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HOMOTOPY TYPE OF AUTOMORPHISM GROUPS OF MANIFOLDS

BY

W. BALCERAK AND B. HAJDUK (WROCŁAW)

1. Introduction. Given a smooth, *PL* or topological manifold, one can consider its group of automorphisms (i.e. diffeomorphisms, *PL*-isomorphisms or homeomorphisms) endowed with a natural topological structure depending on the category under consideration. The homotopy theory of these spaces can be interpreted as a global (parametrized) version of the classification of manifolds. For example, here we came upon the topological space of smooth (or *PL*) structures on a topological manifold and the parametrized h-cobordism theorem. Thus it is not a surprise that the groups have a rich homotopical structure.

This paper is a survey of results on the homotopy type of automorphism groups. We intended it to be comprehensible as much as possible for non-specialists and for that reason we included a number of technical theorems as well as proofs of several simple facts which should clarify some points usually omitted in research papers. An extensive bibliography is attached.

We shall use the following notation. The letter A will always denote one of the three categories of manifolds: D (differentiable), PL (piecewise linear) or Top (topological). If $X \in A$ and $Y \subset X$, then $A(X, Y)$ will be the group of A-isomorphisms of X onto X, equal to the identity on Y. We shall abbreviate $A(X, \partial X)$ to $A(X, \partial)$. Bold-face letters will denote semisimplicial objects, e.g. $\mathbf{A}(X)$ will be the semisimplicial group of A-automorphisms of X. Moreover, we shall use the following symbols:

I — unit closed interval,
D^k — k-dimensional unit disc,
Δ^q — standard q-simplex,
O_k — orthogonal group of R^k,
ΩX — loop space of X,
τM — tangent (micro-) bundle of M,
BG — classifying space of a topological monoid G,
$|S|$ — geometric realization of a semisimplicial complex S.

2. Topological structures on automorphism groups. Assume that M is a smooth compact manifold. The natural structure on the group $D(M)$ of all smooth diffeomorphisms of M is the C^r-topology, where $1 \leqslant r \leqslant \infty$, i.e. uniform convergence of all partial derivatives up to order r. The resulting space is a manifold modelled on a separable Hilbert space if $r < \infty$, the model being a separable Fréchet space otherwise. Let $\Gamma(\tau M)$ be the space of smooth cross-sections of the tangent bundle of M and let $\exp\colon \tau M \to M$ be the exponential map. A chart around id_M is given by the map $\Gamma(\tau M) \to C^\infty(M, M)\colon s \mapsto \exp \circ s$ restricted to an appropriate neighbourhood of the zero section. It is known that for all $r > 1$ the resulting spaces are homotopy equivalent. Recall that a manifold modelled on an infinite-dimensional separable Fréchet space is homotopy equivalent to a CW-complex and two such manifolds are homeomorphic if and only if they are homotopy equivalent.

If M is a topological manifold, then we consider the group $Top(M)$ of all homeomorphisms of M as a topological space with the compact-open topology.

In the PL case, the usual way to introduce topology goes through semisimplicial complexes (see May [2] or Rourke and Sanderson [1]). The same pattern of construction works also in topological and smooth cases and it is often very convenient to work with the semisimplicial groups. If we want to have virtual topological groups, we may pass to geometric realizations.

The semisimplicial complex $\boldsymbol{A}(M)$ of A-automorphisms of an A-manifold M has, as a typical q-simplex, an A-automorphism $F\colon \Delta^q \times M \to \Delta^q \times M$ commuting with the projection onto the simplex Δ^q. The face operator is induced by the restriction of F to faces of Δ^q.

If $A = Top$, then $\boldsymbol{A}(M)$ is isomorphic to the singular complex of $Top(M)$ and $\boldsymbol{D}(M)$ is homotopy equivalent to the singular complex of $D(M)$. Thus $|\boldsymbol{D}(M)| \sim D(M)$ and $|\boldsymbol{Top}(M)|$ is weakly homotopy equivalent to $Top(M)$.

For a PL-manifold we have the semisimplicial complex $\boldsymbol{PD}(M)$ of piecewise smooth homeomorphisms and $\boldsymbol{PD}(M)$ is homotopy equivalent to $\boldsymbol{PL}(M)$. If M is smooth, consider the PL-structure given by a C^1-triangulation. There are natural forgetful maps of semisimplicial complexes

$$\boldsymbol{D}(M) \to \boldsymbol{PD}(M) \sim \boldsymbol{PL}(M) \to \boldsymbol{Top}(M)$$

inducing maps between geometric realizations

$$D(M) \sim |\boldsymbol{D}(M)| \to |\boldsymbol{PL}(M)| \to |\boldsymbol{Top}(M)|.$$

For a non-compact manifold M there is another topology on $Top(M)$ generated by sets

$$U_{a,f} = \{g \in Top(M)\colon d(f(x), g(x)) < a(x) \text{ for } x \in M\},$$

where d is a fixed metric on M and a varies over all positive real-valued continuous functions. This is the majorant topology (and similarly one defines the C^∞-majorant topology). For compact manifolds and for some important special cases it coincides up to homotopy type with the compact-open topology. For a manifold with non-finitely generated homology there is a difference in the local structure — only the majorant topology gives a locally path-connected space. Since the majorant topologies are not fitted to the semisimplicial constructions, we shall not use them.

The local structure of $Top(M)$ is not fully known.

CONJECTURE. *For any compact manifold M, $Top(M)$ is a manifold modelled on the separable Hilbert space.*

The most important result in this direction is due to Černavskiĭ.

THEOREM (Černavskiĭ [1], [2]). *If M is a compact manifold (or the interior of a compact one), then $Top(M)$ is locally contractible.*

Since $Top(M)$ satisfies the disjoint disc property, once one proves $Top(M)$ to be an ANR, the affirmative answer to the conjecture above will follow from a criterium of Toruńczyk. The conjecture was settled by Luke and Mason [1] for compact 2-dimensional manifolds. If it is true, then the set of PL-isomorphisms of a PL-manifold lies in $Top(M)$ like

$$l_2^f = \{\{c_n\}: c_n \in R, n = 1, 2, \ldots; c_n = 0 \text{ for } n \text{ large}\}$$

$$\text{in } l_2 = \Big\{\{c_n\}: \sum_n c_n < \infty\Big\},$$

at least for high-dimensional manifolds.

However, the homotopy type of $PL(M)$ endowed with the compact-open topology does not seem to be interesting, since we have the following discouraging corollary to the theorem of Černavskiĭ:

COROLLARY. *Let M be a compact PL-manifold and consider the set $PL(M)$ of PL-isomorphisms of M with the compact-open topology. Then the inclusion $PL(M) \to Top(M)$ induces a weak homotopy equivalence of the identity components.*

Proof. It follows from the concordance-implies-isotopy theorem that PL-homeomorphisms are dense in the identity component of $Top(M)$ (cf. Kirby and Siebenmann [1]). If $\varphi: (D^n, \partial) \to (Top(M), PL(M))$ is a map such that $\mathrm{Im}\varphi$ is contained in some contractible neighbourhood U of id_M, then we find an extension of $\varphi|\partial D^n$ to a map $\psi: D^n \to PL(M) \cap U$ using the following complement to the theorem of Černavskiĭ:

There exists a deformation of a neighbourhood of $\mathrm{id}_M \in Top(M)$ to $\{\mathrm{id}_M\}$ preserving $PL(M)$. Thus $PL(M)$ with the compact-open topology is locally contractible.

Now, since U is contractible, $\varphi \sim \psi \operatorname{rel} \partial D^n$. A deformation of an arbitrary map $\varphi: (D^n, \partial) \to (Top(M), PL(M))$ to $PL(M)$ is developed easily using the density of $PL(M)$ and such a subdivision of D^n that each simplex is contained in a contractible neighbourhood.

To study the homotopy type of automorphism groups it is necessary to have global ("parametrized" in the semisimplicial language) versions of basic theorems concerning manifolds. We recall now two such tools. One of them is the parametrized version of the extending isotopy theorem.

For two A-manifolds V and M and for a compact subset K of V, let $\boldsymbol{Emb}_A(V, M; K)$ denote the semisimplicial complex with a typical simplex being a fibre-preserving locally flat A-embedding $f\colon \Delta^q \times V \to \Delta^q \times M$ such that $f^{-1}(\Delta^q \times \partial M) = \Delta^q \times \partial V$ and $f(x) = x$ for $x \in \Delta^q \times K$.

