COLLOQUIUM MATHEMATICUM

VOL. XXXVI 1976 FASC. 1

# ON THE PLANARITY OF 2-COMPLEXES

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**1. Introduction.** If $G$ is a graph, it follows from a classical result of Kuratowski [3] that $G$ is *planar*, i.e., may be embedded in the plane, if and only if $G$ contains no subgraph isomorphic to a subdivision of $K_{3,3}$ — the utilities graph, or a subdivision of $K_5$ — the complete graph on five vertices. For a summary of other characterizations of planar graphs see [2], Chap. 11. More generally Kuratowski's result gives a necessary and sufficient condition for a Peano continuum with no cut points to be *spherical*, i.e., embeddable in $S^2$, namely that it contains no subset homeomorphic to $K_{3,3}$ or $K_5$.

Our object is to find suitable methods of determining whether or not a 2-complex is spherical or planar. Preferably such a characterization should depend on the combinatorial structure of the complex without subdivision.

In order to state what is already known in this direction, we must recall the following definition. Denote by $F^2$ that subset of $R^3$ which consists of all points $(x, y, 0)$ with $x^2+y^2 \leqslant 1$ together with all points $(0, 0, z)$ with $0 \leqslant z \leqslant 1$. A homeomorph of $F^2$ is called a *disk with feeler*.

Related to Kuratowski's theorem is the following result, which seems to be part of the "folklore":

A connected complex is spherical if and only if it contains no subset homeomorphic to $K_{3,3}$, $K_5$ or $F^2$. If in addition the complex contains no homeomorph of $S^2$, these conditions are necessary and sufficient for planarity.

While topologically satisfactory this characterization is not of a combinatorial nature, for it requires knowledge of all subsets of the complex. As a goal, then, one might strive to restrict one's attention to subcomplexes; one might start, for example, by considering such canonic subcomplexes as the 1-skeleton and the vertex stars.

In Theorem 1 we do reduce the determination of sphericity for a 2-complex to a property of its 1-skeleton. This property is apparently not in itself purely combinatorial however, since it requires that there be a special kind of embedding of the 1-skeleton in $S^2$. The existence of such a reduction is intrinsically interesting, because it suggests a direction for further investigation.

One line of attack would be to stipulate that the 1-skeleton of the complex be planar. Obviously some further condition is needed. As suggested earlier we might impose this upon the vertex stars by insisting that the complex be locally planar. This leads to a combinatorial description of locally planar complexes in Theorem 3. The combination of local planarity together with the planarity of the 1-skeleton is still not sufficient to ensure the sphericity of a connected complex. Two simple examples are given to illustrate this. We conclude with a question, suggested by these examples, concerning a possible combinatorial characterization of planar 2-complexes.

**2. Definitions.** We shall consider a *complex* to be a collection of geometric simplexes with the usual face properties. If $C$ is a complex, then $|C|$ denotes the associated polyhedron considered as a space with the CW-topology. In line with common usage we shall suppress the notation $|C|$. Usually it will be obvious from the context whether the complex $C$ is being considered as a collection of simplexes or as a polyhedron. The *n-skeleton* of $C$ will be represented by $C^{(n)}$.

For two complexes $C$ and $D$ the *join* $C*D$ is a complex $E$ defined as follows:

There are disjoint copies $C'$ of $C$ and $D'$ of $D$ in $E$ such that each pair of simplexes $s$ of $C'$ and $t$ of $D'$ are skew in some Euclidean space; the simplexes of $E$ are the linear joins $st$ with $s \in C'$ and $t \in D'$, under the convention that the empty set acts as an identity under join: $s\emptyset = s$ and $\emptyset t = t$. If $C$ is the singleton $\{v\}$, then $\{v\}*D$ is called the *cone* over $D$ with vertex $v$.

If $s$ is a simplex of $C$, then the *star of s in C*, $\mathrm{St}(C, s)$, is the subcomplex of $C$ determined by the collection of all simplexes which have $s$ as a face. The *link of s in C*, $\mathrm{Lk}(C, s)$, is the subcomplex of $\mathrm{St}(C, s)$ consisting of all simplexes $t$ such that $s \cap t = \emptyset$. Of course $\{s\}*\mathrm{Lk}(C, s) = \mathrm{St}(C, s)$. The boundary of a simplex $s$ will be written as $\partial s$.

