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ON A GAME OF SIERPIŃSKI

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Introduction. A game of Sierpiński, considered in this paper, is an infinite positional two-person game with perfect information. Lately, there have appeared many papers about such games and their applications to some mathematical constructions.

The first infinite positional game with perfect information was defined by S. Mazur about 1928. But before that, in 1924, a certain infinite positional game was implicitly applied in Sierpiński's paper.

Theorems concerning the existence of winning strategies in the Mazur game were proved by S. Banach and S. Mazur in the interwar period, but they were published only in 1957 by Oxtoby [8].

In 1924, Sierpiński [10] proved that every uncountable Borel set contains a perfect subset. In the proof he made use of some multivariate function of sets, by means of which Telgársky [11] defined a topological game, where the function becomes a winning strategy of Player II. This game will be called the *game of Sierpiński* and denoted by $S(X, Y)$. The game of Sierpiński shares some features with the above-mentioned famous Banach–Mazur game (see [9]) and with its generalizations studied by Morgan II [6]. A certain modification of the Sierpiński game was considered by the author [4].

Definitions and notation. We define the *game of Sierpiński* $S(X, Y)$ as follows. Let X be a subset of a topological space Y. Player I chooses a set $A_1 \subset X$. After that, Player II chooses a set $B_1 \subset A_1$ such that

$$|B_1| > \aleph_0 \text{ if } |A_1| > \aleph_0, \quad B_1 = \emptyset \text{ if } |A_1| \leqslant \aleph_0.$$

Assume inductively that $A_1 \supset B_1 \supset \ldots \supset A_n \supset B_n$ have been chosen. Then Player I chooses a set $A_{n+1} \subset B_n$. After that, Player II chooses a set $B_{n+1} \subset A_{n+1}$ such that

$$|B_{n+1}| > \aleph_0 \text{ if } |A_{n+1}| > \aleph_0, \quad B_{n+1} = \emptyset \text{ if } |A_{n+1}| \leqslant \aleph_0.$$

Player II wins the play $(A_1, B_1, A_2, B_2, \ldots)$ of the game $S(X, Y)$ if

$$\bigcap_n \bar{B}_n \subset X;$$

otherwise Player I wins.

A *strategy of Player* I is a function s defined for all finite (including empty) decreasing sequences $(A_1, B_1, \ldots, A_n, B_n)$ of subsets of X, so that $s(\emptyset) = A_1 \subset X$, and

$$s(A_1, B_1, \ldots, A_n, B_n) = A_{n+1} \subset B_n \quad \text{for each } n \in N.$$

A *stationary strategy of Player* I is a function t defined for all subsets of X and for the empty set such that $t(\emptyset) = A_1 \subset X$ and $t(B) = A \subset B$. A strategy and a stationary strategy of Player II can be defined similarly. For Player II we assume in addition that B_{n+1} (B) is an uncountable set if A_{n+1} (A) is an uncountable one, and B_{n+1} (B) is the empty set if A_{n+1} (A) is a countable set.

In this paper we shall use the following notation:

$\mathrm{I} \uparrow S(X, Y)$ $(\mathrm{II} \uparrow S(X, Y))$ means that Player I (Player II, respectively) has a winning strategy in the game $S(X, Y)$.

$\mathrm{I} \Uparrow S(X, Y)$ $(\mathrm{II} \Uparrow S(X, Y))$ means that Player I (Player II, respectively) has a stationary winning strategy in the game $S(X, Y)$.

The game $S(X, Y)$ is said to be *determined* if either Player I or Player II has a winning strategy in $S(X, Y)$.

First we shall show the following

THEOREM 1. *Let X be a subset of a separable metric space. If* $\mathrm{I} \uparrow S(X, Y)$, *then* $\mathrm{I} \Uparrow S(X, Y)$.

For the proof of Theorem 1 we refer to [3], where the construction of F. Galvin deals with a fairly general class of games.

THEOREM 2. *If* $\mathrm{II} \uparrow S(X, Y)$, *then* $\mathrm{II} \Uparrow S(X, Y)$.

Proof. If Y is a separable metric space, then Theorem 2 results from [3], Corollary 15. The construction presented there was slightly modified and adopted for the modified game of Sierpiński $S_0(X, Y)$ (see [4]). If Y is any topological space, the proof for the game $S(X, Y)$ is analogical to the proof for the game $S_0(X, Y)$, so it is omitted.

It follows that for separable metric spaces it makes no difference whether we investigate the existence of a winning strategy or of a stationary winning strategy.

