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ON TOTALLY UMBILICAL SURFACES IN SOME RIEMANNIAN SPACES

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Introduction. Let V_m be a totally umbilical surface immersed in a Riemannian space V_n satisfying the relation $\nabla_\varepsilon \bar{C}_{\lambda\mu\nu\omega} = \varphi_\varepsilon \bar{C}_{\lambda\mu\nu\omega}$ for some vector φ_ε, where $\bar{C}_{\lambda\mu\nu\omega}$ is the Weyl conformal curvature tensor of V_n. It has been proved [5] that the Weyl conformal curvature tensor C_{lkji} of V_m satisfies the same relation, the vector involved being the projection of the vector φ_ε onto V_m.

In this paper we prove that for V_m we have $H^2 C_{lkji} = 0$, where H is the mean curvature of V_m.

1. Preliminaries. Let V_m be an m-dimensional Riemannian space immersed in an n-dimensional Riemannian space V_n, and let $u^\lambda = u^\lambda(w^i)$ be the parametric representation of the subspace V_m in V_n, where (u^λ) are coordinates in V_n, and (w^i) are coordinates in V_m. Let $B_i^\lambda = \partial_i u^\lambda$, where $\partial_i = \partial/\partial w^i$. If $G_{\lambda\omega}$ is the fundamental tensor of the space V_n, then $g_{ji} = B_j^\lambda B_i^\omega G_{\lambda\omega}$ is the first fundamental tensor of the subspace V_m. In this paper, the Greek indices take on values $1, \ldots, n$, and the Latin indices take on values $1, \ldots, m$ $(m < n)$.

Let N_x^λ $(x = m+1, \ldots, n)$ be pairwise orthogonal unit normals to V_m. Then we have the relations

$$(1) \qquad G_{\lambda\omega} N_x^\lambda N_x^\omega = e_x, \qquad G_{\lambda\omega} N_x^\lambda N_y^\omega = 0 \ (x \neq y), \qquad G_{\lambda\omega} N_x^\lambda B_i^\omega = 0,$$

where e_x is the indicator of the vector N_x^λ.

The *Schouten curvature tensor* H_{ji}^λ of the subspace V_m is defined by

$$(2) \qquad H_{ji}^\lambda = \nabla_j B_i^\lambda,$$

where ∇_j denotes covariant differentiation with respect to the fundamental tensor g_{ji} of V_m. If we put

$$(3) \qquad H_{ji}^\lambda = \sum_x e_x H_{jix} N_x^\lambda,$$

then the second fundamental tensor H_{jix} for N_x^{λ} is given by

(4) $$H_{jix} = H_{ji}^{\lambda} N_{x\lambda}, \quad \text{where } N_{x\lambda} = N_x^{\omega} G_{\omega\lambda}.$$

The Gauss and Codazzi equations for V_m can be written in the form

(5) $$R_{lkji} = \bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} B_i^{\omega} + \sum_x e_x (H_{lix} H_{kjx} - H_{ljx} H_{kix})$$

and

(6) $$\bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} N_x^{\omega} = \nabla_l H_{kjx} - \nabla_k H_{ljx} + \sum_y e_y (L_{lyx} H_{kjy} - L_{kyx} H_{ljy}),$$

respectively (see [3]), where L_{ixy} is the third fundamental tensor with respect to the normals N_x^{λ}, defined by

(7) $$L_{ixy} = (\nabla_i N_x^{\lambda}) N_{y\lambda} \quad (= -L_{iyx}),$$

and R_{lkji} and $\bar{R}_{\lambda\mu\nu\omega}$ are the curvature tensors for V_m and V_n, respectively.

The equations of Weingarten are of the form

(8) $$\nabla_i N_x^{\lambda} = -H_{i\ x}^{r} B_r^{\lambda} + \sum_y e_y L_{ixy} N_y^{\lambda}, \quad \text{where } H_{i\ x}^{r} = H_{ijx} g^{jr}.$$

2. A totally umbilical surface. If H_{ji}^{λ} defined by (2) satisfies the condition

(9) $$H_{ji}^{\lambda} = g_{ji} H^{\lambda},$$

where the vector H^{λ}, called the *mean curvature vector*, is given by

(10) $$H^{\lambda} = \frac{1}{m} g^{ji} H_{ji}^{\lambda},$$

then V_m is called a *totally umbilical surface*.

