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AN AXIOM SYSTEM FOR THE CLASS OF GROUPS OF DILATATIONS IN FANO–PAPPIAN AFFINE PLANES

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Introduction. In this paper an axiom system for the class of Fano–Pappian plane dilatation groups will be given. It will be shown how to find an axiom system for classes of dilatation groups of Fano–Desarguesian affine planes, affine spaces, and Fano–Pappian affine spaces ($\dim \geqslant 3$).

We base on affine geometry with parallelity, either two-dimensional with Szmielew's axioms or of higher dimension with Kusak's axioms (see [2] and [1]). In both of these geometries we assume the axiom of Fano or at least the axiom of Desargues.

In models for such geometries we can define central symmetry as a dilatation which is simultaneously an involution. We know from the Fano axiom and the property of rigidness that a central symmetry has exactly one fixed point. From the Desargues theorem it follows that around each point there exists a dilatation which is an involution. We can therefore identify involutions of a dilatation group with their fixed points. This enables us to create axiom systems for classes of dilatation groups of appropriate affine planes or spaces.

1. Dilatation group of affine plane. By a *plane affine geometry* we shall understand the theory based on the following Szmielew's axioms:

S1.0. $ab \parallel ba$.

S1.1. $ab \parallel cc$.

S1.2. $a \neq b \wedge ab \parallel pq \wedge ab \parallel rs \rightarrow pq \parallel rs$.

S1.3. $ab \parallel ac \rightarrow ba \parallel bc$.

S1.4. $(\exists a, b, c) \sim ab \parallel ac$.

S1.5. $(\forall a, b, p)\ \exists q\ (ab \parallel pq \wedge p \neq q)$.

S1.6. $\sim ab \parallel cd \rightarrow \exists p\ (pa \parallel pb \wedge pc \parallel pd)$.

Let us denote this theory by Af_2; every its model will be called an *affine plane*.

Let us denote by F the sentence

$$ab \parallel cd \wedge ac \parallel bd \wedge ad \parallel bc \rightarrow ab \parallel ac,$$

by D the sentence

$$\sim ab \parallel ap \wedge \sim ab \parallel ar \wedge o \neq a, b, p, q, r, s \wedge oa \parallel ob \wedge op \parallel oq$$
$$\wedge\, or \parallel os \wedge ap \parallel bq \wedge ar \parallel bs \rightarrow pr \parallel qs,$$

and by P the sentence

$$ab \parallel ac \wedge de \parallel df \wedge ae \parallel bd \wedge bf \parallel ce \rightarrow af \parallel cd.$$

If we enrich the axioms of Af_2 by the sentences F, D or P, we shall obtain Fanoian, Desarguesian or Pappian affine geometry, respectively.

Let $A = \langle S, \parallel \rangle$ be a model of Af_2. The set of affine plane dilatations, i.e., the set of all bijections f of the plane S satisfying the condition $ab \parallel f(a)f(b)$ for all $a, b \in S$, will be denoted by $\mathrm{Dil}(A)$.

We recall some basic properties of $\mathrm{Dil}(A)$.

THEOREM 1.1. $\mathrm{Dil}(A)$ *forms a group of transformations.*

THEOREM 1.2 (rigidness). *Let* $f \in \mathrm{Dil}(A)$ *and* $a, b \in S$. *If* $a \neq b$ *and* $f(a) = a$, $f(b) = b$, *then* $f = \mathrm{Id}$.

COROLLARY 1.1. *Every non-identical dilatation has at most one fixed point.*

Let $A = \langle S, \parallel \rangle$ be a Fano–Desarguesian affine plane.

THEOREM 1.3 (homogeneity). *We have*

$$ab \parallel cd \wedge a \neq b, c \neq d \rightarrow \exists f \big(f \in \mathrm{Dil}(A) \wedge f(a) = c \wedge f(b) = d\big).$$

For involutions of such planes the following theorems hold:

THEOREM 1.4. $f \in \mathrm{Dil}(A) \wedge f^2 = \mathrm{Id} \neq f \rightarrow \exists! p \, \big(p \in S \wedge f(p) = p\big)$.