Recall that a subset X of a topological space Y is said to be a *Souslin set* in Y if there is an indexed family

$$\{F(k_1, \ldots, k_n)\colon (k_1, \ldots, k_n) \in N^n,\ n \in N\}$$

of closed subsets of Y such that

$$X = \bigcup \{\bigcap \{F(k_1, \ldots, k_n)\colon n \in N\}\colon (k_1, k_2, \ldots) \in N^N\}.$$

LEMMA 1. *The following conditions are equivalent*:

(a) *X is a Souslin set in Y.*

(b) *There exists a sequence* $(\varepsilon_1, \varepsilon_2, \ldots)$ *of countable partitions of X such* *that* ε_{n+1} *refines* ε_n *for each* $n \in N$, *and* $\bigcap_n \bar{E}_n \subset X$ *for each sequence* $(E_1, E_2, \ldots)$ *with* $E_{n+1} \subset E_n \in \varepsilon_n$ *for each* $n \in N$.

For the proof we refer to [12].

THEOREM 3. *If* X *is a Souslin set in* Y, *then* $\mathrm{II} \uparrow S(X, Y)$.

Proof (cf. [11], Theorem 1.1). By Lemma 1 there exists a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ of countable partitions of X such that ε_{n+1} refines ε_n for each $n \in N$, and $\bigcap_n \bar{E}_n \subset X$ for each sequence $(E_1, E_2, \ldots)$ with $E_{n+1} \subset E_n \in \varepsilon_n$ for each $n \in N$. We define a strategy t for Player II as follows. Let $n \in N$ and let $(A_1, B_1, \ldots, A_n)$ be a sequence of subsets of X such that $A_1 \supset B_1 \supset \ldots \supset A_n$ and $|A_k| > \aleph_0$ for each $k \leqslant n$. Then

$$A_n = \bigcup \{A_n \cap E\colon\ E \in \varepsilon_n\},$$

and thus there exists a set $E_n \in \varepsilon_n$ such that $|A_n \cap E_n| > \aleph_0$. We set

$$t(A_1, B_1, \ldots, A_n) = A_n \cap E_n = B_n.$$

If $n \in N$ and $(A_1, B_1, \ldots, A_n)$ is a sequence of subsets of X such that $A_1 \supset B_1 \supset \ldots \supset A_n$ and $|A_n| \leqslant \aleph_0$, then we set

$$t(A_1, B_1, \ldots, A_n) = \emptyset.$$

The strategy t, defined in this way, is a winning strategy of Player II because

$$\bigcap_n \bar{B}_n \subset \bigcap_n \bar{E}_n \subset X.$$

THEOREM 4. *Let* X *be a subset of an uncountable Polish space* Y. *If* $\mathrm{II} \uparrow S(X, Y)$, *then either* X *is countable or it contains a copy of the Cantor discontinuum.*

The proof of Theorem 4 is analogical to the proof of Sierpiński's theorem (see [10]), so it is omitted.

Remark 1. Moreover, one can easily show the following strengthening of Theorem 4:

If $\mathrm{II} \uparrow S(X, Y)$, *then for each uncountable subset* A_1 *of* X *the set* $\bar{A}_1 \cap X$ *contains a copy of the Cantor discontinuum.*

It is sufficient for Player I to choose a set $A_1 \subset X$ on the first move.

THEOREM 5. *Let* Y *be a Polish space. If* $\mathrm{I} \uparrow S(X, Y)$, *then* $Y - X$ *contains a copy of the Cantor discontinuum.*

The proof of Theorem 5 is similar to the proof of Theorem 4.

A subset X of an uncountable Polish space Y is said to be a *Bernstein set* if neither X nor $Y - X$ contains a copy of the Cantor discontinuum. Assuming the axiom of choice we infer that each uncountable Polish space contains a Bernstein set ([5], p. 514). Since Bernstein sets in Polish spaces are uncountable, from Theorems 4 and 5 we have the following

COROLLARY 1. *If X is a Bernstein set in a Polish space Y, then the game $S(X, Y)$ is undetermined.*

Two subsets X and Z of a topological space Y are said to be *Borel separated* if there is a Borel set B in Y such that $X \subset B$ and $Z \subset Y-B$.

LEMMA 2. *Let X and Z be subsets of a topological space Y and let*

$$Z = \bigcup_n Z_n.$$

If X and Z are not Borel separated, then there is an $n \in N$ such that X and Z_n are not Borel separated.

The proof of Lemma 2 is easy, and so omitted.

LEMMA 3. *Let Y be a Polish space. If $Y-X$ contains an analytic subset Z and X contains a subset W, so that Z and W are not Borel separated, then W contains an uncountable subset D such that each uncountable subset of D is not Borel separated from Z.*

Proof. Since Z is an analytic set, $Y-Z$ is a coanalytic set and $Y-Z \supset W$. If W were contained in some countable union of constituents of $Y-Z$, it would be Borel separated from Z. Thus there are uncountably many ordinals $\alpha < \Omega$ such that $Z_\alpha \cap W \neq \emptyset$, where Z_α are constituents of $Y-Z$. We choose a point w_α from each set $Z_\alpha \cap W$ such that $Z_\alpha \cap W \neq \emptyset$, and put

$$D = \{w_\alpha\colon \alpha < \Omega \text{ and } Z_\alpha \cap W \neq \emptyset\}.$$

Then D is uncountable, D is contained in W, and each its uncountable subset E contains points w_α with any arbitrarily large ordinals $\alpha < \Omega$. Therefore, E is not Borel separated from Z (see [5], p. 501).