If $A \neq PL$ and V is compact, we denote by $Emb_A(V, M; K)$ the space of all locally flat A-embeddings such that $f^{-1}(\partial M) = \partial V$ and f is fixed on K, with the C^∞ or compact-open topology. If V is not compact, we endow the space $Emb_A(V, M; K)$ with the majorant topology. Suppose V is a closed submanifold of M satisfying the conditions above.

THEOREM. *The restriction induces a Kan fibration*

$$\boldsymbol{A}(M, K) \to \boldsymbol{Emb}_A(V, M; K).$$

The following variations of this theorem also hold:

1. *The restriction yields a Kan fibration* $\boldsymbol{A}(M) \to \boldsymbol{A}(\partial M)$.

2. *In C^∞-category we have a locally trivial principal $D(M, V)$-fibration* $D(M) \to Emb_D(V, M)$.

3. $Top(M, K) \to Emb_{Top}(V, M; K)$ *is a Hurewicz fibration.*

4. *Given a base point $x_0 \in \mathrm{Int}M$, $A \neq PL$, the map*

$$A(M) \to \mathrm{Int}M\colon f \mapsto f(x_0)$$

is a Hurewicz fibration.

5. *More generally, if $A \neq PL$ and V is a closed locally flat submanifold in* $\mathrm{Int}M$, *then the map from $A(M)$ to the space of closed embeddings of V into* $\mathrm{Int}M$, *induced by restriction, is a Hurewicz fibration.*

The second tool provides "orthogonalization near a fixed point". Look first on the simple case of R^n.

LEMMA. $D(R^k \times D^n, \{0\} \times D^n \cup R^k \times \partial D^n)$ *contains, as a deformation retract, the loop space* $\Omega^n(O_k, \mathrm{id})$.

Proof. $\Omega^n(O_k, \mathrm{id})$ is realized as the subspace of the considered group consisting of diffeomorphisms which preserve the projection onto D^n and are orthogonal on each fibre $R^k \times \{x\}$. The deformation is given by

$$F(h, t)(x, y) = \begin{cases} \left(\dfrac{h_1(tx, y)}{t}, h_2(tx, y)\right) & \text{for } 0 < t \leqslant 1, \\ \big(dh_1(0, y)(x), y\big) & \text{for } t = 0, \end{cases}$$

where $h(x, y) = \big(h_1(x, y), h_2(x, y)\big) \in R^k \times D^n$.

Now, an application of this orthogonalization process to a neighbourhood of a submanifold (it is particularly easy to proceed with the semisimplicial groups of automorphisms) gives the following (cf. Cerf [2], Antonelli et al. [2], Burghelea and Kuiper [1])

THEOREM. *Let V be a closed smooth submanifold of M such that $V \cap \partial M = \partial V$ and let U be a closed tubular neighbourhood of V. Then $D(M, V \cup \partial M)$ has the homotopy type of the subgroup consisting of such $f \in D(M, V \cup \partial M)$ that $f|U$ is a fibre-preserving map $(U, V) \to (U, V)$, orthogonal on each fibre.*

COROLLARY. *If U is a collaring of ∂M, then $D(M, \partial)$ has the homotopy type of the group $D(M, U)$.*

3. Some important special cases. At the very beginning, the difference between the smooth and PL categories or the topological category is mirrored in the structure of $A(D^n, \partial)$. The group $D(D^n, \partial)$ has many non-trivial homotopy groups (we shall see later that it is contractible only for small n), in contrast to the PL and topological cases.

THEOREM (Alexander trick). *$PL(D^n, \partial)$ and $Top(D^n, \partial)$ are contractible.*

Proof. Topological case. Extend any $f \in Top(D^n, \partial)$ by the identity outside D^n to $f' \in Top(R^n)$. The deformation of $Top(D^n, \partial)$ to $\{\mathrm{id}\}$ is given by $F(t, f) = tf'(t^{-1}x)$, $F(0, f) = \mathrm{id}$.

PL case. Let $D^n \times \Delta^k$ be PL-embedded in R^{n+k} as a convex neighbourhood of $O \in R^{n+k}$. Let us triangulate $D^n \times \Delta^k \times [0, 1]$ as the cone over $D^n \times \Delta^k \times \{0\} \cup \partial(D^n \times \Delta^k) \times [0, 1]$. Any $f\colon D^n \times \Delta^k \to D^n \times \Delta^k$ representing $\pi_k PL(D^n, \partial)$ is equal to id on $\partial(D^n \times \Delta^k)$, hence extends by id to $\partial(D^n \times \Delta^k) \times [0, 1] \cup D^n \times \Delta^k \times \{0\}$, and then linearly to $D^n \times \Delta^k \times [0, 1]$. This yields an isotopy rel $D^n \times \partial\Delta^k$ from f to id.

LEMMA. $D(S^n, *) \sim D(D^n, \partial) \times O_n$.

Proof. Decompose S^n as the sum of two discs: $S^n = D^n_- \cup D^n_+$. By the orthogonalization described in Section 2, $D(S^n, *)$ has the homotopy type of the subspace E of diffeomorphisms orthogonal on D^n_+. The subspace E is the total space of the principal $D(D^n, \partial)$-bundle over O_n given by restriction to D^n_+. The extension of any linear map from D^n to S^n determines a cross-section of the bundle, which must be therefore trivial.

The proof above applied to an exotic sphere Σ^n, $n > 6$, recognizes $D_0(\Sigma, *)$ up to homotopy as the total space of a principal $D(D^n, \partial)$-bundle over $SO(n)$. The classifying map of this bundle is (see Section 7)

$$f\colon SO(n) \to BD(D^n, \partial) \sim \Omega^n(PL_n/O_n).$$

Given Σ^n and an element a of $\pi_i(SO(n))$ we have the composition of maps

$$\tau_{i,n}(\Sigma, a)\colon S^i \xrightarrow{a} SO(n) \xrightarrow{f} \Omega^n(PL_n/O_n) \to \Omega^n(PL/O),$$

hence a homotopy sphere in $\Theta_{n+i} = \pi_{n+i}(PL/O)$, where Θ_k denotes the group of diffeomorphism classes of homotopy k-spheres. The resulting pairing

$$\tau_{i,n}\colon \pi_i\big(SO(n)\big)\times\Theta_n \to \Theta_{n+i}$$

coincides (Hajduk [1]) with the Milnor-Novikov pairing defined geometrically by Novikov [1]. It is known that there are examples of homotopy spheres Σ^n such that $\tau_{1,n}(\Sigma^n, a)$ is non-trivial. If we compare the homotopy exact sequences of the bundles corresponding to S^n and Σ^n, then we see that

$$\pi_i\big(D(S^n)\big) \neq \pi_i\big(D(\Sigma^n)\big) \quad \text{for } i = 0, 1.$$

Thus the homotopy type of $D(M)$ depends on the differential structure imposed on M.

The argument applied in the proof of the preceding lemma gives also

LEMMA. $D(D^n) \sim D(S^{n-1}\times I, S^{n-1}\times\{0\})\times O_n$.

The two lemmas above indicate that two groups should be of some importance: $D(D^n, \partial)$ which shows the primary difference between the topological case and the smooth one, and $D(M\times I, M\times\{0\})$ which measures non-triviality of diffeomorphisms of collars (as well as the difference between the relations of isotopy and pseudoisotopy). We shall say more on these spaces later.

4. Low-dimensional manifolds. The homotopy type of $A(M)$ for a compact 2-dimensional manifold M is quite well understood. First of all, the groups $\pi_0 Top(M)$ are known. In a sequence of papers, Lickorish described generating sets for these groups. In the orientable case it is enough to take one orientation reversing homeomorphism and one twist around each simple closed curve from a set of generators of $H_1(M)$. By the *twist around* c we mean a homeomorphism equal to the identity outside a collar $c\times[0,1]$ of c and rotating $c\times\{t\}$ by $2\pi t$. If M is non-orientable, one should add some other homeomorphisms (see Lickorish [2]). For an orientable closed 2-manifold, Hatcher and Thurston [1] determined relations satisfied by the generators given by Lickorish, providing a presentation of $\pi_0 D(M)$.

