$B = B(1) + B(2) + B(3),$$

where

$$B(1)^h{}_{ijk} = \frac{1}{n(n-1)} S(g_{ij}\delta^h_k - g_{ik}\delta^h_j),$$

$$B(2)^h{}_{ijk} = \frac{1}{n-2}(B_{ij}\delta^h_k - B_{ik}\delta^h_j + g_{ij}B^h{}_k - g_{ik}B^h{}_j) + \frac{2}{n(n-2)} S(g_{ik}\delta^h_j - g_{ij}\delta^h_k),$$

$$B(3)^h{}_{ijk} = B^h{}_{ijk} + \frac{1}{n-2}(\delta^h_j B_{ik} - \delta^h_k B_{ij} + g_{ik}B^h{}_j - g_{ij}B^h{}_k) + \\ + \frac{1}{(n-1)(n-2)} S(g_{ij}\delta^h_k - g_{ik}\delta^h_j),$$

$B_{ij} = B^r{}_{ijr}$ are the components of the Ricci tensor $\mathrm{Ric}(B)$, and $S = S(B) = B^r{}_r$ is the scalar curvature of B. $B(3)$ is called the *Weyl conformal curvature tensor* of B.

One can easily verify that for a proper generalized curvature tensor B the relations

$$B^r{}_{ijk,r} = B_{ij,k} - B_{ik,j}, \quad B^r{}_{j,r} = \frac{1}{2} S_{,j} \tag{11}$$

hold.

THEOREM 2. *Let M be a Riemannian manifold (not necessarily of definite metric form) whose Ricci tensor satisfies the condition*

$$R_{ij,k} - R_{ik,j} = \frac{1}{2(n-1)}(R_{,k}g_{ij} - R_{,j}g_{ik}). \tag{12}$$

If B is a parallel generalized curvature tensor on M, then the scalar curvature of M is constant or $B(1) = B$ and $B(2) = B(3) \doteq 0$.

Proof. Since B is parallel on M, we obtain

$$B_{hijk,lm} - B_{hijk,ml} = 0$$

which, in view of the Ricci identity, can be written as

$$B_{rijk}R^r{}_{hlm} + B_{hrjk}R^r{}_{ilm} + B_{hirk}R^r{}_{jlm} + B_{hijr}R^r{}_{klm} = 0. \tag{13}$$

Differentiating (13) covariantly and contracting with g^{mp}, we get

$$B^r{}_{ijk}R^s{}_{lhr,s} + B^r{}_{hkj}R^s{}_{lir,s} + B^r{}_{khi}R^s{}_{ljr,s} + B^r{}_{jih}R^s{}_{lkr,s} = 0.$$

But the last equation, in view of (12), implies

$$g_{hl}R_{,r}B^r{}_{ijk} + g_{il}R_{,r}B^r{}_{hkj} + g_{lj}R_{,r}B^r{}_{khi} + g_{lk}R_{,r}B^r{}_{jih} - \\ - R_{,h}B_{lijk} - R_{,i}B_{lhkj} - R_{,j}B_{lkhi} - R_{,k}B_{ljih} = 0. \tag{14}$$

Contracting now (14) with g^{hl} and taking into account (9), we obtain

$$(n-1)R_{,r}B^r{}_{ijk} = R_{,k}B_{ij} - R_{,j}B_{ik}, \tag{15}$$

whence

$$R_{,r}B^r{}_k = \frac{1}{n}SR_{,k}. \tag{16}$$

Substituting (15) into (14), we obtain

$$\begin{aligned} g_{hl}(R_{,k}B_{ij}-R_{,j}B_{ik})+g_{il}(R_{,j}B_{hk}-R_{,k}B_{hj})+ \\ +g_{lj}(R_{,i}B_{kh}-R_{,h}B_{ki})+g_{lk}(R_{,h}B_{ij}-R_{,i}B_{jh})- \\ -(n-1)(R_{,h}B_{lijk}+R_{,i}B_{lhkj}+R_{,j}B_{lkhi}+R_{,k}B_{ljih}) = 0, \end{aligned} \tag{17}$$

which, by contracting with g^{ij} and making use of (16) and (15), implies

$$R_{,k}\left(B_{hl}-\frac{1}{n}Sg_{hl}\right)+R_{,h}\left(B_{lk}-\frac{1}{n}Sg_{lk}\right) = 0. \tag{18}$$