THEOREM 6. *Let Y be a Polish space. If $Y-X$ contains an analytic subset Z such that Z and X are not Borel separated, then $\mathrm{I}\uparrow S(X, Y)$.*

Proof. Let f be a continuous map from the set of irrational numbers N^N onto Z, where $Z \subset Y-X$ and let

$$F(j_1, \ldots, j_n) = f(B(j_1, \ldots, j_n)),$$

where

$$B(j_1, \ldots, j_n) = \{(i_1, i_2, \ldots) \in N^N\colon (i_1, \ldots, i_n) = (j_1, \ldots, j_n)\}.$$

Then

$$\bigcup \{F(j)\colon j \in N\} = A,$$
$$\bigcup \{F(j_1, \ldots, j_n, j)\colon j \in N\} = F(j_1, \ldots, j_n),$$

and

$$\operatorname{diam} F(j_1, \ldots, j_n) \to 0 \quad \text{as } n \to \infty$$

for each $(j_1, j_2, \ldots) \in N^N$.

From Lemma 3 it follows that X contains an uncountable subset A_1 such that each its uncountable subset is not Borel separated from Z. We shall define a strategy s for Player I as follows. Let us put $s(\emptyset) = A_1$. Assume that Player II chooses an uncountable subset B_1 such that $B_1 \subset A_1$. Since B_1 and Z are not Borel separated and $Z = \bigcup \{F(j) \colon j \in N\}$, from Lemma 2 it follows that there is $j_1 \in N$ such that B_1 and $F(j_1)$ are not Borel separated. Since B_1 is contained in X and $F(j_1)$ is an analytic subset of $Y - X$, using Lemma 3 for the sets B_1 and $F(j_1)$ we infer the existence of an uncountable set $A_2 \subset B_1$ such that each its uncountable subset is not Borel separated from $F(j_1)$. Let us put $s(A_1, B_1) = A_2$. Assume that $B_2 \subset A_2$ is an uncountable set chosen by Player II. Since B_2 and $F(j_1)$ are not Borel separated and

$$F(j_1) = \bigcup \{F(j_1, j) \colon j \in N\},$$

it follows again from Lemma 2 that there is $j_2 \in N$ such that B_2 and $F(j_1, j_2)$ are not Borel separated. Using once again Lemma 3 for the set $B_2 \subset X$ and the analytic set $F(j_1, j_2) \subset Y - X$, we infer the existence of an uncountable set $A_3 \subset B_2$ such that each its uncountable subset is not Borel separated from $F(j_1, j_2)$. Let us put

$$s(A_1, B_1, A_2, B_2) = A_3.$$

Proceeding by induction, we define the strategy s for Player I and construct the sequence $(j_1, j_2, \ldots) \in N^N$ such that for each $n \in N$ the sets B_n and $F(j_1, \ldots, j_n)$ are not Borel separated, so

$$\bar{B}_n \cap \overline{F(j_1, \ldots, j_n)} \neq \emptyset.$$

Since

$$\bigcap_n F(j_1, \ldots, j_n) = \{p\} \subset Z,$$

where $p = f(j_1, j_2, \ldots)$, also

$$p \in \bigcap_n \bar{B}_n.$$

Indeed, let $n \in N$ and let U be an open neighbourhood of p. Then there is $m \geqslant n$ such that

$$\overline{F(j_1, \ldots, j_m)} \subset U.$$

Since $\overline{F(j_1, \ldots, j_m)} \cap \bar{B}_m \neq \emptyset$, we have $U \cap B_m \neq \emptyset$ and, because $B_n \supset B_m$, we obtain $B_n \cap U \neq \emptyset$. Thus $p \in \bar{B}_n$. Finally, because $p \in Z \subset Y - X$, we have

$$\bigcap_n \bar{B}_n \cap (Y - X) \neq \emptyset,$$

and thus s is a winning strategy of Player I.

COROLLARY 2. *If X is a coanalytic non-Borel subset of a Polish space Y, then* $\mathrm{I}\uparrow S(X, Y)$.

By Theorem 3 and Corollary 2 we have also

COROLLARY 3. *If X is analytic or coanalytic in a Polish space Y, then the game $S(X, Y)$ is determined.*

QUESTION. Let X belong to the σ-algebra generated by analytic subsets of an uncountable Polish space Y. Is then $S(X, Y)$ determined? **(P 1334)**

Remark 2. One can easily notice that if $\mathrm{I}\uparrow S(X, Y)$, then there is an uncountable set $A\subset X$ such that each its uncountable subset B is not Borel separated from $Y-X$. A question arises when the converse implication is true **(P 1335)**. Theorem 6 gives only a partial answer to this question.