We have also the following description of the homotopy type of components of $D(M)$:

THEOREM (Earl and Eells [1], Gramain [1], [2]). *If M is a compact 2-dimensional manifold, then the identity component of $D(M)$ is homotopy equivalent to:*

SO_3 for $M = S^2$ or M being the projective plane,

$S^1\times S^1$ for $M = S^1\times S^1$,

SO_2 for M being the Klein bottle, or the Möbius band, or $M = D^2$, or $M = S^1 \times I$,

a point in any other case.

The proof is based on the following

THEOREM (Smale [1]). *$D(D^2, \partial)$ is contractible.*

Proof. Consider the fibration

$$Emb_D(I \times I, R \times I; \{0\} \times I \cup R \times \partial I) \to Emb_D(\{1\} \times I, R \times I; \{1\} \times \partial I).$$

The fibre of this fibration is $D(D^2, \partial)$ and the total space is contractible by the (parametrized) uniqueness of collars. Thus we have a homotopy equivalence

$$D(D^2, \partial) \sim \Omega Emb_D(\{1\} \times I, R \times I; \{1\} \times \partial I).$$

The latter is a homotopy retract of ΩE, where

$$E = Emb_D(\{*\} \times I, S^1 \times I; \{*\} \times \partial I),$$

since for the map induced by the inclusion $R \subset S^1$ onto a hemisphere we have the left homotopy inverse map given by lifting to the universal cover. The proof will be completed if we show that $\Omega E \sim *$.

Let $S^1 = D^1_- \cup_\partial D^1_+$, let $D_0 \subset D^2$ be the disc of radius 1/2 and consider the fibrations

$$D(D^1_- \times I, \partial) \xrightarrow{i} D(S^1 \times I, \partial) \to Emb_D(D^1_+ \times I, S^1 \times I; D^1_+ \times \partial I),$$

$$D(S^1 \times I, \partial) \xrightarrow{j} D(D^2, \{0\} \cup \partial D^2) \to Emb_D(D_0, D^2; \{0\}).$$

The composition ji is induced by the inclusion

$$D^1_- \times I \subset S^1 \times I = D^2 - \mathrm{Int} D_0 \subset D^2$$

and has a homotopy inverse given by an embedding of D^2 onto $D^1_- \times I \subset D^2$, which is isotopic to the identity. Since $Emb_D(D_0, D^2; \{0\})$ is homotopy equivalent to O_2, Ωj induces a monomorphism on homotopy groups. This implies that Ωi and Ωj are homotopy equivalences. Hence we infer that $\Omega Emb_D(D^1_+ \times I, S^1 \times I; D^1_+ \times \partial I)$ is contractible. Finally, consider

$$Emb_D(D^1_+ \times I, S^1 \times I; \{*\} \times I \cup D^1_+ \times \partial I)$$
$$\to Emb_D(D^1_+ \times I, S^1 \times I; D^1_+ \times \partial I) \to E.$$

Since the fibre is contractible (once more by the uniqueness of collars), we have a homotopy equivalence

$$E \sim Emb_D(D^1_+ \times I, S^1 \times I; D^1_+ \times \partial I),$$

thus $\Omega E \sim *$.

Recently A. Hatcher proved the same to be true for the 3-dimensional disc:

THEOREM (Smale conjecture). *$D(D^3, \partial)$ is contractible.*

The triviality of $\pi_0 D(D^3, \partial)$ was proved by Cerf [4]. In Section 7 we shall see that $D(D^i, \partial) \sim *$ for $i \leqslant 3$ implies

$$D(M) \sim |PL(M)| \sim |Top(M)|$$

if $\dim M \leqslant 3$.

COROLLARY. *$D(S^n) \sim O_{n+1}$ for $n \leqslant 3$; $D(T^n) \sim GL(n, Z) \times T^n$, where T^n is the n-dimensional torus, $n \leqslant 3$.*

There are several results on $D(M^3)$ for some special 3-manifolds (cf. Hatcher [10]). For example, if M is $S^1 \times S^2$ (respectively, an infra-solvmanifold, i.e. a T^2-bundle over S^1 with the gluing map in $SL(2, Z)$ having distinct real eigenvalues), then $D(M)$ is deformed to the fibre-preserving diffeomorphisms (bundle automorphisms). If M is a 3-manifold with non-trivial torus decomposition, then components of $D(M)$ are contractible.

5. Concordance groups. The case of the homotopy trivial pair $(M \times I, M)$ is the only example of a general situation for which computational results are known. In fact, it is proved that the stable part of the homotopy type of $A(M \times I, M)$ depends only on the homotopy type of M and the groups $\pi_i D(M \times I, M)$ are known for $i = 0, 1$. We shall write

$$C_A(M) = A(M \times I, M \times \{0\})$$

calling it the *concordance* (or *pseudoisotopy*) *group*. We say that two automorphisms $f, g \in A(M)$ are *concordant* (or *pseudoisotopic*) if $f = F|M \times \{0\}$ and $g = F|M \times \{1\}$ for some $F \in A(M \times I)$.

The problem of computing $\pi_* C_D(M)$ was reduced by Cerf to a parametrized handle (or Morse function) problem. Let $\mathcal{F}(M)$ denote the space of all C^∞-functions $f: M \times (I; 0, 1) \to (I; 0, 1)$ and let $\mathcal{E}(M)$ be the subset of functions without critical points. The projection $p: M \times I \to I$ induces the fibration

$$\varphi: C_D(M) \to \mathcal{E}(M), \qquad f \mapsto p \cdot f,$$

the fibre being the space of isotopies starting at id_M. Since the fibre is contractible, φ is a homotopy equivalence and contractibility of $\mathcal{F}(M)$ implies that

$$\pi_i(\mathcal{F}(M), \mathcal{E}(M)) = \pi_{i-1}\mathcal{E}(M) = \pi_{i-1} C_D(M).$$

Thus $\pi_0 C_D(M)$ is the obstruction group for the problem of deforming continuously a 1-parameter family of smooth functions to a family of functions with no critical points. One can say that determination of $\pi_k C_D(M)$ is the k-parameter version of the s-cobordism theorem. Therefore, it is no wonder that the higher Whitehead functors will be relevant. The main result is the following

THEOREM (Hatcher and Wagoner [1]). *Let M be a smooth compact manifold of dimension at least* 6 *such that the first Postnikov invariant* $k_1 \in H^3(\pi_1 M; \pi_2 M)$ *is zero. Then*

$$\pi_0 C_D(M) = Wh_2(\pi_1 M) \oplus Wh_1^+(\pi_1 M; Z_2 \times \pi_2 M).$$

Here Wh_2 is the second Whitehead functor and $Wh_1^+(\pi_1 M; Z_2 \times \pi_2 M)$ denotes the quotient of the group $(Z_2 \times \pi_2 M)[\pi_1 M]$ of finite linear combinations $\sum a_i x_i$, $a_i \in Z_2 \times \pi_2 M$, $x_i \in \pi_1 M$, by the subgroup generated by elements of the form $ax - a^y yxy^{-1}$ and $b \cdot 1$, $a, b \in Z_2 \times \pi_2 M$, $x, y \in \pi_1 M$, y acts on a with the result a^y by the trivial action on Z_2 and the usual one on $\pi_2 M$.

For a simply connected manifold M the theorem was proved first by Cerf [8]. In this case $C_D(M)$ is connected (it is known that $Wh_2(0) = 0$ and triviality of $Wh_1^+(0; Z_2 \times \pi_2 M)$ follows easily from the definition). On the other hand, if M satisfies assumptions of the theorem above and $\pi_1 M \neq 0$, then $\pi_0 C_D(M)$ contains the non-trivial subgroup

$$Wh_1^+(\pi_1 M; Z_2) \subset Wh_1^+(\pi_1 M; Z_2 \times \pi_2 M).$$

A similar formula for $\pi_1 C_D(M)$ was announced by Volodin [1] (cf. also Hatcher [9] and Igusa [1]). For $M = D^n$ we have

$$\pi_1 C_D(M) = Z_2 \oplus Wh_3(0).$$

For concordances in the two remaining categories we have

$$\pi_0 C_D(M) = \pi_0 |C_{PL}(M)| = \pi_0 C_{Top}(M),$$

but the higher homotopy groups differ in general (see Section 7).

6. Block automorphisms. The richness of the homotopy structure of automorphism groups arises from its dependence on the manifold (topological, PL or smooth) structure rather than on the homotopy type of the underlying manifold. This manifests in the strength of the isotopy relation and therefore one is tempted to consider first a weaker relation and then to compare it with the isotopy. The concordance relation defined in the preceding section is a natural candidate for this role.