We assume that V_m is a totally umbilical surface.

Putting $H_x = H^{\lambda} N_{x\lambda}$, from (4) by (9) we obtain

(11) $$H_{jix} = g_{ji} H_x,$$

and from (10) by (3) and (11) we get

(12) $$H^{\lambda} = \sum_x e_x H_x N_x^{\lambda}.$$

Hence, using (1), we have

(13) $$H_{\lambda} H^{\lambda} = \sum_x e_x (H_x)^2.$$

The *mean curvature* H of V_m, i.e., the scalar H such that

$$H^2 = \left| \sum_x e_x (H_x)^2 \right|,$$

may then be written as

(14) $$H^2 = |H_{\lambda} H^{\lambda}|.$$

By (8), in view of (11), we have

(15) $$\nabla_i N_x^{\lambda} = -H_x B_i^{\lambda} + \sum_y e_y L_{ixy} N_y^{\lambda}.$$

Differentiating (12) covariantly along V_m and using (15), we obtain

$$\begin{aligned}\nabla_i H^{\lambda} &= \sum_x e_x(\partial_i H_x) N_x^{\lambda} + \sum_x e_x H_x \nabla_i N_x^{\lambda} \\ &= -\sum_x e_x (H_x)^2 B_i^{\lambda} + \sum_x e_x \Big(\partial_i H_x + \sum_y e_y L_{iyx} H_y\Big) N_x^{\lambda}.\end{aligned}$$

Thus, if we adopt the notation

(16) $$A_{ix} = \partial_i H_x + \sum_y e_y L_{iyx} H_y$$

and use (13), then

(17) $$\nabla_i H^{\lambda} = -H_\varrho H^\varrho B_i^{\lambda} + \sum_x e_x A_{ix} N_x^{\lambda}.$$

Substituting (11) into (5) and making use of (13), we get

(18) $$R_{lkji} = \bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} B_i^{\omega} + H_\varrho H^\varrho (g_{li} g_{kj} - g_{lj} g_{ki}).$$

Substituting (11) into (6), we have

(19) $$\begin{aligned}&\bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} N_x^{\omega} \\ &\quad = \Big(\partial_l H_x + \sum_y e_y L_{lyx} H_y\Big) g_{kj} - \Big(\partial_k H_x + \sum_y e_y L_{kyx} H_y\Big) g_{lj}\end{aligned}$$

or, using (16),

(20) $$\bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} N_x^{\omega} = A_{lx} g_{kj} - A_{kx} g_{lj}.$$

Now, considering (12), (19), (7) and (13), we may find

(21) $$\bar{R}_{\lambda\mu\nu\omega} B_l^{\lambda} B_k^{\mu} B_j^{\nu} H^{\omega} = \frac{1}{2}[\nabla_l (H_\varrho H^\varrho) g_{kj} - \nabla_k (H_\varrho H^\varrho) g_{lj}].$$

3. Main results. We suppose that the space V_n satisfies the condition

(22) $$\nabla_\varepsilon \bar{C}_{\lambda\mu\nu\omega} = \varphi_\varepsilon \bar{C}_{\lambda\mu\nu\omega},$$

where $\bar{C}_{\lambda\mu\nu\omega}$ is the *Weyl conformal curvature tensor* of V_n defined by

(23) $$\begin{aligned}\bar{C}_{\lambda\mu\nu\omega} &= \bar{R}_{\lambda\mu\nu\omega} - \frac{1}{n-2}(G_{\lambda\omega}\bar{R}_{\mu\nu} - G_{\lambda\nu}\bar{R}_{\mu\omega} + G_{\mu\nu}\bar{R}_{\lambda\omega} - G_{\mu\omega}\bar{R}_{\lambda\nu}) + \\ &\quad + \frac{1}{(n-1)(n-2)}\bar{R}(G_{\lambda\omega}G_{\mu\nu} - G_{\lambda\nu}G_{\mu\omega}),\end{aligned}$$

φ_ε is a vector in V_n (not necessarily different from zero), and ∇_ε denotes the covariant differentiation with respect to $G_{\lambda\omega}$ in V_n.

