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A CHARACTERIZATION OF SPHERES AMONG COMPACT 3-BODIES SATISFYING CROFTON'S THEOREM

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The aim of this note is to show the following result.

THEOREM. *Let K be a compact subset of E^3 having finite surface area s. Let q be the infimal surface-area-bisecting cross-section of K. If K satisfies Crofton's theorem, then $s \geqslant 4q$, and $s = 4q$ implies that K is a sphere.*

In particular, this result holds if K is assumed to be the closure of a bounded open set B such that ∂B is rectifiable.

In the proof we essentially follow Steinhaus [4]. Steinhaus interpreted Crofton's [2] probabilistic methods for measuring arc length in a modern setting, but instead we use a 3-dimensional approach.

Proof. Consider the set of all lines $\mathscr{L}$ in a Euclidean 3-space Ω. Let $OQ \subset \Omega$ be a fixed segment in a fixed plane π, P the foot of the perpendicular from O to a given line $L \in \mathscr{L}$, P' the perpendicular projection of P onto π. Characterize each line $L \in \mathscr{L}$ by the parameter triple (φ, θ, p), where $0 \leqslant \varphi < \pi$ measures the angle $P'OP$ in a plane perpendicular to π, $0 \leqslant \theta < \pi$ measures the angle QOP' in the plane π, and $-\infty < p < +\infty$ measures the distance OP. The one-to-one correspondence

$$L \in \mathscr{L} \leftrightarrow (\varphi, \theta, p) \in S$$

is evident, where $S = [0, \pi) \times [0, \pi) \times (-\infty, +\infty)$, an infinite rod with square cross-section.

Define the measure $\mu(Z)$ of any set of lines $Z \subset \mathscr{L}$ by setting $\mu(Z)$ equal to the 3-dimensional Lebesgue measure of Z^*, the image of Z in S. This is a modern version of Crofton's basic result. Consider a compact body $K \subset \Omega$, and let A_k $(k = 1, 2, \ldots)$ be the sets of lines $L \in \mathscr{L}$ which cut the surface ∂K in exactly k points. Crofton's theorem in this case takes the form (s being the surface area of K)

$$(1) \qquad s = \frac{1}{\pi} \sum_{k=1}^{\infty} k\mu(A_k),$$

if this sum is finite (for example, if $K = \bar{B}$, where B is bounded and open, and ∂B is rectifiable).

Restrict Ω to the unit sphere U and assume that $K \subset U$. Call $L(\varphi, \theta, p)$ the image $L \in \mathscr{L}$ of the point $(\varphi, \theta, p) \in S$. The number of intersections of $L(\varphi, \theta, p)$ with ∂K in Ω is a function $f(\varphi, \theta, p)$ of three real variables. We can now write (1) in the form

$$s = \frac{1}{\pi} \int_{-\infty}^{+\infty} \int_{0}^{\pi} \int_{0}^{\pi} f(\varphi, \theta, p) \, d\varphi \, d\theta \, dp, \tag{2}$$

where the left-hand side is understood as Jordan content and the right-hand side as a Lebesgue integral. An analogous formula is developed, from the integral geometry viewpoint, in [1], p. 65.

Let $p(\vec{v})$ denote the measure of the 2-dimensional projection of K onto a plane normal to the direction $\vec{v}$, and write

$$p = \inf_{\vec{v}} p(\vec{v}).$$

Similarly, let $q(\vec{v})$ denote the measure of that planar cross-section of K, normal to $\vec{v}$, which bisects the surface area s of K, and write

$$q = \inf_{\vec{v}} q(\vec{v}).$$

Since the number of intersections $f(\varphi, \theta, p)$ with ∂K is not less than 2 for all lines intersecting K (except a measure zero set), we infer, by use of (2), that $s \geqslant 4\pi \geqslant 4p \geqslant 4q$.

If $s = 4q$, then all lines (except a measure zero set) intersecting K must intersect ∂K in exactly 2 points, that is, $f(\varphi, \theta, p) = 2$ there. Hence K is convex, and in the convex case K is already known to be a sphere [3].

REFERENCES

[1] W. Blaschke, *Vorlesungen über Integralgeometrie*, 3-rd edition, Berlin 1955.

[2] M. W. Crofton, *On the theory of local probability, applied to straight lines drawn at random in a plane*, Philosophical Transactions of the Royal Society 158 (1868), p. 181-199.

[3] H. Herda, *A characterization of spheres among convex 3-bodies*, Colloquium Mathematicum 33 (1975), p. 103-104.

[4] H. Steinhaus, *Length, shape and area*, ibidem 3 (1954), p. 1-13.

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