By $\tilde{A}(M, \partial)$, the *complex of block automorphisms* of M, we mean the Δ-set whose standard k-simplex is an automorphism

$$\varphi \colon M \times \Delta^k \to M \times \Delta^k$$

such that $\varphi(M \times \partial_i \Delta^k) \subset M \times \partial_i \Delta^k$ and $\varphi | \partial M \times \Delta^k = \mathrm{id}$ ($\partial_i \Delta^k$ denotes the i-th face of Δ^k). The face operators are given by restrictions to the faces of Δ^k. Similarly we define $\tilde{A}(M)$ and the Δ-set $\tilde{G}(M)$ of block homotopy equivalences of M. The Δ-sets $\tilde{A}(M)$, $\tilde{A}(M, \partial)$ and $\tilde{G}(M)$ satisfy the Kan condition.

It is easy to see that $\pi_0\tilde{A}(M)$ is the set of equivalence classes of the concordance relation and $\pi_k\tilde{A}(M)$ is isomorphic to the factor of $A(M\times S^k)$ by the subgroup generated by automorphisms extendable to $M\times D^{k+1}$ and this factor group is in turn the k-th concordance homotopy group $\pi_k(A; M)$ defined by Antonelli et al. [3].

We have the following inclusions of semisimplicial complexes:

$$\begin{array}{ccc} A(M) & \subset & \tilde{A}(M) \\ \cap & & \cap \\ G(M) & \subset & \tilde{G}(M). \end{array}$$

The homotopy types $A(M)$ and $\tilde{A}(M)$ usually differ, but for $G(M) \subset \tilde{G}(M)$ the situation is simple.

LEMMA. *The inclusion $G(M) \subset \tilde{G}(M)$ is a homotopy equivalence.*

Proof. Let $\Lambda_k = \partial\Delta^k - \partial_i\Delta^k$ and assume that $p_1\colon M\times\Lambda_k \to M$ and $p_2\colon M\times\Lambda_k \to \Lambda_k$ are the projections. Suppose that $f\colon M\times\Lambda_k \to M\times\Lambda_k$ is a homotopy equivalence preserving faces and that $f|M\times\partial\Lambda_k$ commutes with the projection onto $\partial\Lambda_k$. Using the homotopy

$$f_t(x) = \big(tp_2(x)+(1-t)p_2f(x),\, p_1f(x)\big)$$

we extend f over $\Delta^k = \Lambda_k\times I$ so that the restriction over $\partial_i\Delta^k$ commutes with the projection onto $\partial_i\Delta^k$.

The block automorphisms are compatible, via surgery, with block homotopy equivalences. Let $f\in\tilde{G}(M,\partial)$ represent an element of $\pi_k\big(\tilde{G}(M,\partial), \tilde{A}(M,\partial)\big)$. It determines an element of $h\mathscr{S}_A(M\times D^k, \partial)$, the group of homotopy A-structures on $M\times D^k$ $\big(\mathrm{mod}\,\partial(M\times D^k)\big)$. This correspondence induces a bijection

$$\pi_k\big(\tilde{G}(M,\partial), \tilde{A}(M,\partial)\big) \to h\mathscr{S}_A(M\times D^k, \partial) \qquad \text{for } \dim M + k \geqslant 6.$$

Let G/A denote the homotopy fibre of the natural map $BA\to BG$, where BA is the classifying space for stable bundles in the category A (i.e. vector bundles, PL-bundles or Top-bundles) and BG classifies stable spherical fibrations. We have the exact sequence of surgery for $\dim M + k \geqslant 6$,

$$\to [\Sigma^{k+1}(M/\partial), G/A] \xrightarrow{\theta} L^s_{n+k+1}(\pi_1M) \to \pi_k\big(\tilde{G}(M,\partial), \tilde{A}(M,\partial)\big)$$
$$\to [\Sigma^k(M/\partial), G/A] \xrightarrow{\theta} L^s_{n+k}(\pi_1M), \qquad \dim M = n,$$

where $L_i^s(\pi_1 M)$ is the Wall obstruction group for simple homotopy equivalences and Θ is the surgery obstruction map.

Since $\tilde{G}(M)$ is homotopy equivalent to $G(M)$, we have

COROLLARY. *The homotopy type of $\tilde{A}(M, \partial)$ is determined, up to extension, by the homotopy type of M.*

Since the Wall groups of the trivial group are known to be ($k \geqslant 5$)

$$L_k^s(0) = \pi_k(G/PL) = \begin{cases} 0 & \text{for } k \text{ odd}, \\ Z_2 & \text{for } k \equiv 2 \,(\mathrm{mod}\, 4), \\ Z & \text{for } k \equiv 0 \,(\mathrm{mod}\, 4), \end{cases}$$

the exact sequence above for M simply connected and $A = PL$ or Top splits into short exact sequences

$$0 \to \pi_k\big(\tilde{G}(M, \partial), \tilde{A}(M, \partial)\big) \to [S^k(M/\partial), G/A] \to \pi_{n+k}(G/PL) \to 0.$$

If M is smooth and simply connected, then we get exact sequences

$$bP_{n+k+1} \to \pi_k\big(\tilde{G}(M, \partial), \tilde{D}(M, \partial)\big) \to [S^k(M/\partial), G/O] \to \pi_{n+k}(G/PL),$$

where bP_{n+k+1} is the group of isomorphism classes of homotopy spheres that bound π-manifolds.

Computing $\pi_k\big(\tilde{A}(M), A(M)\big)$ is equivalent to deciding whether an automorphism of $M \times \Delta^k$ which preserves fibres $M \times \{x\}$ for $x \in \partial\Delta^k$ can be deformed mod $M \times \partial\Delta^k$ to a fibre-preserving automorphism. This k-concordance-implies-k-isotopy problem was studied by Burghelea, Lashof and Rothenberg [1] (see also Millett [1] and [2]).

Let $N \subset \mathrm{Int} M$ be a compact submanifold with boundary, $\dim N = \dim M$. Then there is an injection

$$a\colon \big(\tilde{A}(N, \partial), A(N, \partial)\big) \to \big(\tilde{A}(M, \partial), A(M, \partial)\big)$$

defined by extension of all automorphisms by the identity to M.

THEOREM (Burghelea, Lashof and Rothenberg [1]). *Suppose that $n = \dim M = \dim N \geqslant 5$, $\pi_i \partial N = \pi_i N$ for $i = 0, 1$, $\pi_i(M, N) = 0$ for $i \leqslant r$, $r \leqslant n-4$ and $\pi_i N = 0$ for $i \leqslant k \leqslant r$. Then*

$$a_\#\colon \pi_j\big(\tilde{A}(N, \partial), A(N, \partial)\big) \to \pi_j\big(\tilde{A}(M, \partial), A(M, \partial)\big)$$

is an isomorphism for

$$j \leqslant \begin{cases} \min(2r-3, r+k-2) & \textit{for } A = D, \\ \min(2r-3, r+k-2, r+3) & \textit{for } A = PL \textit{ or } Top. \end{cases}$$

COROLLARY. *$\pi_j\big(\tilde{A}(M), A(M)\big)$ depends only on the $(j+2)$-dimensional skeleton of M.*

Applying this theorem to $N = D^n$ and using the Alexander trick we have

COROLLARY. *If M^n is k-connected, $k \leqslant n-4$, then*

$$\pi_i(\tilde{D}(M, \partial), D(M, \partial)) = \pi_i(\tilde{D}(D^n, \partial), D(D^n, \partial)) \quad \textit{for } i \leqslant 2k-3$$

and

$$\pi_i(\tilde{A}(M, \partial), A(M, \partial)) = 0 \quad \textit{for } i \leqslant \min(2k-3, k+2),\ A = PL \textit{ or } Top.$$

