

*THE SPLITTING OF ZERO-DIMENSIONAL AUTOMORPHISMS
AND ITS APPLICATION*

BY

NOBUO AOKI (TOKYO)

0. Introduction. Group automorphisms of zero-dimensional compact metric groups are said to be *zero-dimensional automorphisms*. For such automorphisms, we shall prove a splitting theorem which is sharper than a result (Theorem 11.7) of Yuzvinskiĭ [10] and a result (Theorem 2) of the present author [1]. From our result it can be derived what kind of zero-dimensional automorphisms has the specification property, and prove easily that every zero-dimensional automorphism has the pseudo-orbit tracing property. For the abelian case, the above results are proved by the present author [2].

In the remainder of this section, we shall give some definitions which are used in the proofs of our results. Let X be a compact metric group with the invariant metric d and let σ be an automorphism of X . Let X split into a direct product

$$X = \prod_{-\infty}^{\infty} \sigma^i H$$

of normal subgroups $\sigma^i H$ (the term *subgroup* will be applied to closed subgroups). Then X is said to be a *Bernoulli group under σ* (in abbreviation, *B. group*). If, in particular, H is a simple subgroup (the term *simple* will be applied only to algebraic subgroups), then X is said to be a *simple Bernoulli group under σ* (in abbreviation *S.B. group*). Let X_1 and X_2 be arbitrary subgroups of a compact group. We denote by $[X_1, X_2]$ the subgroup generated by elements of the forms $[x_1, x_2] = x_1^{-1} x_2^{-1} x_1 x_2$, $x_1 \in X_1$ and $x_2 \in X_2$. If X_1 is normal, $[X_1, X_2]$ is contained in X_1 , and if further X_2 is also normal, $[X_1, X_2]$ is a normal subgroup contained in $X_1 \cap X_2$. Since every automorphism σ preserves the normalized Haar measure m of X , we can consider ergodic theoretical properties of (X, σ) . The Kolmogorov entropy of σ under m will be denoted by $h(\sigma)$. The set of equivalent classes of irreducible unitary representations of X will be denoted by X^* . Let $\gamma: X^* \rightarrow X^*$ be the map defined by $(\gamma f)(x) = f(\sigma x)$, $f \in X^*$ and $x \in X$. It is proved in

[7] that (X, σ) is ergodic under m iff $\gamma^n f = f$ ($f \in X^*$, $n > 0$) implies that f maps X to the identity matrix of some order. We say that (X, σ) is *expansive* if there is an open neighborhood U of the identity e in X such that

$$\bigcap_{-\infty}^{\infty} \sigma^n U = \{e\}.$$

It is known that every expansive automorphism has finite entropy. For materials on topological dynamics the reader may refer to [5].

Throughout this paper, the restriction and the factor of σ will be denoted by the same symbol if there is no possibility of confusion.

1. A main result. In this section X will be a zero-dimensional compact metric group and σ will be an automorphism of X . Our main result is the following

THEOREM. *Let X_0 denote the center of X . If (X, σ) is ergodic under the normalized Haar measure, then X contains a sequence $\{X_j: j \geq 0\}$ of completely σ -invariant normal subgroups satisfying the following conditions:*

- (i) (X_0, σ) is ergodic;
- (ii) for every $j \geq 1$, X_j is an S.B. non-abelian group under σ ;
- (iii) X splits into the direct product

$$X = \bigtimes_{j \geq 0} X_j$$

of the subgroups X_j ($j \geq 0$).

The following results (I)-(V) are proved in [10].

(I) (Theorem 3.4). *If X splits into a direct product*

$$X = \bigtimes_{-\infty}^{\infty} L_i$$

of simple non-abelian groups L_i , then this direct product splitting is unique, and an arbitrary normal subgroup of X is equal to the direct product of some collection of the groups L_i .

(II) (Theorem 3.7). *Let Z be the center of X . Assume that Z is finite and X/Z is an S.B. non-abelian normal subgroup under σ . Then X contains a completely σ -invariant normal subgroup H such that X splits into a direct product $X = Z \times H$.*

(III) (Theorem 11.5). *Let A be an open normal subgroup of X with*

$$\bigcap_{-\infty}^{\infty} \sigma^j A = \{e\}.$$

If X/A is simple, then either X is finite or X is an S.B. group under σ .