Since the tensor

$$T_{ij} = B_{ij}-\frac{1}{n}Sg_{ij}$$

is symmetric and parallel, as an immediate consequence of Lemma 3 we get R = const or $T_{ij} = 0$ on M. If $T_{ij} = 0$, equation (17) takes the form

$$\begin{aligned} R_{,h}\left[(n-1)B_{lijk}-\frac{1}{n}S(g_{ij}g_{lk}-g_{ik}g_{lj})\right]+ \\ +R_{,i}\left[(n-1)B_{lhkj}-\frac{1}{n}S(g_{hk}g_{lj}-g_{lk}g_{jh})\right]+ \\ +R_{,j}\left[(n-1)B_{lkhi}-\frac{1}{n}S(g_{hk}g_{il}-g_{hl}g_{ik})\right]+ \\ +R_{,k}\left[(n-1)B_{ljih}-\frac{1}{n}S(g_{ij}g_{hl}-g_{il}g_{hj})\right] = 0. \end{aligned}$$

The assertion follows now from Lemma 4 and from the fact that the tensor

$$D_{lijk} = B_{lijk}-\frac{1}{n(n-1)}S(g_{ij}g_{lk}-g_{ik}g_{lj})$$

satisfies (8) and is parallel on M.

Remark 4. It is easy to verify that the ordinary Weyl conformal curvature tensor C is a generalized curvature tensor with $\mathrm{Ric}(C) = 0$ and, therefore, $S(C) = 0$. If M is now conformally symmetric, then C is parallel and, in view of

$$C^r{}_{ijk,r} = \frac{n-3}{n-2}\left[(R_{ij,k}-R_{ik,j})-\frac{1}{2(n-1)}(R_{,k}g_{ij}-R_{,j}g_{ik})\right] = 0$$

(see [3], p. 91), condition (12) is satisfied. Hence, as an immediate consequence of Theorem 2, we have $R = \text{const}$ or the tensor C vanishes. Tanno's theorem is therefore a consequence of Theorem 2.

4. Conditions for a Riemannian manifold to be locally symmetric. In this section we obtain some necessary and sufficient conditions for a Riemannian manifold with a positive-definite metric to be locally symmetric.

We shall use the following notation:

$$(19)\qquad \begin{aligned} -S_{hijklm} &= B_{hijk,lm}-B_{hijk,ml}\\ &= -B_{rijk}R^r{}_{hlm}-B_{hrjk}R^r{}_{ilm}-B_{hirk}R^r{}_{jlm}-B_{hijr}R^r{}_{klm},\end{aligned}$$

$$(20)\qquad \begin{aligned} -S^*_{hijklm} &= C_{hijk,lm}-C_{hijk,ml}\\ &= -C_{rijk}R^r{}_{hlm}-C_{hrjk}R^r{}_{ilm}-C_{hirk}R^r{}_{jlm}-C_{hijr}R^r{}_{klm},\end{aligned}$$

$$(21)\qquad \begin{aligned} -P_{hijklm} &= R_{hijk,lm}-R_{hijk,ml}\\ &= -R_{rijk}R^r{}_{hlm}-R_{hrjk}R^r{}_{ilm}-R_{hirk}R^r{}_{jlm}-R_{hijr}R^r{}_{klm}.\end{aligned}$$

LEMMA 5. *If B is a proper generalized curvature tensor on a Riemannian manifold M, then on M the equation*

$$(22)\qquad \frac{1}{2}\Delta Q = g^{lm}B^{hijk}{}_{,m}B_{hijk,l}+2B^{hijk}B^r{}_{ijk,rh}-2B^{hijk}S_{hij}{}^r{}_{kr}$$

holds, where $Q = B^{hijk}B_{hijk}$ and Δ denotes the Laplace operator.

Proof. Since B is proper on M, we have

$$B_{hijk,lm} = -B_{hilj,mk}-B_{hikl,mj}+S_{hikljm}+S_{hiljkm}.$$

This relation, in view of (9) and (10), yields

$$g^{lm}B^{hijk}B_{hijk,lm} = 2B^{hijk}B^r{}_{ijk,rh}-2B^{hijk}S_{hij}{}^r{}_{kr}$$

which, in view of the obvious formula

$$\frac{1}{2}\Delta Q = g^{lm}B^{hijk}{}_{,m}B_{hijk,l}+g^{lm}B^{hijk}B_{hijk,lm},$$

leads immediately to (22). Thus the lemma is proved.