THEOREM 1.5. $\forall p \in S \; \exists! f \big(f \in \mathrm{Dil}(A) \wedge f^2 = \mathrm{Id} \neq f \wedge f(p) = p\big)$.

DEFINITION 1.1. Let σ_p be a transformation from $\mathrm{Dil}(A)$ such that $\sigma_p^2 = \mathrm{Id} \neq \sigma_p$ and $\sigma_p(p) = p$.

COROLLARY 1.2. $f \in \mathrm{Dil}(A) \rightarrow \big(f^2 = \mathrm{Id} \neq f \Leftrightarrow \exists p \; (p \in S \wedge f = \sigma_p)\big)$.

THEOREM 1.6. *If* $f \in \mathrm{Dil}(A)$ *and* $p \in S$, *then* $f \sigma_p f^{-1} = \sigma_{f(p)}$.

Let $\mathrm{Tr}(A)$ be the set of dilatations without fixed points and identity, and $J_p(A)$ the set of dilatations with a fixed point p. Notice that $\sigma_p \in J_p(A)$.

2. Dilatation group theory. We shall give here the axiom system of Fano–Pappian affine plane dilatation groups.

Let Dl_2 be a theory based on the following axioms:

G2.1. $a1 = 1a = a$.

G2.2. $\forall a \; \exists x \; (xa = ax = 1)$.

G2.3. $(ab)c = a(bc)$.

G2.4. $(a^2 = 1 \wedge b^2 = 1 \wedge c^2 = 1)$
$\rightarrow \big((abc)^2 = 1 \vee a = 1 \vee b = 1 \vee c = 1\big)$.

G2.5. $(ab = ba \wedge ac = ca) \rightarrow (bc = cb \vee a = 1)$.

G2.6. $(\forall a, b, c)\big((a^2 = b^2 = c^2 = 1 \wedge a, b, c \neq 1)$
$\rightarrow \big(\exists d\ (d \neq 1 \wedge ad = da \wedge bd = dc)$
$\rightarrow \exists d\ (d \neq 1 \wedge dc = cd \wedge ad = db)\big)\big)$.

G2.7. $(\exists a, b, c)\forall d\ \big(a^2 = b^2 = c^2 = 1 \wedge a, b, c \neq 1 \wedge a \neq b, c$
$\wedge (ad = da \rightarrow db \neq cd)\big)$.

G2.8. $(\forall a, b, c, d)(\exists e, f, g)\big[(a^2 = b^2 = c^2 = d^2 = 1 \neq a, b, c, d$
$\wedge \neq (a, b, c) \wedge d \neq a, c) \rightarrow \big((fa = ef \wedge fb = df$
$\wedge (ga = ag \wedge ge = cg \vee a = e)) \vee (fa = af \wedge fc = df)$
$\vee (fa = af \wedge fb = cf)\big)\big]$.

THEOREM 2.1. *Let* $M \models \mathrm{Af}_2 + P + F$. *Then* $\mathrm{Dil}(M) \models \mathrm{Dl}_2$.

Proof. Take M satisfying $\mathrm{Af}_2 + P + F$ and consider $\mathrm{Dil}(M)$.

1° $\mathrm{Dil}(M) \models$ G2.1, G2.2 and G2.3. G2.1–G2.3 state that the discussed structure forms a group, so they are a consequence of Theorem 1.1.

2° $\mathrm{Dil}(M) \models$ G2.4. Assume f, g, h are involutions, so from Corollary 1.2 we obtain $f = \sigma_p$, $g = \sigma_q$, $h = \sigma_r$ for some $p, q, r \in |M|$. Then $\sigma_p\sigma_q \in \mathrm{Tr}(M)$. Therefore for some s we have $\sigma_p\sigma_q = \sigma_s\sigma_r$. Thus $\sigma_p\sigma_q\sigma_r = \sigma_s$.

3° $\mathrm{Dil}(M) \models$ G2.5. It is easily seen that if $f \in J_p(M)$, $f \neq \mathrm{Id}$ and $fg = gf$ for $g \in \mathrm{Dil}(M)$, then $g \in J_p(M)$.