Let C_{lkji} denote the Weyl conformal curvature tensor for V_m.

THEOREM. *Let V_m be a totally umbilical surface in a space V_n satisfying condition* (22). *Then, for V_m, the relation $H^2 C_{lkji} = 0$ holds.*

Proof. From (22) and (23) we find

$$(24)\qquad \nabla_\varepsilon \bar{R}_{\lambda\mu\nu\omega} = \varphi_\varepsilon \bar{R}_{\lambda\mu\nu\omega} + \frac{1}{n-2}[G_{\lambda\omega}(\nabla_\varepsilon \bar{R}_{\mu\nu} - \varphi_\varepsilon \bar{R}_{\mu\nu}) - G_{\lambda\nu}(\nabla_\varepsilon \bar{R}_{\mu\omega} - \varphi_\varepsilon \bar{R}_{\mu\omega}) + G_{\mu\nu}(\nabla_\varepsilon \bar{R}_{\lambda\omega} - \varphi_\varepsilon \bar{R}_{\lambda\omega}) - G_{\mu\omega}(\nabla_\varepsilon \bar{R}_{\lambda\nu} - \varphi_\varepsilon \bar{R}_{\lambda\nu})] - \frac{1}{(n-1)(n-2)}(\nabla_\varepsilon \bar{R} - \varphi_\varepsilon \bar{R})(G_{\lambda\omega}G_{\mu\nu} - G_{\lambda\nu}G_{\mu\omega}).$$

Since $G_{\lambda\omega}B_l^{\lambda}H^{\omega} = 0$, which follows from (12) and (1), by summing (24) with $B_h^{\varepsilon}B_l^{\lambda}B_k^{\mu}B_j^{\nu}H^{\omega}$ we obtain

$$(25)\qquad (\nabla_\varepsilon \bar{R}_{\lambda\mu\nu\omega})B_h^{\varepsilon}B_l^{\lambda}B_k^{\mu}B_j^{\nu}H^{\omega} = \varphi_h \bar{R}_{\lambda\mu\nu\omega}B_l^{\lambda}B_k^{\mu}B_j^{\nu}H^{\omega} + \frac{1}{n-2}[-g_{lj}(\nabla_\varepsilon \bar{R}_{\mu\omega} - \varphi_\varepsilon \bar{R}_{\mu\omega})B_h^{\varepsilon}B_k^{\mu}H^{\omega} + g_{kj}(\nabla_\varepsilon \bar{R}_{\lambda\omega} - \varphi_\varepsilon \bar{R}_{\lambda\omega})B_h^{\varepsilon}B_l^{\lambda}H^{\omega}],$$

where $\varphi_h = \varphi_\varepsilon B_h^{\varepsilon}$.

We differentiate (21) covariantly with respect to g_{ji}. Then, by (9), (17), and the antisymmetry of the tensor $\bar{R}_{\lambda\mu\nu\omega}$ with respect to the indices ν and ω, we have

$$(\nabla_\varepsilon \bar{R}_{\lambda\mu\nu\omega})B_h^{\varepsilon}B_l^{\lambda}B_k^{\mu}B_j^{\nu}H^{\omega} + g_{hl}\bar{R}_{\lambda\mu\nu\omega}H^{\lambda}B_k^{\mu}B_j^{\nu}H^{\omega} + g_{hk}\bar{R}_{\lambda\mu\nu\omega}B_l^{\lambda}H^{\mu}B_j^{\nu}H^{\omega} - H_\varrho H^\varrho \bar{R}_{\lambda\mu\nu\omega}B_l^{\lambda}B_k^{\mu}B_j^{\nu}B_h^{\omega} + \sum_x e_x A_{hx}\bar{R}_{\lambda\mu\nu\omega}B_l^{\lambda}B_k^{\mu}B_j^{\nu}N_x^{\omega} = \frac{1}{2}[\nabla_h \nabla_l (H_\varrho H^\varrho)g_{kj} - \nabla_h \nabla_k (H_\varrho H^\varrho)g_{lj}].$$