For spaces $X$ and $Y$, $X \approx Y$ will mean that $X$ and $Y$ are homeomorphic. The disjoint union of $X$ and $Y$ will be written as $X + Y$. It may be seen that $F^2 \approx \{v\}*(S^1 + \{w\})$. For a subset $A$ of a space $X$, $\mathrm{Cl}\, A$ will denote the closure of $A$ in $X$.

All complexes hereafter considered will be assumed to be finite unless otherwise stated.

Let $C$ be an $n$-complex and $M$ be an $n$-manifold. An embedding $f\colon C^{(n-1)} \to M$ will be called *regular* if for every $n$-simplex $s \in C$, $f(C^{(n-1)} - s)$

is contained in one component of $M-f(\partial s)$. This is equivalent to requiring that for each $n$-simplex $s$, $f|(C^{(n-1)}-s)$ is null homologous in dimension 0 reduced homology with respect to $M-f(\partial s)$. A special class of regular embeddings was studied in [4].

**3. A characterization of planarity.** In this section we shall carry out the reduction of determining when a 2-complex is spherical or planar to a requirement upon the 1-skeleton.

THEOREM 1. *A 2-complex $C$ may be embedded in $S^2$ if and only if there is a regular embedding of $C^{(1)}$ in $S^2$.*

THEOREM 2. *A 2-complex $C$ is planar if and only if $C \not\approx S^2$ and there is a regular embedding of $C^{(1)}$ in $R^2$.*

Proof. We demonstrate both theorems by first proving Theorem 1, and then building upon this to complete the proof of Theorem 2.

The necessity is obvious in both cases, for if $C \subseteq S^2$ (or $R^2$), the identity map is regular on $C^{(1)}$.

Assume $f$: $C^{(1)} \rightarrow S^2$ is a regular embedding. Let $t_1, \ldots, t_k$ be the 2-simplexes of $C$. Then for each $i$, $1 \leqslant i \leqslant k$, $S^2-f(\partial t_i)$ is the union of disjoint domains $U_i$ and $V_i$ such that $\mathrm{Cl}\, U_i$ and $\mathrm{Cl}\, V_i$ are 2-cells. Since $f$ is regular, either $U_i$ or $V_i$ is free of points of the image of $f$; suppose that in each case $U_i$ does not meet $f(C^{(1)})$. Accordingly, $f$ may be extended over $t_i$ to an embedding $f_i$ of $C^{(1)} \cup t_i$ onto $f(C^{(1)}) \cup U_i$. If $i \neq j$, then, because $f(\partial t_j) \cap U_i = \emptyset$, the image of $f_j$ does not meet $U_i$. From this it follows that the image of $f_i$ intersects the image of $f_j$ precisely in the image of $f$. Thus the extensions $f_1, \ldots, f_k$ and their inverses are respectively mutually consistent, so that

$$F = \bigcup_{i=1}^{k} f_i$$

embeds $C$ in $S^2$. This completes the proof of Theorem 1.

Now consider $S^2$ as the one-point compactification of $R^2$. Then if $f$ is a regular embedding of $C^{(1)}$ in $R^2$, it is regular with respect to $S^2$ as well. By Theorem 1, $C$ may be embedded in $S^2$. Since $C \not\approx S^2$, the embedding leaves a point $x \in S^2$ free, hence the image of $C$ lies in $S^2-\{x\} \approx R^2$.

Remark 1. Note that the proof works equally well if $C$ were assumed to be a countable locally finite 2-complex.

Remark 2. The methods of Theorems 3 and 4 of [4] may be adapted to give a proof, analogous to the proof of Theorem 1 above, of the following

THEOREM 1′. *A countable locally finite 3-complex $C$ may be embedded in $S^3$ if and only if there is a regular embedding of $C^{(2)}$ in $S^3$.*

The proof requires extensive use of piecewise linear topology, including the 2-complex approximation theorem of Bing (see [1]). Similarly, there is a 3-dimensional analogue to Theorem 2.

We illustrate how Theorem 1 may be used by applying it to the case of a 2-complex which is a triangulation of a connected 2-manifold $M$ which cannot be embedded in $S^2$. For example, $M$ could be a closed surface of genus different from that of the 2-sphere. We assume that $M$ has been triangulated in an arbitrary manner. Let $s$ be a 2-simplex of this triangulation. From classical results, it follows that $M-s$ is connected. This implies that $M^{(1)}-\partial s$ is connected. Therefore any embedding of $M^{(1}$ in a 2-manifold is regular. By Theorem 1, $M^{(1)}$ can never be embedded in $S^2$. Thus by Kuratowski's criterion, $M^{(1)}$ must contain $K_{3,3}$ or $K_5$.