Assume $f \neq \mathrm{Id}$. Consider two cases:

(i) $f \in J_p(M)$. Then $g, h \in J_p(M)$, which gives $gh = hg$ as a consequence of P.

(ii) $f \in \mathrm{Tr}(M)$. Then $g, h \in \mathrm{Tr}(M)$, so $gh = hg$, which is a consequence of D.

4° $\mathrm{Dil}(M) \models$ G2.6. By the assumption there are points p, q, r such that $f = \sigma_p$, $g = \sigma_q$, $h = \sigma_r$. Take $j \in \mathrm{Dil}(M)$ such that $j \neq \mathrm{Id}$ and $fj = jf$ and $gj = jh$. For such j we have $j(p) = p$ and $j(r) = q$. We search $k \in \mathrm{Dil}(M)$ such that $k \neq \mathrm{Id}$, $k(r) = r$, and $k(q) = p$. Assume that $\neq (p, q, r)$ (possibly taking f for k). We have $pr \parallel pq$, so $qr \parallel qp$. The existence of k follows from Theorem 1.3.

5° $\mathrm{Dil}(M) \models$ G2.7. By S1.4 there exist points a, b, c such that $\sim ab \parallel ac$ (thus $a \neq b, c$ and $b \neq c$). Therefore, there is no $f \in \mathrm{Dil}(M)$ such that $f(a) = a$ and $f(b) = c$. We take involutions $\sigma_a, \sigma_b, \sigma_c$.

6° $\mathrm{Dil}(M) \models$ G2.8. Notice the following fact remains true for every affine plane. For all points a, b, c, d such that $\sim ab \parallel ac$ there exists a point e such that $ab \parallel de$ and $ac \parallel ae$. So if we have involutions $\sigma_a, \sigma_b, \sigma_c, \sigma_d$, then the existence of the dilatations we seek follows from Theorem 1.3.

The following sentences can be proved in Dl_2.

LEMMA 2.1. $(ab = ba \wedge a^2 = 1 \wedge b^2 = 1) \to (a = b \vee a = 1 \vee b = 1)$.

Proof. Let $a, b \neq 1$. Denote aba^{-1} by c. We have then $ab = ca$ and $a \neq 1$. By G2.6 there exists f such that $fc = cf$ and $fb = bf$. From the equality $ab = ba$ we obtain $c = b$, $fb = bf$ and $fb = af$. This gives $a = b$.

LEMMA 2.2. *We have*

$$(\forall a, b)\ \exists c\ (a^2 = b^2 = 1 \neq a, b \to ac = cb \wedge c^2 = 1 \wedge c \neq 1).$$

Proof. When $a = b$, for c take a. Assume $a \neq b$. Denote bab^{-1} by d. We have $ba = db$ and $b \neq 1$. By G2.6 there exists f such that $fa = af$ and $fd = bf$. Let $c = fbf^{-1}$; then $c^2 = 1$, $c \neq 1$, and $ab = bd$. Then $fabf^{-1} = fbdf^{-1}$; but we have $fabf^{-1} = faf^{-1}fbf^{-1} = ac$ and $fbdf^{-1} = fbf^{-1}fdf^{-1} = cb$, so $ac = cb$.

LEMMA 2.3. *We have*

$$(a^2 = b^2 = 1 \neq a, b \wedge fa = af \wedge fb = bf) \to (a = b \vee f = 1).$$

Proof. Let $a \neq b$. Assume that $f \neq 1$. By c we denote fbf^{-1}. By G2.6 there exists g such that $gc = cg$ and $gb = ag$. From the assumption $bf = fb$ it follows that $b = c$; so $gc = gb = bg$, which means that $ag = bg$ and, finally, $a = b$.

Given a group satisfying axioms G2.1–G2.8 we may construct an affine plane.

Let $G = \langle G, 1, \cdot \rangle$ be a group.

DEFINITION 2.1.

$$S(G) := \mathrm{Inv}(G) = \{a \in G\colon\ a^2 = 1 \neq a\},$$

$$ab \parallel_G cd :\Leftrightarrow \big(\exists f\ (f \in G \wedge fa = cf \wedge fb = df) \vee a = b \vee c = d\big) \wedge a, b, c, d \in S(G),$$

$$A(G) := \langle S(G), \parallel_G \rangle.$$

THEOREM 2.2. *Let* $M \models \mathrm{Af}_2 + F + P$. *Then* $M \cong A(\mathrm{Dil}(M))$.