If we substitute (25), (18), (20) into this equation and consider (21), then

$$g_{hl}\bar{R}_{\lambda\mu\nu\omega}H^{\lambda}B_k^{\mu}B_j^{\nu}H^{\omega} - g_{hk}\bar{R}_{\mu\lambda\nu\omega}H^{\mu}B_l^{\lambda}B_j^{\nu}H^{\omega} = H_\varrho H^\varrho[R_{lkjh} - H_\lambda H^\lambda(g_{lh}g_{kj} - g_{lj}g_{kh})] - \frac{1}{n-2}[-g_{lj}(\nabla_\varepsilon \bar{R}_{\mu\omega} - \varphi_\varepsilon \bar{R}_{\mu\omega})B_h^{\varepsilon}B_k^{\mu}H^{\omega} + g_{kj}(\nabla_\varepsilon \bar{R}_{\lambda\omega} - \varphi_\varepsilon \bar{R}_{\lambda\omega})B_h^{\varepsilon}B_l^{\lambda}H^{\omega}] - \frac{1}{2}[\varphi_h \nabla_l (H_\varrho H^\varrho)g_{kj} - \varphi_h \nabla_k (H_\varrho H^\varrho)g_{lj}] + \frac{1}{2}[\nabla_h \nabla_l (H_\varrho H^\varrho)g_{kj} - \nabla_h \nabla_k (H_\varrho H^\varrho)g_{lj}] - \sum_x e_x A_{hx}(A_{lx}g_{kj} - A_{kx}g_{jl})$$

or, if we use the notation

$$E_{hk} = \frac{1}{n-2}(\nabla_\varepsilon \bar{R}_{\mu\omega} - \varphi_\varepsilon \bar{R}_{\mu\omega}) B_h^{\varepsilon} B_k^{\mu} H^{\omega} -$$

$$-\frac{1}{2}[\nabla_h \nabla_k (H_\varrho H^\varrho) - \varphi_h \nabla_k (H_\varrho H^\varrho)] + \sum_x e_x A_{hx} A_{kx},$$

(26) $$g_{hl} \bar{R}_{\lambda\mu\nu\omega} H^\lambda B_k^{\mu} B_j^{\nu} H^{\omega} - g_{hk} \bar{R}_{\mu\lambda\nu\omega} H^\mu B_l^{\lambda} B_j^{\nu} H^{\omega}$$
$$= H_\varrho H^\varrho [R_{lkjh} - H_\lambda H^\lambda (g_{lh} g_{kj} - g_{lj} g_{kh})] + g_{lj} E_{hk} - g_{kj} E_{hl}.$$

Contracting (26) with g^{hl} and assuming $E = g^{hk} E_{hk}$, we obtain

(27) $$(m-1)\bar{R}_{\lambda\mu\nu\omega} H^\lambda B_k^{\mu} B_j^{\nu} H^{\omega} = H_\varrho H^\varrho [R_{kj} - (m-1) H_\lambda H^\lambda g_{kj}] + E_{jk} - E g_{kj}.$$

Equation (27) shows that E_{jk} is symmetric. Thus we have

(28) $$\bar{R}_{\lambda\mu\nu\omega} H^\lambda B_k^{\mu} B_j^{\nu} H^{\omega}$$
$$= H_\varrho H^\varrho \left(\frac{1}{m-1} R_{kj} - H_\lambda H^\lambda g_{kj}\right) + \frac{1}{m-1}(E_{kj} - E g_{kj}).$$

Substituting (28) into (26), after an easy computation we can find

(29) $$H_\varrho H^\varrho \left[R_{lkjh} - \frac{1}{m-1}(g_{lh} R_{kj} - g_{kh} R_{lj})\right]$$
$$= \frac{1}{m-1}[g_{lh} E_{kj} - g_{kh} E_{lj} - E(g_{lh} g_{kj} - g_{lj} g_{hk})] - g_{lj} E_{hk} + g_{kj} E_{hl}.$$