There is a generalization of the theorem to the case of a tangential homotopy equivalence $N \to M$. If N is not a submanifold of M, then there is no natural map

$$(\tilde{A}(N, \partial), A(N, \partial)) \to (\tilde{A}(M, \partial), A(M, \partial)),$$

but we may compare the homotopy groups of these pairs assuming M and N to be of the same tangential homotopy type. Two spaces X and Y are said to be of *the same r-homotopy type* if X^r and Y^r, the r-th Postnikov terms in the Postnikov towers, are homotopy equivalent. A chosen homotopy equivalence $h\colon X^r \to Y^r$ is called an *r-homotopy equivalence*. If $h\colon X^r \to Y^r$ is an r-homotopy equivalence, there exist an $(r+1)$-dimensional CW-complex K and continuous maps $f\colon K \to X$ and $g\colon K \to Y$ with $\pi_i(f) = \pi_i(g)$ for $i \leqslant r+1$; further, if $u\colon X \to X^r$ and $v\colon Y \to Y^r$ are natural maps in the Postnikov tower, then $g_\# \cdot f_\#^{-1} = v_\#^{-1} \cdot h_\# \cdot u_\#$ on $\pi_i X$, $i \leqslant r$. Two manifolds M and N are said to be of *the same tangential r-type* if they are r-equivalent and if $f^*\tau_M = g^*\tau_N$, $f\colon K \to M$, $g\colon K \to N$ as above. If $r+1 < \frac{1}{2}n$, we may assume above that f and g are embeddings. Let P and R be regular neighbourhoods of $f(K)$ and $g(K)$, respectively. Then there is a homotopy equivalence $a\colon P \to R$ which is covered by a bundle map $\tau a\colon \tau P \to \tau R$. Hence we may assume that a is an immersion of P in $\operatorname{Int} R$. By the general position we may take $a\colon P \to \operatorname{Int} R$ as an embedding. Then we have the inclusions

$$\bar{a}\colon (\tilde{A}(P, \partial), A(P, \partial)) \to (\tilde{A}(R, \partial), A(R, \partial))$$

induced by a and

$$\beta\colon (\tilde{A}(P, \partial), A(P, \partial)) \to (\tilde{A}(M, \partial), A(M, \partial)),$$

$$\gamma\colon (\tilde{A}(R, \partial), A(R, \partial)) \to (\tilde{A}(N, \partial), A(N, \partial))$$

induced by injections $P \to N$ and $R \to N$, respectively. Thus, for all k for which the map

$$\gamma_\#\colon \pi_k(\tilde{A}(R, \partial), A(R, \partial)) \to \pi_k(\tilde{A}(N, \partial), A(N, \partial))$$

is an isomorphism, we have a map

$$\beta_\# \cdot a_\# \cdot \gamma_\#^{-1}\colon \pi_k(\tilde{A}(N, \partial), A(N, \partial)) \to \pi_k(\tilde{A}(M, \partial), A(M, \partial)).$$

An immediate consequence of the previous theorem is the following

THEOREM (Burghelea, Lashof and Rothenberg [1]). *If M^n and N are k-connected compact manifolds of the same tangential r-type*, $\dim M = \dim N \geqslant 5$, $n/2 > r+1 \geqslant k$, *then*

$$\pi_i\big(\tilde{A}(N, \partial), A(N, \partial)\big) = \pi_i\big(\tilde{A}(M, \partial), A(M, \partial)\big)$$

for

$$i \leqslant \begin{cases} \min(2r-1, r+k-1) & \textit{for } A = D, \\ \min(2r-1, r+k-1, r+3) & \textit{for } A = PL \textit{ or } Top. \end{cases}$$

These results may be applied to get similar informations on the group $C_A(M)$ of concordances of M. The map

$$p\colon C_A(M, \partial) \to A(M, \partial)$$

defined by restriction to $M \times \{1\}$ is a fibration with $A(M \times I, \partial)$ as a fibre. Similarly,

$$\tilde{p}\colon \tilde{C}_A(M, \partial) \to \tilde{A}(M, \partial)$$

is a fibration with a fibre $\tilde{A}(M \times I, \partial)$. Since the diagram

$$\begin{array}{ccc} C_A(M, \partial) & \to & \tilde{C}(M, \partial) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \tilde{p}} \\ A(M, \partial) & \to & \tilde{A}(M, \partial) \end{array}$$

commutes, we have a long exact sequence

$$\to \pi_k\big(\tilde{A}(M \times I, \partial), A(M \times I, \partial)\big) \to \pi_k\big(\tilde{C}_A(M, \partial), C_A(M, \partial)\big)$$
$$\to \pi_k\big(\tilde{A}(M, \partial), A(M, \partial)\big) \to \pi_{k-1}\big(\tilde{A}(M \times I, \partial), A(M \times I, \partial)\big) \to .$$

Since $\tilde{C}_A(M)$ is contractible (any concordance is concordant to the identity), the sequence above takes the form

$$\to \pi_k\big(\tilde{A}(M \times I, \partial), A(M \times I, \partial)\big) \to \pi_{k-1} C_A(M, \partial)$$
$$\to \pi_k\big(\tilde{A}(M, \partial), A(M, \partial)\big) \to \pi_{k-1}\big(\tilde{A}(M \times I, \partial), A(M \times I, \partial)\big) \to .$$

Let $N \subset \operatorname{Int} M$ be a compact submanifold (with boundary) of codimension 0. Then there is an injection $\alpha\colon C_A(N, \partial) \to C_A(M, \partial)$ defined in the same way as for $\tilde{A}(N, \partial) \to \tilde{A}(M, \partial)$ and the following diagram is commutative:

$$\begin{array}{ccccc} \to \pi_k\big(\tilde{A}(M \times I, \partial), A(M \times I, \partial)\big) & \to & \pi_{k-1} C_A(M, \partial) & \to & \pi_k\big(\tilde{A}(M, \partial), A(M, \partial)\big) \to \\ \uparrow & & \uparrow{\scriptstyle \alpha_\#} & & \uparrow \\ \to \pi_k\big(\tilde{A}(N \times I, \partial), A(N \times I, \partial)\big) & \to & \pi_{k-1} C_A(N, \partial) & \to & \pi_k\big(\tilde{A}(N, \partial), A(N, \partial)\big) \to \end{array}$$

Thus the preceding theorem implies analogous results for $\pi_{k-1}C_A(M, \partial)$ as well as the following

COROLLARY. *The i-th homotopy group of $C_A(M^n)$ for $i \leqslant n-7$ depends only on the $(i+3)$-dimensional skeleton of M.*

Note that by the Alexander trick we know now that $\pi_1 C_{PL}(M) = 0$ for any 2-connected compact manifold M, while $\pi_1 C_D(M) = Z_2 \oplus Wh_3(0)$ (cf. Section 5).

7. Comparing automorphisms in various categories. If we know one of the spaces $D(M)$, $Top(M)$ (e.g. for $M = R^n$ or D^n), then we certainly will appreciate information on the relation between the smooth and topological automorphisms. This is the general pattern of the theory of structures on manifolds: passing from the homotopy type to the homeomorphism class, then to the PL and smooth isomorphism classes. With except for homotopy equivalences, it works also for automorphisms.

We would like to mention one possible application of the comparison theory. Since the homotopy structure of the smooth concordance group is closely related to singularities of smooth functions, the connection is given, via the restriction map $C_D(M) \to D(M)$, $h \mapsto h|M \times \{1\}$, by the Cerf approach to pseudoisotopies. Since by the Alexander trick, PL and Top singularities are considerably simpler, the difference between automorphisms reflects the behaviour of smooth singularities.

Consider first the case of block automorphisms of a smooth compact manifold M. Homeomorphisms are simple equivalences, hence using the s-cobordism theorem one can deduce that the natural map

$$\pi_k\big(Top^{\sim}(M, \partial), \tilde{D}(M, \partial)\big) \to \mathcal{S}_D(M \times D^k, \partial)$$

is a bijection for $n+k \geqslant 6$, $\mathcal{S}_D(M \times D^k, \partial)$ being smooth structures on the topological manifold $M \times D^k$ fixed on $\partial(M \times D^k)$. Thus from the theorems of Hirsch, Mazur and Kirby, and Siebenmann we have

THEOREM (Antonelli et al. [3]). *The stable differential induces an isomorphism*

$$\pi_k\big(Top^{\sim}(M, \partial), \tilde{D}(M, \partial)\big) \xrightarrow{\cong} [\Sigma^k(M/\partial), Top/O] \quad \textit{for } k+n \geqslant 6,$$

and similarly for $\big(Top^{\sim}(M, \partial), PL^{\sim}(M, \partial)\big)$ *and* $PL^{\sim}(M, \partial), \tilde{D}(M, \partial)$.