We shall prove below some properties of set families for which the players have winning strategies in the Sierpiński game $S(X, Y)$.

THEOREM 7. *The family $\{X\subset Y\colon \mathrm{II}\uparrow S(X, Y)\}$ is closed under $\mathscr{A}$-operation for every topological space Y.*

Proof. We should prove that if $\mathrm{II}\uparrow S\big(X(k_1, \ldots, k_n), Y\big)$ for each $(k_1, \ldots, k_n)\in N^n$, $n\in N$, and

$$X = \bigcup\{\bigcap\{X(k_1, \ldots, k_n)\colon n\in N\}\colon (k_1, k_2, \ldots)\in N^N\},$$

then $\mathrm{II}\uparrow S(X, Y)$.

First we define a secondary system

$$\{X^{(k_1,\ldots,k_n)}\colon (k_1, \ldots, k_n)\in N^n,\ n\in N\}$$

in the following way:

$$X^{(k_1,\ldots,k_n)} = \bigcup\{X(k_1)\cap\ldots\cap X(k_1, \ldots, k_n)\cap X(k_1, \ldots, k_n, j_1)$$
$$\cap X(k_1, \ldots, k_n, j_1, j_2)\cap\ldots\colon (j_1, j_2, \ldots)\in N^N\}.$$

Then we have

$$X = \bigcup\{X^{(k_1)}\colon k_1\in N\},$$
$$X(k_1)\supset X^{(k_1)} = \bigcup\{X^{(k_1,k_2)}\colon k_2\in N\},$$
$$X(k_1, k_2)\supset X^{(k_1,k_2)} = \bigcup\{X^{(k_1,k_2,k_3)}\colon k_3\in N\}, \quad \ldots,$$

and so on.

Let $s^{(k_1,\ldots,k_n)}$ be a winning strategy of Player II in the game $S\big(X(k_1, \ldots, k_n), Y\big)$. Without loss of generality, we may assume that X is uncountable. We shall define a strategy s for Player II in the game $S(X, Y)$ as follows. Let $A_1\subset X$ be an uncountable set chosen by Player I in $S(X, Y)$. There is $k_1\in N$ such that

$$|A_1\cap X^{(k_1)}| > \aleph_0.$$

Now, in the game $S(X(k_1), Y)$ let Player I choose $B_1^{(k_1)} = s^{(k_1)}(A_1^{(k_1)})$ in reply. We have $B_1^{(k_1)} \subset A_1^{(k_1)}$ and $|B_1^{(k_1)}| > \aleph_0$.

In the game $S(X, Y)$ we set $s(A_1) = B_1 = B_1^{(k_1)}$. Let $A_2 \subset B_1$ be an uncountable set chosen by Player I. In the game $S(X(k_1), Y)$ let Player I choose $A_2^{(k_1)} = A_2$ and let Player II choose

$$B_2^{(k_1)} = s^{(k_1)}(A_1^{(k_1)}, B_1^{(k_1)}, A_2^{(k_1)})$$

in reply. We have $B_2^{(k_1)} \subset A_2^{(k_1)}$ and $|B_2^{(k_1)}| > \aleph_0$. There is $k_2 \in N$ such that

$$|B_2^{(k_1)} \cap X^{(k_1,k_2)}| > \aleph_0.$$

Let Player I in the game $S(X(k_1, k_2), Y)$ choose

$$A_1^{(k_1,k_2)} = B_2^{(k_1)} \cap X^{(k_1,k_2)}$$

and let Player II choose

$$B_1^{(k_1,k_2)} = s^{(k_1,k_2)}(A_1^{(k_1,k_2)})$$

in reply. We have $B_1^{(k_1,k_2)} \subset A_1^{(k_1,k_2)}$ and $|B_1^{(k_1,k_2)}| > \aleph_0$.

In the game $S(X, Y)$ we set $B_2 = s(A_1, B_1, A_2) = B_1^{(k_1,k_2)}$. Let $A_3 \subset B_2$ be an uncountable set chosen by Player I. In the game $S(X(k_1), Y)$ let Player I choose $A_3^{(k_1)} = A_3$ and let Player II choose

$$B_3^{(k_1)} = s^{(k_1)}(A_1^{(k_1)}, B_1^{(k_1)}, A_2^{(k_1)}, B_2^{(k_1)}, A_3^{(k_1)})$$

in reply. Then $B_3^{(k_1)} \subset A_3^{(k_1)}$ and $|B_3^{(k_1)}| > \aleph_0$. In the game $S(X(k_1, k_2), Y)$ let Player I choose $A_2^{(k_1,k_2)} = B_3^{(k_1)}$ and let Player II choose

$$B_2^{(k_1,k_2)} = s^{(k_1,k_2)}(A_1^{(k_1,k_2)}, B_1^{(k_1,k_2)}, A_2^{(k_1,k_2)})$$