As a consequence of Lemma 5 we have

COROLLARY 1. *Let B be a proper generalized curvature tensor on a compact Riemannian manifold M.*

(a) *If $B^r{}_{ijk,r} = 0$ and $B^{hijk}S_{hij}{}^r{}_{kr}\leqslant 0$, then B is parallel on M.*

(b) *B is parallel on M if and only if $B^r{}_{ijk,r} = 0$ and $S_{hijklm} = 0$.*

It is known that the Weyl conformal curvature tensor C satisfies the following equation (see [1], (3.7), and [3], p. 91):

$$C_{hijk,l}+C_{hikl,j}+C_{hilj,k}$$
$$=\frac{1}{n-3}\left(g_{ik}C^{r}{}_{hjl,r}+g_{hj}C^{r}{}_{ikl,r}+g_{il}C^{r}{}_{hkj,r}+g_{hk}C^{r}{}_{ilj,r}+g_{ij}C^{r}{}_{hlk,r}+g_{hl}C^{r}{}_{ijk,r}\right).$$

Hence C is a proper generalized curvature tensor if and only if $C^{r}{}_{ijk,r}=0$.

Therefore, as a consequence of Corollary 1, we have the following result obtained in [7]:

COROLLARY 2 ([7], Theorem 1). *Let M be a compact Riemannian manifold of dimension $n>3$.*

(a) *If $C^{r}{}_{ijk,r}=0$ and $C^{hijk}S^{*}_{hij}{}^{r}{}_{kr}\leqslant 0$, then M is conformally symmetric.*

(b) *M is conformally symmetric if and only if $C^{r}{}_{ijk,r}=0$ and $S^{*}_{hijklm}=0$.*

Combining Corollary 2 with Theorem A, we get

COROLLARY 3. *Let M be a compact Riemannian manifold of dimension $n>4$.*

(a) *If $C^{r}{}_{ijk,r}=0$ and $C^{hijk}S^{*}_{hij}{}^{r}{}_{kr}\leqslant 0$, then M is conformally flat or locally symmetric.*

(b) *If M is not conformally flat, then M is locally symmetric if and only if $C^{r}{}_{ijk,r}=0$ and $S^{*}_{hijklm}=0$.*

Remark 5. Replacing in Corollary 1 the conditions $S_{hijklm}=0=B^{r}{}_{ijk,r}$ by $P_{hijklm}=0=R^{r}{}_{ijk,r}$, we obtain another characteristic of a compact locally symmetric manifold (cf. [5], [13], p. 44).

LEMMA 6. *Let B be a proper generalized curvature tensor on a compact Riemannian manifold M. If*

$$B_{ij,k}+B_{jk,i}+B_{ki,j}=0 \quad \text{and} \quad S_{hijklm}=0, \tag{23}$$

then B is parallel on M.

Proof. Applying the Laplace operator to $Q^{*}=B^{ij}B_{ij}$ and using (23), we get

$$\frac{1}{2}\Delta Q^{*}=g^{lm}B^{ij}{}_{,l}B_{ij,m}-2g^{lm}B^{ij}B_{il,jm}. \tag{24}$$

But, by (23) and (11), $S_{,j}=B^{r}{}_{j,r}=0$. The last relation, together with (24) and $B_{il,jm}-B_{il,mj}=0$, implies

$$\frac{1}{2}\Delta^{*}Q=g^{lm}B^{ij}{}_{,l}B_{ij,m}.$$

B_{ij} is therefore parallel on M. Our assertion follows now from (11) and Corollary 1.

As an immediate consequence of Lemma 6, we get

COROLLARY′4. *A compact Riemannian manifold is locally symmetric if and only if*

$$R_{ij,k}+R_{jk,i}+R_{ki,j} = 0 \quad \textit{and} \quad P_{hijklm} = 0.$$

Remark 6. Lemma 6 has been inspired by results of Sumitomo ([11], p. 129) and Simon [10].

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