Proof. From Theorems 1.4 and 1.5 it follows that $S(\mathrm{Dil}(M)) = \{\sigma_p\colon p \in |M|\}$ and σ transforms $|M|$ one-to-one onto $S(\mathrm{Dil}(M))$. From Theorems 1.3 and 1.6 it follows that σ preserves parallelity, which means that σ is an isomorphism.

THEOREM 2.3. *If* $G \models \mathrm{Dl}_2$, *then* $A(G) \models \mathrm{Af}_2$.

Proof. Take $G = \langle G, 1, \cdot \rangle$ such that $G \models \mathrm{Dl}_2$. Let $S = S(G)$ and $\parallel = \parallel_G$. We shall check if in $\langle S, \parallel \rangle$ axioms S1.0–S1.6 are true:

0° $ab \parallel ba$. $a, b \in S$. By Lemma 2.2 there exists $c \in S$ such that $ac = cb$.

1° $ab \parallel cc$. This follows from the definition of $\parallel$.

2° $a \neq b \wedge ab \parallel pq \wedge ab \parallel rs \to pq \parallel rs$. Let $a, b, p, q, r, s \in S$. If $p = q$ or $r = s$, then $pq \parallel rs$ from 1°. If $p \neq q$ and $r \neq s$, then there exist f and g such that

$fa = pf$, $fb = qf$ and $ga = rg$, $gb = sg$. Therefore, $gf^{-1}p = rgf^{-1}$ and $gf^{-1}q = sgf^{-1}$.

3° $ab \| ac \to ba \| bc$. Let $a, b, c \in S$ and $ab \| ac$. If $a = b$ or $b = c$ or $a = c$, then $ba \| bc$. If $a \neq b$, $b \neq c$, $a \neq c$, then there is f such that $fa = af$ and $fb = cf$, $f \neq 1$. By G2.6 there exists g such that $gb = bg$ and $ga = cg$.

4° $(\exists a, b, c) \sim ab \| ac$. By G2.7 there exist $a, b, c \in S$ such that $a \neq b, c$ and, for any d, $ad = da$ implies $db \neq cd$. This gives $\sim ab \| ac$.

5° $(\forall a, b, p)\, \exists q\, (ab \| pq \wedge p \neq q)$. Let $a, b, p \in S$. If $a = b$, then choose arbitrary $q \in S$, $q \neq p$. If $a \neq b$, take $r \in S$ such that $rp = br$ and put $q = rar$.

Remark. We have $rp = br$ and $ra = qr$; in other words, $pa \| bq$. Therefore, q and a, b, p form a parallelogram. Moreover, we do not need G2.8 to prove the existence of q.

6° $\sim ab \| cd \to \exists p\, (pa \| pb \wedge pc \| pd)$. Let $a, b, c, d \in S$ and $\sim ab \| cd$; then $a \neq b$ and $c \neq d$. From 5° we have $q \neq 0$ such that $ba \| cq$. So $\sim cq \| cd$. Now we choose p in accordance with axiom G2.8.

THEOREM 2.4. *If* $G \models \mathrm{Dl}_2$, *then* $G \cong \mathrm{Dil}(A(G))$.

Proof. Let $S = S(G)$ and let $\lambda\colon G \to S^S$ be defined as follows: for each $a \in S$, $\lambda g(a) = gag^{-1}$; in other words, λ is the conjugation corresponding to g, restricted to S. Thus λ preserves the group operation and is well defined, i.e., $a \in S$ implies $\lambda g(a) \in S$ for every $g \in G$.

1° λ is one-to-one. Take $g, h \in G$ and let $\lambda_g = \lambda_h$. Pick $a, b \in S$ such that $a \neq b$. Then $\lambda_g(a) = \lambda_h(a)$ and $\lambda_g(b) = \lambda_h(b)$; in other words, $h^{-1}gag^{-1} = a$ and $h^{-1}gbg^{-1}h = b$, so $g = h$ by Lemma 2.3.