in reply. We have $B_2^{(k_1,k_2)} \subset A_2^{(k_1,k_2)}$ and $|B_2^{(k_1,k_2)}| > \aleph_0$. There is $k_3 \in N$ such that

$$|B_2^{(k_1,k_2)} \cap X^{(k_1,k_2,k_3)}| > \aleph_0.$$

In the game $S(X(k_1, k_2, k_3), Y)$ let Player I choose

$$A_1^{(k_1,k_2,k_3)} = B_2^{(k_1,k_2)} \cap X^{(k_1,k_2,k_3)}$$

and let Player II choose

$$B_1^{(k_1,k_2,k_3)} = s^{(k_1,k_2,k_3)}(A_1^{(k_1,k_2,k_3)})$$

in reply. Then

$$B_1^{(k_1,k_2,k_3)} \subset A_1^{(k_1,k_2,k_3)} \quad \text{and} \quad |B_1^{(k_1,k_2,k_3)}| > \aleph_0.$$

In the game $S(X, Y)$ we set

$$B_3 = s(A_1, B_1, A_2, B_2, A_3) = B_1^{(k_1, k_2, k_3)}, \quad \ldots,$$

and so on.

In order to prove that the just defined s is a winning strategy for Player II in $S(X, Y)$, it suffices to show that $\bigcap_k \bar{B}_k \subset X$. Since

$$X = \bigcup \{\bigcap \{X(k_1, \ldots, k_n): n \in N\}: (k_1, k_2, \ldots) \in N^N\},$$

so if we take the above-constructed sequence $(k_1, k_2, \ldots) \in N^N$, it is sufficient to prove that

$$\bigcap_k \bar{B}_k \subset \bigcap \{X(k_1, \ldots, k_n): n \in N\}.$$

Let us take any arbitrary $n \in N$. Then

$$\bigcap_{k=1}^{\infty} \bar{B}_k \subset \bigcap_{k=n}^{\infty} \bar{B}_k \subset \bigcap_{k=1}^{\infty} \overline{B_k^{(k_1, \ldots, k_n)}} \subset X(k_1, \ldots, k_n),$$

because the sets $B_k^{(k_1, \ldots, k_n)}$ are chosen in the game $S(X(k_1, \ldots, k_n), Y)$ by Player II according to the (winning) strategy $s^{(k_1, \ldots, k_n)}$. Thus for each $n \in N$ we have

$$\bigcap_{k=1}^{\infty} \bar{B}_k \subset X(k_1, \ldots, k_n),$$

and therefore

$$\bigcap_{k=1}^{\infty} \bar{B}_k \subset \bigcap_{n=1}^{\infty} X(k_1, \ldots, k_n) \subset X.$$

Remark 3. If F is a closed subset of a topological space Y, then, clearly, II $\uparrow S(F, Y)$. From Theorem 7 it follows that II $\uparrow S(X, Y)$ if X is a Souslin set in Y. Hence Theorem 3 is a consequence of Theorem 7.

By Theorem 7 we get

Corollary 4. *For every fixed topological space Y the family $\{X \subset Y$: II $\uparrow S(X, Y)\}$ is closed for countable unions and countable intersections.*

Remark 4. One can easily check that, in opposition to Corollary 4, for $Y = [0, 1]$ the family $\{X \subset Y$: I $\uparrow S(X, Y)\}$ is neither additive nor multiplicative.

Remark 5. It is not difficult to prove that if $Z \subset X \subset Y$ and Z is a Borel set in Y, then the games $S(X, Y)$ and $S(X-Z, Y)$ are equivalent. In particular, if Y is a T_1-space and Z is a countable subset of $X \subset Y$, then the games $S(X, Y)$ and $S(X-Z, Y)$ are equivalent.

Further, we shall use the following two lemmas, whose simple proofs will be omitted.

LEMMA 4. *If* $X \subset Z \subset Y$ *and* $\mathrm{II} \uparrow S(X, Y)$, *then* $\mathrm{II} \uparrow S(X, Z)$.

LEMMA 5. *If* $X \subset Z \subset Y$ *and* $\mathrm{II} \uparrow S(X, Z)$ *and* $\mathrm{II} \uparrow S(Z, Y)$, *then* $\mathrm{II} \uparrow S(X, Y)$.

THEOREM 8. *Let* X *be a subset of a complete metric space* Y. *If* $\mathrm{II} \uparrow S(X, Y)$ *and* Z *is an arbitrary space which contains* X, *then* $\mathrm{II} \uparrow S(X, Z)$.