**4. Locally planar complexes.** The existence of a regular embedding of $C^{(1)}$ in $S^2$ allows a straightforward exploitation of the Jordan-Schoenflies Theorem. On the other hand, these results suggest that planarity for a 2-complex should depend on the planarity of its 1-skeleton together with some additional global combinatorial hypothesis. As discussed in Section 1, the assumption of local planarity is a natural condition to consider.

A complex $C$ will be called *locally planar* if every vertex $v$ of $C$ has a neighborhood $N_v$ in $C$ such that $N_v$ is planar. Since the star of $v$ in $C$ is topologically invariant under subdivision, it may be seen that $C$ is locally planar if and only if every vertex star of $C$ is planar. The following lemma can be proved in dimensions 2 and 3 by use of piecewise linear topology. It seems useful, however, to have an elementary combinatorial demonstration of the lemma and the consequent theorem.

LEMMA. *A cone $C = \{v\} * D$ may be embedded in $R^2$ or $S^2$ if and only if $D$ may be embedded in $S^1$.*

Proof. $C$ may be embedded in $R^2$ if and only if it may be embedded in $S^2$. This is because a cone is contractible, so an embedding in $S^2$ must leave a point free.

The sufficiency is, of course, trivial. Therefore we assume that $C$ is planar. Then $D$ cannot contain a triod $T$, since $\{v\} * T$ would then contain a homeomorph of $F^2$. Similarly, if $D$ contains a circle $S \approx S^1$, it follows that $D = S$; otherwise $C$ would again contain a disk with feeler. Suppose then that $D$ contains no circle. As a polyhedron, $D$ has a finite number of components. Each nonempty component is a compact connected acyclic and atriodic polyhedron of dimension $\leqslant 1$, that is, a closed arc or a point. Therefore once again $D$ embeds in $S^1$.

This leads immediately to a simple combinatorial determination of local planarity.

THEOREM 3. *A complex $C$ is locally planar if and only if for every vertex $v$ of $C$,* $\mathrm{Lk}(C, v)$ *embeds in $S^1$.*

Proof. This theorem is a consequence of the lemma since $\mathrm{St}(C, v) = \{v\} * \mathrm{Lk}(C, v)$.

It might seem reasonable to conjecture that a complex $C$ is planar if and only if the following three conditions hold:

(I) $C$ contains no homeomorph of $S^2$,

(II) $C^{(1)}$ is planar,

(III) $C$ is locally planar.

But this is immediately disproved by the following counterexamples.

The first counterexample is depicted in Fig. 1. In $R^3$ we select a 2-simplex $|abc|$ lying in a plane $R^2$, and points $d$ above and $e$ below $R^2$.