(IV) (p. 73). *If H is a completely σ -invariant normal subgroup of X , then $h(\sigma) = h(\sigma_{X/H}) + h(\sigma_H)$.*

(V) (Theorem 3.5). *If H is a completely σ -invariant finite normal subgroup of X and if further (X, σ) is ergodic, then H is central in X .*

(VI) (Theorem 2, [1]). *If (X, σ) is ergodic, then X contains a sequence $X = F_0 \supset F_1 \supset \dots$ of completely σ -invariant normal subgroups such that $\bigcap F_n = \{e\}$ and, for every $n \geq 0$,*

(i) *if F_n/F_{n+1} is non-abelian, it is an S.B. group under σ ;*

(ii) *if F_n/F_{n+1} is abelian, in F_n there is a sequence $F_n = Y_{n,0} \supset Y_{n,1} \supset \dots$ of completely σ -invariant normal subgroups such that*

$$\bigcap_i Y_{n,i} = F_{n+1}$$

and, for every $i \geq 1$, $F_n/Y_{n,i}$ is an S.B. group under σ .

From now on, it will be assumed that $X \neq \{e\}$ and (X, σ) is ergodic under the normalized Haar measure.

LEMMA 1. *$([X, X], \sigma)$ is ergodic. If $[X, X] \neq \{e\}$, then it is infinite.*

Proof. Since (X, σ) is ergodic, (X, σ) is topologically mixing (i.e., for any two non-empty open sets U and V there is an $M > 0$ such that $U \cap \sigma^n V \neq \emptyset$ for all $n \geq M$). Hence for every $k \geq 1$ both (X, σ^k) and $(X \times X, \sigma^k \times \sigma^k)$ are also topologically mixing, so that there is a point $\dot{x}_0 \in X \times X$ such that the set $\{(\sigma^k \times \sigma^k)^j \dot{x}_0: -\infty < j < \infty\}$ is dense in $X \times X$. Let $\varphi: X \times X \rightarrow [X, X]$ be a continuous map defined by

$$\varphi(x, y) = [x, y], \quad (x, y) \in X \times X.$$

Since $\varphi(\sigma^k \times \sigma^k) \dot{x} = \sigma^k \varphi \dot{x}$ for $\dot{x} \in X \times X$, the set $\{\sigma^{kj} \varphi \dot{x}_0: -\infty < j < \infty\}$ is dense in $\varphi(X \times X)$. Now, assume that $\gamma^k f = f$ on $[X, X]$ for some $f \in [X, X]^*$ and some $k > 0$. Consequently, $f = I$ on $\varphi(X \times X)$, where I denotes the identity matrix. Hence, for every $n \geq 1$, $f = I$ on $\{x_1 \dots x_n: x_i \in \varphi(X \times X), 1 \leq i \leq n\}$, which implies that $([X, X], \sigma)$ is ergodic. Hence $[X, X]$ is infinite if $[X, X] \neq \{e\}$ (because every non-trivial ergodic automorphism of a compact group has a completely positive entropy under the normalized Haar measure).

LEMMA 2. *Assume that X contains a σ -invariant normal subgroup $K \neq \{e\}$ such that (K, σ) is ergodic and X/K is an S.B. group under σ . Then there is an open normal subgroup F such that $X = FK$. In particular, let X_1 and $\{e\} \neq K_1$ ($\subset X_1$) be completely σ -invariant normal subgroups of X . If (X_1, σ) and (K_1, σ) are both ergodic and if further X_1/K_1 is an S.B. group under σ , then there is a normal subgroup F_1 of X such that $X_1 = F_1 K_1$.*

Proof. Since (K, σ) is ergodic, so is (K, σ^k) for all $k \neq 0$. Hence there is a point $x_0 \in K$ such that for every $k \geq 1$ the set $\{\sigma^{kj} x_0: -\infty < j < \infty\}$ is dense in K . Choose $g \in X^*$ with $g(x_0) \neq I$ (the existence of such a representation g is a consequence of Peter-Weyl's theorem). Let F be the kernel of g in X . Then X/F is finite and $F \not\subset K$ because X/K is an S.B. group under σ ; i.e.,

$$X/K = \bigcap_{-\infty}^{\infty} \sigma^j(B/K),$$

where B is a normal subgroup of X . Since

$$\sigma^j B / (F \cap \sigma^j B) \cong F \sigma^j B / F \subset X/F \quad \text{for all } j,$$