Proof. Let us consider the completion $\tilde{Z}$ of the space Z. By the Lavrentieff theorem (see [5], p. 429), the homeomorphism $\mathrm{id}_X\colon X \to X$ between subspaces of the spaces Y and Z is extendable to a homeomorphism $h\colon Y_1 \to Z_1$, where $X \subset Y_1 \subset Y$ and $Y_1 \in G_\delta(Y)$ as well as $X \subset Z_1 \subset \tilde{Z}$ and $Z_1 \in G_\delta(Z)$. Let us put $Z_2 = Z_1 \cap Z$ and $Y_2 = h^{-1}(Z_2)$. Then $X \subset Z_2 \subset Z$ and $Z_2 \in G_\delta(Z)$.

Since $\mathrm{II} \uparrow S(X, Y)$ and $X \subset Y_2 \subset Y$, from Lemma 4 we infer that $\mathrm{II} \uparrow S(X, Y_2)$. Since h is a homeomorphism and $Z_2 = h(Y_2)$, it follows that $\mathrm{II} \uparrow S(X, Z_2)$. From Theorem 3 we infer that $\mathrm{II} \uparrow S(Z_2, Z)$, as $Z_2 \in G_\delta(Z)$. Finally, since $\mathrm{II} \uparrow S(X, Z_2)$ and $\mathrm{II} \uparrow S(Z_2, Z)$, by Lemma 5 we obtain $\mathrm{II} \uparrow S(X, Z)$.

Remark 6. One can easily notice that if Y is any topological space, then the games $S(X, Y)$ and $S(X, \bar{X}^Y)$ are equivalent.

Below we shall give some examples of set families for which the game $S(X, Y)$ is undetermined.

By Corollary 1 we see that the game $S(X, Y)$ is undetermined if X is a Bernstein set in a Polish space Y. In order to give an example of another set for which the Sierpiński game is undetermined, we shall use the Banach–Mazur game.

Oxtoby [8] considered the following variant of the Banach–Mazur game. Let Y be any topological space and let $\mathbf{G}$ be a family of subsets of the space Y. It is assumed that this family satisfies the following conditions: $\mathrm{Int}\, G \neq \emptyset$ for every set $G \in \mathbf{G}$, and for each non-empty open subset U there is $G \in \mathbf{G}$ such that $G \subset U$. Let X be a subset of Y. We define the *Banach–Mazur game* $BM(X, Y, \mathbf{G})$ as follows. Players I and II choose alternately subsets $U_i, V_i \in \mathbf{G}$ such that $U_1 \supset V_1 \supset U_2 \supset V_2 \supset \ldots$ Player I wins the play $(U_1, V_1, U_2, V_2, \ldots)$ of the game $BM(X, Y, \mathbf{G})$ if

$$X \cap \bigcap_i U_i \neq \emptyset;$$

otherwise Player II wins.

Oxtoby proved the following

THEOREM 9. (a) $\mathrm{II} \uparrow BM(X, Y, \mathbf{G})$ *if and only if* X *is of first category in* Y.

(b) *Let* Y *be a complete metric space.* $\mathrm{I} \uparrow BM(X, Y, \mathbf{G})$ *if and only if* $Y - X$ *is of first category at some point of* Y.

For the proof of Theorem 9 we refer to [8] (Theorems 1 and 2).

Recall that a subset X of a topological space Y is said to be a *Lusin set* if X is uncountable and $|X \cap F| \leqslant \aleph_0$ for every nowhere dense set $F \subset Y$.

Let Y be a metric space. A subset $X \subset Y$ is said to be a *Lusin set in every ball* if for each open ball $K \subset Y$ the set $K \cap X$ is a Lusin set. We shall use the following notation. Let A be any set. We put

$$A^0 = \{x \in A\colon |U \cap A| > \aleph_0 \text{ for each open neighbourhood } U \text{ of } x\}.$$

LEMMA 6. *If X is a Lusin set in Y and $A \subset X$ is an arbitrary uncountable set, then* $\operatorname{Int}\overline{A^0} \neq \emptyset$.

The proof is easy, and thus it is omitted.

THEOREM 10. *Let Y be a Polish space. If $X \subset Y$ is a Lusin set in every ball, then* $\neg\,[\mathrm{I} \uparrow S(X, Y)]$.

Proof. Assume that Player I has a winning strategy s. Player I chooses a subset $A_1 = s(\emptyset)$. Let $U = \operatorname{Int}\overline{A_1^0}$. By Lemma 6, U is non-empty. To this set U we introduce a metric d such that the metric space (U, d) is complete (see [2], p. 342).