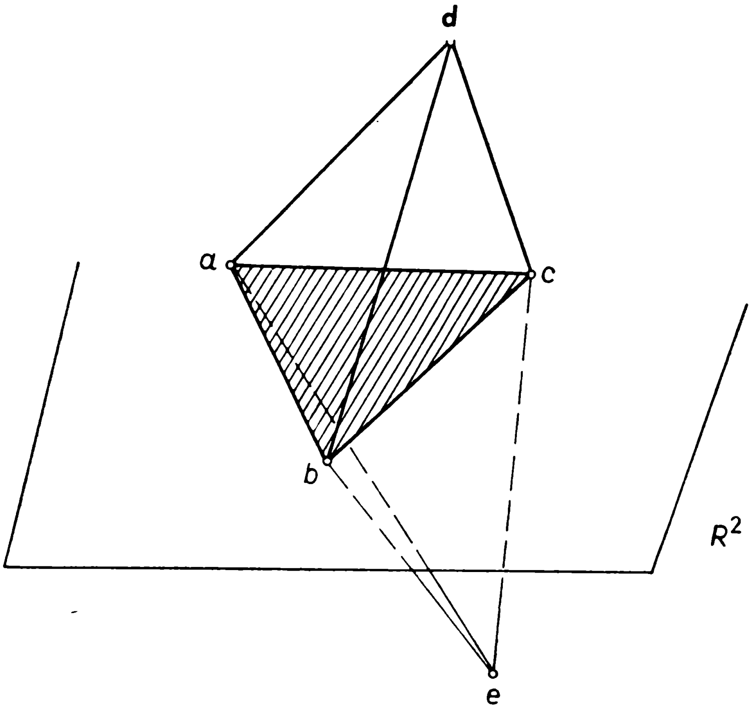


Fig. 1. A counterexample $C_1$ with $K_{3,3}$

Consider the polyhedron $C_1$ which is the union of the 2-simplex $|abc|$ and the six edges $|da|$, $|db|$, $|dc|$, $|ea|$, $|eb|$ and $|ec|$. By Theorem 3, it may be seen that $C_1$ is locally planar. It is obvious that $C_1$ satisfies the other two conditions of the statement. Taking the straight segments joining the barycenter of $|abc|$ with the vertices $a$, $b$ and $c$, we can exhibit a subset of $C_1$ which is homeomorphic to $K_{3,3}$.

For the second counterexample we require a more elaborate construction. Once again let $R^2$ be a plane in $R^3$. We select six maximally independent points in $R^3$, labeled $a$ through $f$, so that the following properties hold: $a$, $b$, $c$ and $f$ lie in $R^2$ with $f$ exterior to the 2-simplex $|abc|$; and again $d$ is above and $e$ is below $R^2$. As illustrated in Fig. 2, the polyhedron $C_2$ is the union of three 2-simplexes $|abc|$, $|acd|$ and $|abe|$ together with four edges $|af|$, $|bf|$, $|bd|$ and $|de|$. By inspection of this figure it may be observed that, for each vertex, the components of the link are arcs or points. Hence $C_2$ is locally planar by Theorem 3. An inspection of Fig. 2 taking only the vertices and edges as drawn, enables one to see an embedding of $C_2^{(1)}$ in the plane. We already have an embedding

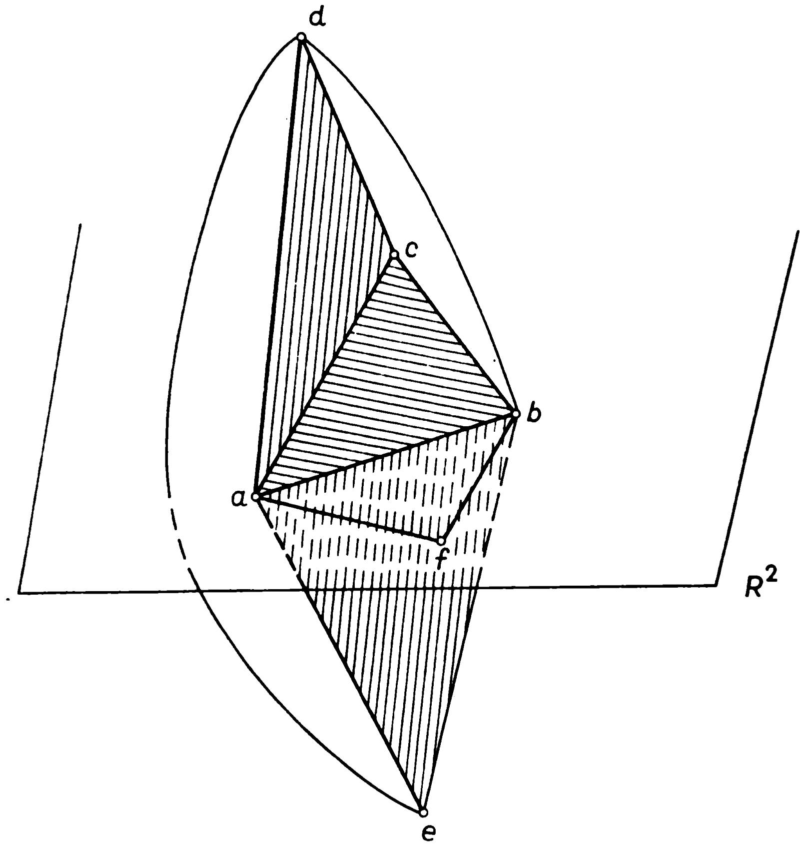


Fig. 2. A counterexample $C_2$ with $K_5$

of $K_4$ in the 1-skeleton in the following way. Consider the four vertices $a, b, d$ and $e$ together with the six paths $|af| \cup |fb|$, $|ad|$, $|ae|$, $|be|$, $|bd|$ and $|de|$.