$\sigma^j B / (F \cap \sigma^j B)$ is finite. It is easy to see that $F \cap \sigma^j B \not\subset K$ for all j . Indeed, assuming $F \cap \sigma^j B \subsetneq K$ for some j , we infer that $K / (F \cap \sigma^j B)$ is finite. Since the set

$$\{\sigma^n x_0 : -\infty < n < \infty\} (F \cap \sigma^j B) / (F \cap \sigma^j B)$$

is a finite subset of $K / (F \cap \sigma^j B)$, there are n and m ($n > m \geq 0$) such that

$$\sigma^n x_0 (F \cap \sigma^j B) = \sigma^m x_0 (F \cap \sigma^j B).$$

Put $y = \sigma^m x_0$. Since $(\sigma^{n-m} y)^{-1} y \in F$, we have $g(\sigma^{n-m} y) = g(y)$ and $g = I$ on K , which is a contradiction. Since $F \cap \sigma^j B \not\subset K$ for all j , we obtain $(F \cap \sigma^j B)K \not\subsetneq K$ and, by simplicity of $\sigma^j B/K$, $\sigma^j B = (F \cap \sigma^j B)K$. Therefore, $X = FK$. The second statement is easily obtained by the above argument, and so we omit the proof.

LEMMA 3. *Let K be a completely σ -invariant normal subgroup of X . Then:*

(i) *If K is finite and X/K is abelian, then X is abelian.*

(ii) *Assume that there is a proper normal subgroup F such that $X = FK$. If K is an S.B. abelian group under σ and X/K is abelian, then X is abelian.*

Proof. (i) Notice that (X, σ) is ergodic. Since X/K is abelian, we get $[X, X] \subset K$, and so $[X, X] = \{e\}$ if K is finite (by Lemma 1). Thus (i) is proved.

(ii) If $F \cap K = \{e\}$, then $X = F \times K$ is clearly abelian. If $F \cap K \neq \{e\}$, we have $[X, X] \subset F \cap K \subsetneq K$ since $X / (F \cap K)$ is abelian. Since (X, σ) is ergodic, $([X, X], \sigma)$ is also ergodic and $[X, X]$ is infinite unless $[X, X] = \{e\}$ (by Lemma 1). By (IV) we have $h(\sigma_K) = h(\sigma_{[X, X]}) + h(\sigma_{K/[X, X]})$. Since K is an S.B. abelian group, $h(\sigma_K) = \log p$ for some prime p . Therefore, we get $[X, X] = \{e\}$ (since the entropy of every zero-dimensional automorphism equals $\log k$ for some integer $k \geq 0$ (cf. [10], p. 87)).

LEMMA 4. *Assume that X contains an open normal subgroup A such that*

$$\bigcap_{-\infty}^{\infty} \sigma^j A = \{e\}.$$

Then there is a finite sequence $X = F_0 \supset \dots \supset F_n \not\subsetneq F_{n+1} = \{e\}$ of completely σ -invariant normal subgroups such that for every i ($0 \leq i \leq n$) F_i / F_{i+1} is an S.B. group under σ .

Proof. By assumption, (X, σ) is ergodic. Hence there is a sequence $X = F_0 \supset F_1 \supset \dots$ of normal subgroups such that $\bigcap F_n = \{e\}$ and $\sigma F_n = F_n$

for every $n \geq 0$, and F_n/F_{n+1} satisfies the condition (i) or (ii) of (VI). Since A is open in X and $\bigcap_{-\infty}^{\infty} \sigma^j A = \{e\}$, we get $F_n \not\supseteq F_{n+1} = \{e\}$ for some $n \geq 0$. If

F_n/F_{n+1} is non-abelian, then it is an S.B. group under σ . When F_n/F_{n+1} is abelian, it remains only to show that F_n/F_{n+1} is an S.B. group under σ .

Let $\{Y_{n,i}\}$ be a sequence of normal subgroups as in (VI). Since $F_n/Y_{n,i}$ is an S.B. abelian group under σ , we have $h(\sigma_{F_n/Y_{n,i}}) = \log p$, where p is a prime. Consequently, $h(\sigma_{Y_{n,i}/F_{n+1}}) = 0$ (by (IV)). In fact, (X, σ) is expansive since $\bigcap_{-\infty}^{\infty} \sigma^j A = \{e\}$, and so is $(Y_{n,i}, \sigma)$. We see by Theorem 2 in [3] that $(Y_{n,i}/F_{n+1}, \sigma)$ is expansive. Hence $Y_{n,i}/F_{n+1}$ is finite (by Lemma 8 in [3]); i.e., $Y_{n,k} = F_{n+1}$ for some $k > 0$. The conclusion is obtained.

LEMMA 5. Let K be as in Lemma 2 and let K and X/K be both S.B. groups under σ . Assume that one of the following conditions holds:

- (i) K is abelian and X/K is non-abelian;
- (ii) K is non-abelian and X/K is abelian;
- (iii) K and X/K are both non-abelian.