2° For each $g \in G$, $\lambda_g \in \mathrm{Dil}(A(G))$. This fact follows from the definitions of $\|$ and λ.

3° λ transforms G onto $\mathrm{Dil}(A(G))$. Let us take $a, b \in S$, $a \neq b$, and $\gamma \in \mathrm{Dil}(A(G))$. We have $ab \| \gamma(a)\gamma(b)$ and $\gamma(a) \neq \gamma(b)$. Therefore, there exists g such that $ga = \gamma(a)g$ and $gb = \gamma(b)g$. Thus $\gamma(a) = \lambda_g(a)$ and $\gamma(b) = \lambda_g(b)$.

Now the equality $\gamma = \lambda_g$ is a consequence of Theorem 1.2 and 2°.

THEOREM 2.5. $G \models \mathrm{Dl}_2 \to A(G) \models F$.

Proof. Take $a, b, c, d \in S(G)$ such that $ad \| cb$, $ac \| bd$ and let $\sim ac \| cb$. Pick $p \in S(G)$ such that $pa = bp$. Denote pdp^{-1} by c'. We have $pd = c'p$, $ad \| c'b$, and $ac' \| bd$. From axiom S1.2 we obtain $cb \| c'b$ and $ac \| ac'$, and from S1.3 we get $cc' \| cb$ and $cc' \| ac$. If $c \neq c'$, then we would have $cb \| ac$. Hence $c = c'$ because $\sim cb \| ac$. Thus we obtain $ap \| ab$ and $cp \| cd$.

THEOREM 2.6. $G \models \mathrm{Dl}_2 \to A(G) \models D$.

Proof. On the affine plane the sentence D holds if and only if the groups of homotheties with fixed center are transitive. Let $a, b, c \in S$, $a \neq b, c$, and $ab \| ac$. By the definition of parallelity there exists f such that $fa = af$ and $fb = cf$. We take λ_f for the dilatation we look for.

THEOREM 2.7. $G \models \mathrm{Dl}_2 \to A(G) \models P$.

Proof. On the Desarguesian affine plane the sentence P holds if and only if the groups of homotheties with fixed center are abelian.

Let $\beta, \gamma \in J_a(A(G))$, where $a \in S$. There exist h and k such that $\beta = \lambda_h$ and $\gamma = \lambda_k$. We have $hah^{-1} = a$ and $kak^{-1} = a$. Therefore $ha = ah$ and $ka = ak$. By axiom G2.5 we have $hk = kh$, so $\beta\gamma = \gamma\beta$.

THEOREM 2.8 (representation). *We have*

$$G \models \mathrm{Dl}_2 \Leftrightarrow \exists M \ (M \models \mathrm{Af}_2 + P + F \wedge G \cong \mathrm{Dil}(M)).$$

Proof. $\to$ is a direct consequence of Theorems 2.3–2.5 and 2.7 (for M take $A(G)$).

$\leftarrow$ is a consequence of Theorem 2.1.

Below we give a new "group" theorem about representation for 2-dimensional affine geometry.

THEOREM 2.9. $M \models \mathrm{Af}_2 + P + F \Leftrightarrow \exists G \ (G \models \mathrm{Dl}_2 \wedge M \cong A(G))$.

Proof. $\to$ is a consequence of Theorems 2.1 and 2.2 (for G take $\mathrm{Dil}(M)$).

$\leftarrow$ is a consequence of Theorems 2.3, 2.5 and 2.7.

3. Remarks on the axiom system Dl_2. Now we show to which geometrical facts some of the axioms of Dl_2 correspond. Let Dl_2^D be the theory based upon axioms G2.1–G2.4 and G2.6–G2.8.

THEOREM 3.1. $G \models \mathrm{Dl}_2^D \Leftrightarrow \exists M \ (M \models \mathrm{Af}_2 + D + F \wedge G \cong \mathrm{Dil}(M))$.