Now we pass to the main theorem concerning usual groups of automorphisms. Let $\tau_M[O_n]$ be the principal O_n-bundle associated with the tangent vector bundle of M, and let $\tau_M[Top_n] = \tau_M[O_n] \times_{O_n} Top_n$ be the associated principal Top_n-bundle. Let

$$\tau_M[Top_n/O_n] = \tau_M[Top_n] \times_{Top_n} Top_n/O_n,$$

the associated bundle with fibre Top_n/O_n. If $\partial M \neq 0$, then the restriction of the bundle $\tau_M[Top_n/O_n]$ to ∂M contains the bundle $\tau_{\partial M}[Top_{n-1}/O_{n-1}]$. The bundle $\tau_M[Top_n/O_n]$ has the natural section

$$\sigma\colon M = \tau_M[O_n]\times_{O_n} O_n/O_n \to \tau_M[O_n]\times_{O_n} Top_n/O_n = \tau_M[Top_n/O_n].$$

By smoothing theory (see Kirby and Siebenmann [1]) the isotopy classes of smoothings of the topological manifold M are in one-to-one correspondence with the homotopy classes of sections of $\tau_M[Top_n/O_n]$. If N is a compact smooth submanifold of ∂M of codimension 0, denote by

$$\Gamma^N(\tau_M[Top_n/O_n],\, \tau_{\partial M}[Top_{n-1}/O_{n-1}],\, \sigma)$$

the space of continuous cross-sections of $\tau_M[Top_n/O_n]$ which agree with σ on N and whose restrictions to ∂M are cross-sections of $\tau_{\partial M}[Top_{n-1}/O_{n-1}]$, with the compact-open topology. If $h\colon M \to M$ is a homeomorphism of M, then the topological differential dh is an equivalence of the tangent vector bundle τ_M to itself. Thus dh defines a vector bundle reduction of the underlying Top_n bundle. The space of vector bundle reductions of τ_M may be identified with the space $\Gamma(\tau_M[Top_n/O_n])$. Since the composition with the vector bundle equivalence does not change vector bundle reductions, the differential d induces a semisimplicial map

$$\boldsymbol{Top}(M, N)/\boldsymbol{D}(M, N) \to S\Gamma^N(\tau_M[Top_n/O_n],\, \tau_{\partial M}[Top_{n-1}/O_{n-1}],\, \sigma),$$

where S denotes the singular complex functor.

The following comparison theorem was proposed in 1969 by Morlet [2] and proved by Burghelea and Lashof [1] (cf. also Kirby and Siebenmann [1]):

THEOREM (Morlet). *Let M be a smooth compact manifold,* $\dim M = n \neq 4$. *If $N \neq \partial M$, assume also* $\dim M \neq 5$. *The map*

$$\boldsymbol{Top}(M, N)/\boldsymbol{D}(M, N) \to \Gamma^N(\tau_M[Top_n/O_n],\, \tau_{\partial M}[Top_{n-1}/O_{n-1}],\, \sigma),$$

induced by the differential, induces an injective correspondence for connected components and a weak homotopy equivalence on any connected component. Analogous results hold for Top/PL and PL/D cases.

COROLLARY 1. $BD(D^n, \partial) \sim \Omega^n(PL_n/O_n)$.

Proof. The Alexander trick implies that

$$BD(D^n, \partial) = |\boldsymbol{PL}(D^n, \partial)/\boldsymbol{D}(D^n, \partial)|.$$

By Morlet's theorem,

$$|\boldsymbol{PL}(D^n, \partial)/\boldsymbol{D}(D^n, \partial)| \sim \Gamma^{S^{n-1}}(\tau_{D^n}[PL_n/O_n]).$$

But the bundle $\tau_{D^n}[PL_n/O_n]$ is trivial, whence

$$\Gamma^{S^{n-1}}(\tau_{D^n}[PL_n/O_n]) = \mathrm{Maps}(D^n, \partial D^n; PL_n/O_n, *) = \Omega^n(PL_n/O_n),$$

where $*$ denotes the base point of PL_n/O_n.

Similarly, $BD(D^n, \partial) \sim \Omega^n(Top_n/O_n)$ for $n \neq 4$.

COROLLARY 2. *$\Omega^n(Top_n/PL_n)$ is homotopically trivial for $n \neq 4$.*

Thus the homotopy groups of $D(D^n, \partial)$ are the unstable homotopy groups of PL_n/O_n. Antonelli et al. [2] have shown that the group $D(D^n, \partial)$ does not have finite type but it is not known yet whether $\pi_k(PL_n/O_n)$ are finitely generated. Some non-trivial elements of $\pi_k D(D^n, \partial)$ were found by Antonelli et al. [2], [3] (cf. also Burghelea and Lashof [1]). These elements often give non-trivial elements in $\pi_k D(M^n)$ under the natural map $\pi_k D(D^n, \partial) \to \pi_k D(M^n)$ induced by an embedding $D^n \to M^n$.

One can prove (cf. Kirby and Siebenmann [1]), using contractibility of $D(D^n, \partial)$ for $n \leqslant 3$, that Top_n/PL_n and Top_n/O_n are contractible for $n \leqslant 3$.

COROLLARY 3. $D(M) \sim Top(M) \sim |PL(M)|$ *if* $\dim M \leqslant 3$.

Following Burghelea and Lashof [2] we can apply Morlet's theorem to concordances. If $U, V \subset X$, let $P(X; V, U)$ be the space of paths in X beginning in U and ending in V, with the compact-open topology. Write

$$P_n = P(Top_{n+1}/O_{n+1}; Top_n/O_n, *)$$

and

$$R_n = P(Top_{n+1}/O_{n+1}; Top_n/O_n, Top_n/O_n).$$

We have the fibration

$$P_n \to R_n \xrightarrow{p} Top_n/O_n, \quad p(w) = w(0).$$

Let $\tau_M[R_n] = \tau_M[Top_n] \times_{Top_n} R_n$ be the associated fibration with fibre R_n. Then we have the fibration

$$P_n \to \tau_M[R_n] \xrightarrow{q} \tau_M[Top_n/O_n], \quad q[x, w] = [x, w(0)].$$

Let $\tau_M[P_n] = \sigma^*\tau_M[R_n]$ be the bundle induced from $\tau_M[R_n]$ under the section σ of $\tau_M[Top_n/O_n]$ corresponding to the isotopy class of the given smooth structure on M. Morlet's theorem may be paraphrased in this case as an equivalence

$$C_{Top}(M)/C_D(M) \sim S\Gamma(\tau_M[P_n]).$$

Analogous descriptions are obtained also for $C_{Top}(M)/C_{PL}(M)$ and $C_{PL}(M)/C_D(M)$.

The same arguments as in Corollary 1 applied to $C_D(D^n, \partial)$ give the following

COROLLARY 4. $BC_D(D^n) \sim \Omega^n P(PL_{n+1}/O_{n+1}; PL_n/O_n, *)$.

It is known that $\pi_i(PL_{n+1}/O_{n+1}, PL_n/O_n) = 0$ for $i \leqslant n+1$. Since $\pi_0 C_D(D^n) = 0$ by Cerf's theorem,

$$\pi_{n+2}(PL_{n+1}/O_{n+1}, PL_n/O_n) = 0$$

and we have

COROLLARY 5. *The natural homomorphism $\pi_i C_D(M) \to \pi_i C_{PL}(M)$ is an isomorphism for $i = 0$ and an epimorphism for $i = 1$.*

The groups $\pi_1 C_D(M)$ and $\pi_1 C_{PL}(M)$ differ in general (see the remark at the end of Section 6).

Since $\pi_i(Top_{n+1}/PL_{n+1}, Top_n/PL_n) = 0$ for all $i, n \geqslant 5$, we have

COROLLARY 6. *$\pi_i C_{PL}(M^n) = \pi_i C_{Top}(M^n)$ for all $i, n \geqslant 5$.*

A difference between smooth and PL cases is distinctly noticeable for concordances of the n-sphere. Chenciner [1] has shown that $C_D(S^n) \sim C_D(D^n)$ and, consequently, from Corollary 2 we obtain

$$C_D(S^n) \sim \Omega^{n+1} P(PL_{n+1}/O_{n+1}, PL_n/O_n).$$

It is known (cf. Kuiper and Lashof [1]) that

$$|C_{PL}(S^n)| \sim P(PL_{n+1}/O_{n+1}, PL_n/O_n).$$

Thus we have

COROLLARY 7. $C_D(S^n) \sim \Omega^{n+1} C_{PL}(S^n)$.