We shall show that Player II has a winning strategy t in the Banach–Mazur game $BM(X \cap U, U, G)$, where G is a family of non-empty regular open subsets. In the Banach–Mazur game let Player I choose a non-empty regular open subset $U_1 \subset U$. Without loss of generality we may assume that $\operatorname{diam} U_1 \leqslant 1$ in the metric d. The set $B_1 = A_1^0 \cap U_1 \subset A_1$ is uncountable. If in the Sierpiński game $S(X, Y)$ Player II chooses a set B_1, then, in reply, Player I will choose $s(A_1, B_1) = A_2$ such that $A_2 \subset B_1$ and $|A_2| > \aleph_0$. Let us put $V_1 = \operatorname{Int}\overline{A_2^0}$. Then $V_1 \neq \emptyset$. We will show that $V_1 \subset U_1$. Indeed, we have

$$V_1 = \operatorname{Int}\overline{A_2^0} \subset \overline{A_2^0} \subset \overline{B_1} = \overline{U_1 \cap A_1^0} \subset \overline{U_1},$$

and therefore

$$V_1 = \operatorname{Int} V_1 \subset \operatorname{Int}\overline{U_1} = U_1$$

because U_1 is a regular open set. In the Banach–Mazur game we define a reply of Player II as $t(U_1) = V_1$. Let Player I choose a non-empty regular open subset $U_2 \subset V_1$ such that $\operatorname{diam} U_2 \leqslant \frac{1}{2}$. The set $B_2 = A_2^0 \cap U_2 \subset A_2$ is uncountable. If in the Sierpiński game $S(X, Y)$ Player II chooses a set B_2, then, in reply, Player I will choose

$$s(A_1, B_1, A_2, B_2) = A_3$$

such that $A_3 \subset B_2$ and $|A_3| > \aleph_0$. Let us put $V_2 = \operatorname{Int}\overline{A_3^0}$. Similarly as above, we have $V_2 \neq \emptyset$ and $V_2 \subset U_2$. We define $t(U_1, V_1, U_2) = V_2$.

Assume that in the Sierpiński game $S(X, Y)$ sets $A_1, B_1, \ldots, A_n, B_n$ were chosen while in the Banach–Mazur game $BM(X \cap U, U, G)$ sets $U_1, V_1, \ldots, U_n$ were chosen. Then in the Sierpiński game Player I chooses

$$s(A_1, B_1, \ldots, A_n, B_n) = A_{n+1},$$

so that $A_{n+1} \subset B_n$ and $|A_{n+1}| > \aleph_0$. Let us put $V_n = \mathrm{Int}\,\overline{A^0_{n+1}}$. Then $V_n \neq \emptyset$ and $V_n \subset U_n$. We define

$$t(U_1, V_1, \ldots, U_n) = V_n.$$

Then in the Banach–Mazur game Player I chooses a non-empty regular open subset $U_{n+1} \subset V_n$ such that

$$\operatorname{diam} U_{n+1} \leqslant \frac{1}{n+1}.$$

In the Sierpiński game Player II chooses $B_{n+1} = A^0_{n+1} \cap U_{n+1}$. In this manner we have inductively defined the sets A_{n+1}, V_n, U_{n+1} and B_{n+1}.

Since s is a winning strategy of Player I in the Sierpiński game $S(X, Y)$ and diameters of the sets B_n converge to zero when $n \to \infty$, we have

$$\emptyset \neq \bigcap_n \bar{B}_n = \bigcap_n \bar{A}_n \subset Y - X.$$

We shall show that t is a winning strategy for Player II in the Banach–Mazur game $BM(X \cap U, U, G)$. Indeed,

$$\emptyset \neq \bigcap_n \bar{U}_n = \bigcap_n \bar{V}_n = \bigcap_n \overline{\mathrm{Int}(\overline{A^0_n})} \subset \bigcap_n \overline{A^0_n}$$
$$\subset \bigcap_n \bar{A}_n \subset Y - X.$$

By Theorem 9 (a) we see that $X \cap U$ is of first category; i.e.,

$$X \cap U = \bigcup_i F_i,$$

where F_i for $i = 1, 2, \ldots$ are nowhere dense sets. Since $|X \cap U| > \aleph_0$, there is a positive integer k such that $|F_k| > \aleph_0$. Then

$$|(X \cap U) \cap F_k| > \aleph_0.$$

Since the space Y is separable, we have

$$U = \bigcup_{i=1}^{\infty} K_i,$$

where $K_1, K_2, \ldots$ is a sequence of open balls contained in U. There is a positive integer n such that

$$|(X \cap K_n) \cap F_k| > \aleph_0.$$

Since F_k is a nowhere dense subset, this contradicts X being a Lusin set in every ball.

Thus Player I does not have a winning strategy in the game $S(X, Y)$.

By Theorems 10 and 4 we get

COROLLARY 5. *Let Y be a Polish space. If $X \subset Y$ is a Lusin set in every ball, then the game $S(X, Y)$ is undetermined.*

Consider the case where $Y-X$ is a Lusin set.

THEOREM 11. *Let Y be a Polish space and let $Y-X$ be a Lusin set in every ball. Then the game $S(X, Y)$ is undetermined.*

Proof. First, we shall show that X is of second category in Y. Indeed, assume that

$$X \subset \bigcup_n F_n,$$

where F_n are closed nowhere dense sets in Y. Then, for any positive integer n, $F_n \cap (Y-X)$ is a countable set. Hence $F_n \cap X = F_n - (Y-X)$ is countable, so $F_n \cap X$ is a G_δ-set in Y. Thus

$$X = \bigcup_n (F_n \cap X)$$

is a $G_{\delta\sigma}$-set in Y and $Y-X$ is a Borel set (even $F_{\sigma\delta}$) in Y. Therefore $Y-X$ contains a copy of the Cantor discontinuum. This contradicts $Y-X$ being a Lusin set.