Proof. It suffices to notice that if $G \models \mathrm{Dl}_2^D$, then

$$A(G) \models \mathrm{Af}_2 + F + D \quad \text{and} \quad G \cong \mathrm{Dil}(A(G))$$

(in the proofs of Theorems 2.3–2.6 we did not use G2.5).

If $M \models \mathrm{Af}_2 + F + D$, then $\mathrm{Dil}(M) \models \mathrm{Dl}_2^D$.

COROLLARY 3.1. *If* $G \models \mathrm{Dl}_2^D$, *then* $G \models \mathrm{G2.5} \Leftrightarrow A(G) \models P$.

Axiom G2.5 is therefore equivalent to the axiom of Pappus.

LEMMA 3.1. *If* $G \models$ G2.1–G2.7, *then*

$$A(G) \models \mathrm{S1.1–S1.5} + P + F \quad \textit{and} \quad G \cong \mathrm{Dil}(A(G)).$$

Proof. Notice that in the proofs of Theorems 2.3 ($0°$–$5°$) and 2.4–2.7 axiom G2.8 was not used.

Remark 3.1. In the proof of Theorem 2.4 the rigidness of the plane dilatation group was used, but this fact holds for dimension-free affine structures satisfying S1.1–S1.5 only.

THEOREM 3.2. *If* $G \models$ G2.1–G2.7, *then*

$$G \models \mathrm{G2.8} \Leftrightarrow A(G) \models \mathrm{S1.6}.$$

Proof. $\to$ is evident (cf. Theorem 2.8).

$\leftarrow$. From the assumption and Lemma 3.1 we obtain $A(G) \models \mathrm{Af}_2$ and $G \cong \mathrm{Dil}(A(G))$. By Theorem 2.1, $\mathrm{Dil}(A(G)) \models$ G2.8.

We can say that axiom G2.8 is equivalent to the upper axiom of dimension.

Let $\mathrm{Af}_{\geqslant 3}$ be a theory of affine spaces of dimension 3 or higher based upon Kusak's axioms, i.e., S1.1–S1.3, S1.6,

$$\exists d \ (ab \| cd \wedge ac \| bd), \quad \sim ab \| ac \wedge ad \| ac \rightarrow \exists p \ (ab \| dp \wedge bc \| bp).$$

LEMMA 3.2. *If* $G \models$ G2.1–G2.7, *then*

(i) $A(G) \models \exists d \ (ab \| cd \wedge ac \| bd)$,

(ii) $A(G) \models \ \sim ab \| ac \wedge ad \| ac \rightarrow \exists p \ (ab \| dp \wedge bc \| bp)$.

Proof. (i) has been already shown in the proof of Theorem 2.3 (see the Remark to 5°).

(ii) Take $a, b, c, d \in S(G)$. From the assumption that $ad \| ac$ we obtain $ca \| cd$. By the assumption $\sim ab \| ac$ we have $a \neq c$; then there exists an f such that $fc = cf$ and $fa = df$. Take $p = fbf^{-1}$; in other words, $fb = pf$. Thus $ab \| dp$ and $cb \| cp$, whence $ab \| dp$ and $bc \| bp$. (If $c = d$, then $p = c$.)

THEOREM 3.3. *We have*

$$G \models \text{G2.1–G2.7}, \ \sim \text{G2.8} \Leftrightarrow \exists M \ \big(G \cong \mathrm{Dil}(M) \wedge M \models \mathrm{Af}_{\geqslant 3} + F + P\big).$$

This theorem is a consequence of Theorem 3.2 and Lemmas 3.1 and 3.2.

COROLLARY 3.2. *We have*

$$G \models \text{G2.1–G2.7}$$
$$\Leftrightarrow \exists M \ \big(G \cong \mathrm{Dil}(M) \wedge (M \models \mathrm{Af}_2 + P + F \vee M \models \mathrm{Af}_{\geqslant 3} + P + F)\big).$$

Remark 3.2. In $\mathrm{Af}_{\geqslant 3}$, the sentence D can be proved (see [1]).

Remark 3.3. In $\mathrm{Af}_{\geqslant 3} + F$, axiom G2.5 is equivalent to the sentence P.

Remark 3.4. In Dl_2^D, axiom G2.8 is also equivalent to the upper axiom of dimension.

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