The following diagram of fibrations shows that comparison theorems and computations of homotopy groups of $\tilde{A}(M)/A(M)$ are parallel in some sense:

$$\begin{array}{ccccc}
X(M) & \to & \tilde{D}(M)/D(M) & \to & Top^{\sim}(M)/Top(M) \\
\downarrow & & \downarrow & & \downarrow \\
Top(M)/D(M) & \to & BD(M) & \to & BTop(M) \\
\downarrow & & \downarrow & & \downarrow \\
Top^{\sim}(M)/\tilde{D}(M) & \to & B\tilde{D}(M) & \to & BTop^{\sim}(M)
\end{array}$$

Note that $Top^{\sim}(M)/\tilde{D}(M)$ is a sum of components of the space of smooth structures on M, thus

$$\pi_i\big(Top^{\sim}(M), \tilde{D}(M)\big) \cong [\Sigma^i M, Top/O] \quad \text{for } i > 0.$$

A similar diagram can be written for $\big(G(M), D(M)\big)$:

$$\begin{array}{ccccc}
Y(M) & \to & \tilde{D}(M)/D(M) & \to & \{*\} \\
\downarrow & & \downarrow & & \downarrow \\
G(M)/D(M) & \to & BD(M) & \to & BG(M) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{G}(M)/\tilde{D}(M) & \to & B\tilde{D}(M) & \to & B\tilde{G}(M)
\end{array}$$

Hsiang and Sharpe succeeded in computing the π_1-part of the homotopy sequence of the first vertical fibration and used this to determine $\pi_0 D(T^n)$ for high-dimensional tori T^n.

THEOREM (Hsiang and Sharpe [1]). *If the first k-invariant of M vanishes and* $\dim M > 5$, *then*

$$\pi_1\big(\tilde{D}(M)/D(M)\big) \cong \pi_0\big(C_D(M)\big)/\{c+\varepsilon\bar{c}\}.$$

Here $\varepsilon = (-1)^{\dim M}$ and $^-$ is an involution on $Wh_2(\pi_1 M)\oplus \oplus Wh_1^+(\pi_1 M; \pi_2 M \times Z_2)$ induced by the inverse map of $\pi_1 M$ and Stiefel-Whitney classes. It seems probable that this isomorphism is induced by the natural map

$$C_D(M) \to P\big(\tilde{D}(M); D(M), \mathrm{id}\big).$$

COROLLARY. *If $n \geqslant 6$, then there is an exact sequence*

$$0 \to Wh_1(Z^n; Z_2)/\{c+\varepsilon\bar{c}\} \oplus h\mathscr{S}_D(T^n \times S^1; T^n \times \{1\}) \to \pi_0 D(T^n) \to GL_n(Z) \to 0.$$

There is a generalization of this theorem to higher homotopy groups. Note first that we have an involution on $C_A(M)$ that arises from the involution τ on $A(M \times I, \partial)$ defined by $\tau(\varphi)(x, t) = \varphi(x, 1-t)$ for any φ in $A(M \times I, \partial)$, $x \in M$, $t \in I$. The involution is defined first on the identity component of $C_A(M)/I_A(M)$, where $I_A(M)$ is the space of isotopies of id_M. In any class $a \in \big(C_A(M)/I_A(M)\big)_0$ choose $\varphi \in C_A(M)$ such that $\varphi | M \times \{1\} = \mathrm{id}_{M\times\{1\}}$ and write $\tau(a) = [\tilde{\varphi}]$, where $\tilde{\varphi}(x, t) = \varphi(x, 1-t)$. Since $I_A(M)$ is contractible, we have a homotopy equivalence $C_A(M)_0 \sim \big(C_A(M)/I_A(M)\big)_0$, hence a homotopy involution on $C_A(M)_0$, being a homomorphism of H-spaces. In the smooth case the involution is induced by the involution on $\mathscr{E}(M)$ (the space of smooth functions on $M \times I$ without critical points) given by $f \mapsto \bar{f}$, $\bar{f}(x, t) = 1 - f(x, 1-t)$.

Let G be an abelian 2-local group. Since multiplication by 2 is an isomorphism of G, any involution τ on G decomposes G into a direct sum $G^+ \oplus G^-$, where $G^+ = \{g \in G \colon \tau(g) = g\}$ and $G^- = \{g \in G \colon \tau(g) = -g\}$. A similar decomposition exists for an involution τ on a commutative CW H-group X localized at odd numbers. For any CW-complex K, $[K, X_{\mathrm{odd}}]$ is an abelian 2-local group with involution. Thus

$$[K, X_{\mathrm{odd}}] = [K, X_{\mathrm{odd}}]^+ \times [K, X_{\mathrm{odd}}]^-.$$

It follows from the Brown theorem that functors $[\cdot, X_{\mathrm{odd}}]^+$ and $[\cdot, X_{\mathrm{odd}}]^-$ are representable. If X_{odd}^+ and X_{odd}^- denote the classifying spaces, then $X_{\mathrm{odd}} \sim X_{\mathrm{odd}}^+ \times X_{\mathrm{odd}}^-$. In particular, we have a decomposition

$$\bar{C}_A(M)_{\mathrm{odd}} \sim \bar{C}_A(M)_{\mathrm{odd}}^+ \times \bar{C}_A(M)_{\mathrm{odd}}^-.$$

THEOREM (Lashof). *There exists an s-equivalence*

$$\Omega\big(A(M^n, \partial)/A(M^n, \partial)\big)_{\mathrm{odd}} \to \bar{C}_A(M^n)_{\mathrm{odd}}^{(-1)^n},$$

where

$$s = \begin{cases} [n/12-5] & \textit{for } A = D, \\ [n/6-5] & \textit{for } A = PL \textit{ or } Top. \end{cases}$$

8. Stable concordances and algebraic K-theory. One can now think of the following program to determine homotopy groups of automorphism groups: compute the block automorphism group using surgery, then find the homotopy of the concordance group and put these two pieces together. It is the second step which makes this program rather hard to execute. Only π_0 and π_1 of the concordance group are more or less known and there is no hope that the method of computation extends to higher homotopy groups. However, we know that low homotopy groups of $C_A(M)$ depend only on the homotopy type of M and it will be helpful to have homotopy functors containing those "homotopical parts" of $C_A(M)$ and properly related to the algebraic K-theory. In fact, such functors are now described to be the stabilized (with respect to dimension) concordance groups. The *stabilization map* $t\colon C(M) \to C(M\times I)$ is defined by $\varphi \mapsto \varphi \times \mathrm{id}_I$ and by the *stable concordance group* of M we mean the group

$$\bar{C}_A(M) = \lim_k C_A(M\times D^k).$$

Remark. We have been decided to work with concordances equal to the identity on $\partial M\times I$, therefore we must slightly improve the definition of t. This makes no problem since $(M\times I,\ M\times\{0\})$ is isomorphic to $(M\times I,\ M\times\{0\}\cup\partial M\times I)$.

THEOREM (Hatcher [7]). *The stabilization map*

$$t\colon\ C_{PL}(M^n) \to C_{PL}(M^n\times I)$$

is $[n/6-5]$*-connected and the stable concordance group functor* $\bar{C}_{PL}$ *extends to a homotopy functor on the category of compact polyhedra.*

The comparison theorem of Morlet allows us to transfer this result to the smooth category. Note first that the homotopy equivalence

$$C_{PL}(X)/C_D(X) \to S\Gamma^{\partial X}(\tau_X[P_{\dim X}])$$

applied to $X = M^n\times I$ yields a homotopy equivalence

$$C_{PL}(M^n\times I)/C_D(M^n\times I) \to S\Gamma^{\partial M}(\tau_M[\Omega P_{n+1}]).$$

The stabilization map induces a map

$$t\colon\ C_{PL}(M)/C_D(M) \to C_{PL}(M\times I)/C_D(M\times I).$$

This map is related to the suspension map: if $s\colon P_n \to \Omega P_{n+1}$ is adjoint to the usual suspension map $\sum P_n \to P_{n+1}$, then there is a commutative diagram

$$\begin{array}{ccc} C_{PL}(M^n)/C_D(M^n) & \xrightarrow{d} & S\Gamma^{\partial M}(\tau_M[P_n]) \\ \downarrow{\scriptstyle i} & & \downarrow \\ C_{PL}(M^n \times I)/C_D(M^n \times I) & \xrightarrow{d} & S\Gamma^{\partial M}(\tau_M[\Omega P_{n+1}]), \end{array}$$

where the right vertical arrow is induced by s.