We shall show that $\neg[\mathrm{II} \uparrow S(X, Y)]$. Assume that Player II has a winning strategy s. Since X is of second category in Y, we may construct (assuming CH) a Lusin set A_1 in Y which is contained in X (see [5], p. 525).

In the game $S(X, Y)$ let Player I choose a set A_1. We shall show that Player I has a winning strategy t in the Banach–Mazur game $BM(X, Y, G)$, where G is a family of non-empty regular open sets. Let $B_1 = s(A_1)$. By Lemma 6 we have $U = \operatorname{Int}\overline{B_1^0} \neq \emptyset$. There exists a non-empty regular open subset U_1 contained in U. In the Banach–Mazur game we put $t(\emptyset) = U_1$. Let $V_1 \subset U_1$ be a non-empty regular open subset chosen by Player II. Without loss of generality we may assume that $\operatorname{diam} V_1 \leqslant 1$. The set $A_2 = B_1^0 \cap V_1 \subset B_1$ is uncountable. In the game $S(X, Y)$ Player I chooses A_2. In this game Player II chooses $B_2 = s(A_1, B_1, A_2)$ in reply, so that $B_2 \subset A_2$ and $|B_2| > \aleph_0$. Let $U_2 = \operatorname{Int}\overline{B_2^0}$. Similarly as in the proof of Theorem 10, we have $U_2 \neq \emptyset$ and $U_2 \subset V_1$. We define a reply for Player I in the Banach–Mazur game as $t(U_1, V_1) = U_2$. Let Player II choose in this game a non-empty regular open subset $V_2 \subset U_2$ such that $\operatorname{diam} V_2 \leqslant \frac{1}{2}$. The set $A_3 = B_2^0 \cap V_2 \subset B_2$ is uncountable. In the game $S(X, Y)$ Player I chooses A_3. In this game Player II chooses

$$B_3 = s(A_1, B_1, A_2, B_2, A_3)$$

in reply, so that $B_3 \subset A_3$ and $|B_3| > \aleph_0$. Let $U_3 = \operatorname{Int}\overline{B_3^0}$. Then $U_3 \neq \emptyset$ and $U_3 \subset V_2$. We define

$$t(U_1, V_1, U_2, V_2) = U_3, \quad \ldots,$$

and so on.

Since s is a winning strategy of Player II in the game $S(X, Y)$, we have

$$\{x\} = \bigcap_n \bar{B}_n \subset X.$$

Next, t is a winning strategy for Player I in the game $BM(X, Y, G)$. Indeed,

$$\emptyset \neq \bigcap_n \overline{U_n} = \bigcap_n \overline{\mathrm{Int}(\overline{B_n^0})} \subset \bigcap_n \overline{B_n^0} \subset \bigcap_n \overline{B_n} \subset X.$$

By Theorem 9 (b), the set $Y-X$ is of first category at some point, i.e., there exists an open set W such that $W \cap (Y-X)$ is of first category. Similarly as in the proof of Theorem 10, we obtain a contradiction with the assumption that $Y-X$ is a Lusin set in every ball.

By Theorem 5 we have $\neg [\mathrm{I} \uparrow S(X, Y)]$. Thus the game $S(X, Y)$ is undetermined.

Remark 7. Davis [1] considers the following game. Let X be a subset of the Cantor discontinuum $\{0, 1\}^N$. Two players construct a sequence

$$(x_0, x_1, x_2, \ldots) \in \{0, 1\}^N$$

choosing alternately a bit

$$x_i \in \{0, 1\} \quad \text{(Player I)}$$

and a finite binary sequence

$$(x_{i+1}, \ldots, x_{i+k}) \in \{0, 1\}^k \quad \text{(Player II)}.$$

Player I wins if $(x_0, x_1, x_2, \ldots) \in X$; otherwise Player II wins. In paper [7] of Mycielski this game is denoted by $G_2^*(X)$ and the following sentence is called the *axiom* P: "every uncountable set $X \subset \{0, 1\}^N$ contains a copy of the Cantor discontinuum" (of course, axiom P is inconsistent with the axiom of choice). Axiom P is equivalent to the game $G_2^*(X)$ being determined for each $X \subset \{0, 1\}^N$.

Notice that axiom P is equivalent to the statement that for each subset $X \subset \{0, 1\}^N$ the Sierpiński game $S(X, \{0, 1\}^N)$ is determined, besides we always have $\mathrm{II} \uparrow S(X, \{0, 1\}^N)$.

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