Contracting this with g^{kj}, by the symmetry of E_{hk} we obtain

$$H_\varrho H^\varrho \left(\frac{m}{m-1} R_{lh} - \frac{1}{m-1} R g_{lh}\right) = \frac{m(m-2)}{m-1} E_{lh} - \frac{m-2}{m-1} E g_{lh}.$$

Hence

$$E_{lh} = \frac{1}{m} E g_{lh} + H_\varrho H^\varrho \left[\frac{1}{m-2} R_{lh} - \frac{1}{m(m-2)} R g_{lh}\right].$$

In virtue of the last relation, (29) leads to

$$H_\varrho H^\varrho \left[R_{lkjh} - \frac{1}{m-2}(g_{lh} R_{kj} - g_{lj} R_{kh} + g_{kj} R_{lh} - g_{kh} R_{lj}) + \right.$$
$$\left. + \frac{1}{(m-1)(m-2)} R (g_{lh} g_{kj} - g_{lj} g_{kh})\right] = 0,$$

that is to

$$H_\varrho H^\varrho C_{lkjh} = 0,$$

which, by (14), completes the proof.

A Riemannian space V_n is called *symmetric in the sense of E. Cartan* if

$$\nabla_\varepsilon \bar{R}_{\lambda\mu\nu\omega} = 0.$$

We see that, for every symmetric space, the relation

$$\nabla_\varepsilon \bar{C}_{\lambda\mu\nu\omega} = 0 \tag{30}$$

holds.

A Riemannian space V_n $(n > 3)$ whose Weyl's conformal tensor satisfies (30) is called *conformally symmetric* [2]. It has been proved [6] that there exist conformally symmetric spaces which are neither Cartan-symmetric nor conformally flat.

A Riemannian space V_n is called *recurrent* [8] if its curvature tensor satisfies the relation

$$\nabla_\varepsilon \bar{R}_{\lambda\mu\nu\omega} = \varphi_\varepsilon \bar{R}_{\lambda\mu\nu\omega}$$

for some vector $\varphi_\varepsilon \neq 0$.

It is easy to verify that for every recurrent space the condition

$$\nabla_\varepsilon \bar{C}_{\lambda\mu\nu\omega} = \varphi_\varepsilon \bar{C}_{\lambda\mu\nu\omega}, \quad \text{where } \varphi_\varepsilon \neq 0, \tag{31}$$

is satisfied.

A Riemannian space V_n $(n > 3)$ whose Weyl's conformal tensor satisfies (31) for some vector $\varphi_\varepsilon \neq 0$ is said to be *conformally recurrent* [1]. Roter [7] has proved the existence of conformally recurrent spaces which are neither conformally flat nor recurrent.

In the sequel we assume $n > m > 3$.

As a consequence of our theorem we obtain the following

COROLLARY 1. *If V_m is a totally umbilical surface in a Cartan-symmetric space V_n, then V_m is conformally flat or its mean curvature H vanishes.*

If $H = 0$, then V_m is necessarily Cartan-symmetric ([4], Theorem 5.2).

COROLLARY 2. *If V_m is a totally umbilical surface in a conformally symmetric space V_n, then V_m is conformally flat or its mean curvature H vanishes.*

If $H = 0$, then V_m is conformally symmetric ([5], Theorem 1).

COROLLARY 3. *If V_m is a totally umbilical surface in a recurrent space V_n, then V_m is conformally flat or its mean curvature H vanishes.*

If $H = 0$ and φ_ε is not orthogonal to V_m, then V_m is recurrent ([4], Theorem 3.3). *If $H = 0$ and φ_ε is orthogonal to V_m, then V_m is necessarily flat* ([4], Theorem 4.1).

COROLLARY 4. *If V_m is a totally umbilical surface in a conformally recurrent space V_n, then V_m is conformally flat or its mean curvature H vanishes.*

Let $H = 0$. If φ_ε is not orthogonal to V_m, then V_m is conformally recurrent [([5], Theorem 2). *If φ_ε is orthogonal to V_m, then V_m is conformally symmetric* [(5], Theorem 3).

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