Elaboration of this diagram for $M = S^n \times D^k$ gives

PROPOSITION (Burghelea and Lashof [2]). *The map* $s\colon P_n \to \Omega P_{n+1}$ *is* $[n/12 + n - 3]$*-connected.*

This implies in turn the following stability theorem for smooth concordances:

THEOREM (Burghelea and Lashof [2]). *The stabilization map* $C_D(M) \to C_D(M \times I)$ *is* $[n/12 - 5]$*-connected.*

The stabilization map generalizes to a transfer map

$$\tau\colon C_D(M) \to C_D\big(\delta(\xi)\big),$$

where ξ is a smooth vector bundle over M and $\delta(\xi)$ is the disc bundle associated with ξ. This map can be obtained as follows. Recall that $C_D(M)$ and $C_D\big(\delta(\xi)\big)$ are homotopy equivalent to $\mathscr{E}(M)$ and $\mathscr{E}\big(\delta(\xi)\big)$, respectively. We define the map $\tau\colon \mathscr{E}(M) \to \mathscr{E}\big(\delta(\xi)\big)$ by $\tau(f) = f \circ p$, where p is the projection of the bundle $\delta(\xi)$. It is easy to see that for ξ being the trivial line bundle the stabilization map and the transfer coincide up to homotopy.

Now we have a general stability theorem:

THEOREM (Burghelea and Lashof [2]). *For any vector bundle over* M^n, *the map* $\tau\colon C_D(M^n) \to C_D\big(\delta(\xi)\big)$ *is* $[n/12 - 5]$*-connected.*

Remark. There is no reason to expect that the stability range is the best possible.

We can extend the stable concordance group functor to a functor on the category of compact polyhedra in the standard way, defining

$$\bar{C}_A(K) = \lim_n C_A\big(N_n(K)\big),$$

where $N_n(K)$ is a regular neighbourhood of K in S^n for sufficiently large n. For any compact smooth manifold M the transfer induces a homotopy equivalence

$$\lim_n C_D(M \times D^n) \to \bar{C}_D(M).$$

The following is a consequence of the theorem of Burghelea, Lashof and Rothenberg:

THEOREM. *$\bar{C}_D$ is a homotopy functor.*

Morlet's theorem can be used once more to compare the PL and smooth stable concordance functors. If

$$D^n_* \subset M^n \subset D^n \quad \text{and} \quad C^0_A(M) = C_A(M)/C_A(D^n_*),$$

then there is a splitting $C_A(M) \sim C^0_A(M) \times C_A(D^n_*)$, whence

$$\bar{C}_A(M) \sim \bar{C}^0_A(M) \times \bar{C}_A(*).$$

PROPOSITION (Burghelea and Lashof [2]). *$\pi_*\big(\bar{C}^0_{PL}(K)/\bar{C}^0_D(K)\big)$ is a homology theory that comes from the spectrum $\mathscr{P} = \{P_n\}$, i.e.*

$$\pi_i\big(\bar{C}^0_{PL}(K)/\bar{C}^0_D(K)\big) = \lim_n \pi_{i+n}(K \wedge P_n).$$

Another homology theory can be obtained by stabilization of the functor $\bar{C}_A$. If we let

$$\bar{C}^s_A(K) = \lim_n \Omega^n \bar{C}_A(\Sigma^n K),$$

then simple computations show that $\pi_* \bar{C}^s_{PL}$ is a homology theory that comes from the spectrum $\mathscr{P} = \{P_n\}$.

COROLLARIES. 1. *The groups $\pi_* \bar{C}^0_D(K)$ are determined up to extension by stable PL concordance groups.*

2. $\pi_* \bar{C}^s_D K = 0$.

The next problem is to relate the stable functors $\bar{C}_A$ to algebraic K-theory as is prompt by the computation of $\pi_0 C$. Note that by the corollaries above it is sufficient to work only with the functor $\bar{C}_{PL}$.

Look first on the Whitehead functor Wh_*. The groups $Wh_i(G)$ for $i = 1, 2$ were defined algebraically long time ago and it is known (cf. Loday [1]) that $Wh_i(G)$ is the cokernel of a natural map $h_i(BG, K_Z) \to K_i(Z[G])$, where $h(\cdot, K_Z)$ is the homology theory given by the Gersten-Wagoner spectrum for $K(Z)$. In general, the Whitehead groups may be defined (cf. Waldhausen [2]) as the homotopy groups of a space $Wh(G)$ such that there is a homotopy fibration

$$h(BG) \to K(Z[G]) \to Wh(G)$$

with h being a functor from the category of spaces to itself such that $\pi_* h$ is a homology theory.

The Whitehead groups describe only a part of the homotopy structure of $C_{PL}(M)$ (in $\pi_0 C_{PL}(M)$ the remaining part is $Wh_1^+(\pi_1 M; Z_2 \times \pi_2 M)$, a factor of $\pi_1^s(\Omega M)$), thus the functor K should be enlarged analogously to passing from $Wh(\pi_1 M)$ to $\bar{C}_{PL}(M)$. Therefore, we would like to have an algebraic K-theory W defined for spaces rather than for rings, together with a map

$$\Omega^2 W(M) \to \bar{C}(M)$$

which is a homology approximation (i.e. the homotopy fibre of this map yields a homology theory). Such a theory was constructed by Waldhausen [3].

The functor K is, as was indicated by Waldhausen and rigorously treated by May, a special case of a functor defined on rings up to homotopy which generalizes Quillen's algebraic K-theory (cf. May [1]). We shall sketch this construction.

Let H be $\Omega^\infty \Sigma^\infty(\Omega X \cup *)$ or a topological ring, which are, in our context, the most important examples of rings up to homotopy. Then $\pi_0 H$ is a ring and we have the natural map $d\colon H \to \pi_0 H$. Let $M_n(H)$ be the H_--space of all $(n \times n)$-matrices with entries in H (with the juxtaposition given by the usual composition of matrices). The map d induces a map $\bar{d}\colon M_n(H) \to M_n(\pi_0 H)$ of H-spaces. Let $\widehat{GL}_n(H)$ be the space of matrices from $M_n(H)$ with image under $\bar{d}$ invertible in $M_n(\pi_0 H)$, and define

$$\widehat{GL}(H) = \lim_n \widehat{GL}_n(H).$$

Then put

$$\bar{K}(H) = B^+ \widehat{GL}(H),$$

where B^+ is a superposition of the classifying space functor and the $+$ construction of Quillen. For a discrete ring R, $\bar{K}(R)$ coincides with Quillen's $K(R)$ and d induces a map

$$\bar{K}(H) \to K(\pi_0 H).$$

Definition. $W(X) = \bar{K}\big(\Omega^\infty \Sigma^\infty(\Omega X \cup *)\big)$.

Waldhausen has constructed a homology approximation

$$\Omega^2 W(X) \to |\bar{C}_{PL}(X)|$$

with fibre being a homology theory given by a spectrum obtained from an infinite delooping of $\bar{K}(\Omega^\infty \Sigma^\infty)$. In fact, the original definition of W, used to construct the homology approximation, differs from the one described above, but Waldhausen states that they are equivalent.

We may now come back to the computation of $\bar{C}(M)$. The last step to be done is to determine the homotopy groups of $W(M)$. There is hope (and some examples of effective computations) that this may be performed after localization at Q. It is caused by the fact that homotopy type of $\Omega^\infty \Sigma^\infty$ is known (to be trivial) only after localization at Q. In particular, this implies (cf. Waldhausen [3]) that $\bar{K}(\Omega^\infty\Sigma^\infty(\Omega M \cup *))_Q$ is homotopy equivalent to $\bar{K}(|Z[\Lambda SM]|)_Q$, where ΛSM is the loop group of Kan of the singular complex SM of M and $Z[\Lambda SM]$ is the simplicial group ring. For example, $W(*)_Q$ is homotopy equivalent to $K(Z)_Q$, hence

$$\pi_k W(*)_Q = \begin{cases} Q & \text{if } k = 4i+1, \\ 0 & \text{for other } k \end{cases}$$

by the calculation of $\pi_* K(Z)_Q$ due to Borel.

There are also more general results concerning $W(M)_Q$ with applications to computation of homotopy groups of $D(M)_Q$ (cf. Burghelea [4] and [5]). The more detailed exposition of Waldhausen's work and its applications will be given by the first-named author in a subsequent paper.

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WROCŁAW

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY

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