Finally Fig. 3 shows how the above embedding of $K_4$ may be extended to an embedding of $K_5$ in $C_2$ by using the barycenter of $|abc|$ to correspond to the fifth vertex; the required extra paths are also shown.

That the third canonic pathological figure cannot appear is demonstrated in our final result.

THEOREM 4. *A complex is locally planar if and only if it contains no disk with feeler.*

Proof. As previously observed, we have a cone representation $F^2 \approx \{v\} * (S^1 + \{w\})$. It may thus be observed that $v$ lies in disks with a feeler, of arbitrarily small diameter.

Now suppose $h$ embeds $F^2$ in $C$. Let $x = h(v)$, using the representation above. The interiors of the vertex stars form an open covering of $C$. Therefore one vertex star contains $x$ in its interior. From our previous observation, this star contains a disk with feeler. Consequently, $C$ is not locally planar.

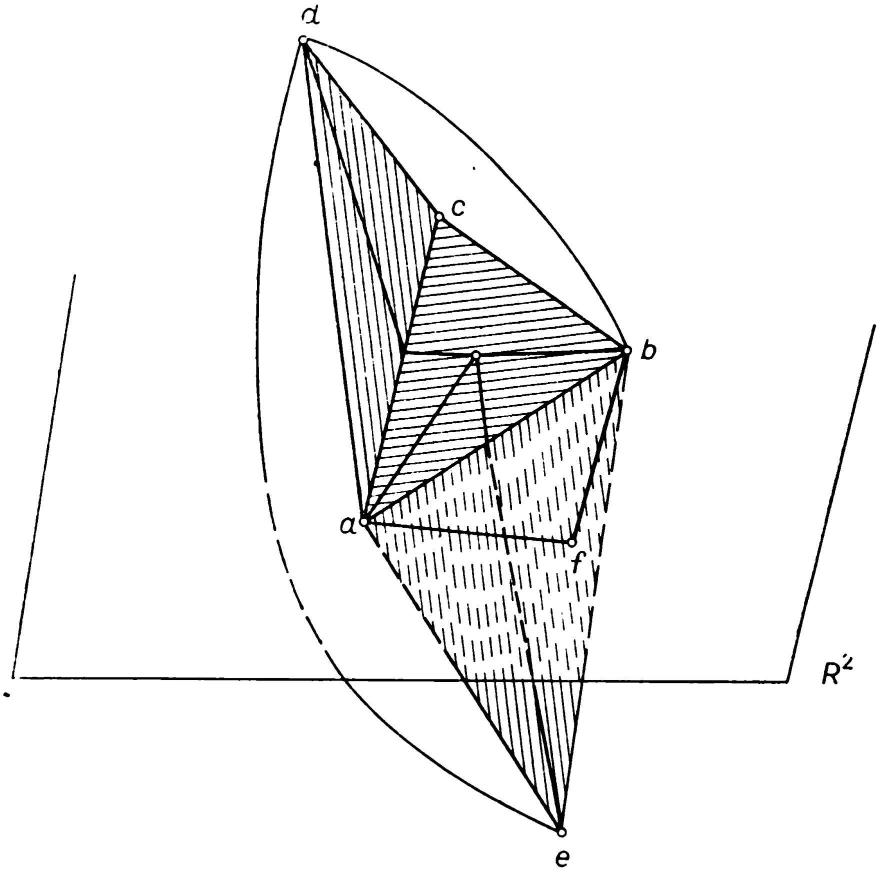


**Fig. 3. An embedding of $K_5$ in $C_2$**

To establish the converse, we assume that $C$ contains no disk with feeler. Let $v$ be a vertex of $C$. Then, as in the proof of the Lemma, we recognize that $L = \mathrm{Lk}(C, v)$ contains no triod. If $L$ contains a circle $S$, again we must have $L = S$. Lastly, if $L$ contains no circle, then as in the proof of the Lemma we conclude that $L$ embeds in $S^1$. Hence $C$ is locally planar.

The results of this section lead to the posing of our closing query.

Question. What natural combinatorial property must be added to properties (I), (II), and (III) above, in order to characterize planar 2-complexes? (**P 988**)

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*Reçu par la Rédaction le 5. 5. 1975*