Then there is an S.B. normal subgroup B under σ such that X splits into a direct product $X = K \times B$.

Remark. B is non-abelian when (i) or (iii) holds, and B is abelian when (ii) holds.

Proof. Choose a normal subgroup F' such that X/F' is simple, $X = KF'$ and $F' \not\subset K$ (the existence of such an F' is a consequence of Lemma 2). Then $B = \bigcap_{-\infty}^{\infty} \sigma^j F'$ is normal in X . By (III), X/B is an S.B. group under σ . First, we shall give a proof for the case (i). Since K is abelian, X/F' is also abelian, and so is X/B . Since X/K is an S.B. non-abelian group under σ , so is X/BK by (I). But X/BK is abelian since it is a factor of X/B . Hence $X = BK$. We shall now show that there is an S.B. non-abelian normal subgroup B' under σ such that $X = K \times B'$. Since $h(\sigma_K) = \log p$ for some prime p and $B \cap K \subsetneq K$, $h(\sigma_{B \cap K}) = 0$ by (IV). Since (K, σ) is expansive, $B \cap K$ is finite. By (V), $B \cap K$ is central in X . Since $X/K \cong B/(B \cap K)$ is an S.B. non-abelian group under σ , $B \cap K$ is the center of B . Using (II), we see that in B there is an S.B. abelian subgroup B' such that $B = (B \cap K) \times B'$. Obviously, B' contains no center in B and $x^{-1} B' x$ ($x \in X$) is normal in B . Applying (I), we see that $B'(x^{-1} B' x)$ contains no center in B . This implies that $x^{-1} B' x \subset B'$; i.e. B' is normal in X . Therefore, $X = K \times B'$. Assuming (ii), we get the conclusion in the same way. Under the assumption (iii), we infer that X/B is non-abelian. Since X/K is an S.B. non-abelian group under σ , by (I) we have $X/K = BK/K$, and so $X = BK = B \times K$.

LEMMA 6. Assume that X contains a finite sequence

$$X = F_0 \supset F_1 \supset \dots \not\supseteq F_{n+1} = \{e\}$$

of normal subgroups with the conditions of Lemma 4. If $k \geq 2$ is the greatest integer such that X/F_{k-1} is abelian, then in X there exists a completely σ -invariant normal subgroup C such that C/F_k is abelian, F_{k-1}/F_k is non-abelian when F_{k-1}/F_k is non-trivial, and such that $X/F_k = (C/F_k) \times (F_{k-1}/F_k)$.

Proof. Since $(X/F_2)/(F_1/F_2) \cong X/F_1$ is an S.B. group under σ and F_1/F_2 is ergodic, by Lemma 2 there is an open normal subgroup C_1/F_2 with $X/F_2 = (C_1/F_2)(F_1/F_2)$. In fact, C_1 is normal in X . Apply the same argument for $(F_1/F_3, F_2/F_3)$. Then there is a normal subgroup C_2/F_3 of X/F_3 with $F_1/F_3 = (C_2/F_3)(F_2/F_3)$, and so

$$X/F_3 = (C_1 C_2/F_3)(F_2/F_3).$$

Remark that $C_1 C_2$ is normal in X . After a finite number ($k-1$, say) of steps, we get a non-trivial normal subgroup C' such that

$$X/F_k = (C'/F_k)(F_{k-1}/F_k),$$

and hence $X = C' F_{k-1}$. Since $X/C' \cong F_{k-1}/(C' \cap F_{k-1})$, X/C' is a factor of F_{k-1}/F_k . By Lemma 3, F_{k-1}/F_k is non-abelian when it is non-trivial. Hence F_{k-1}/F_k is an S.B. non-abelian group under σ , i.e., it has no center. From (I) it follows that X/C' has also no center. It is easy to see that, for every $n \geq 1$,

$X/\bigcap_{i=-n}^n \sigma^i C'$ is isomorphic to some subgroup of the direct product group $\bigtimes_{i=-n}^n (X/\sigma^i C')$. Put

$$C = \bigcap_{i=-\infty}^{\infty} \sigma^i C'.$$

Then we infer easily that X/C has no center. Since X/F_{k-1} is abelian, we have $X = CF_{k-1}$. Since

$$(C \cap F_{k-1})/F_k \subset F_{k-1}/F_k,$$

we have $C \cap F_{k-1} = F_k$ by (I) and (VI) (i).

LEMMA 7. Assume that X is non-abelian and X contains an open normal subgroup A with

$$\bigcap_{j=-\infty}^{\infty} \sigma^j A = \{e\}.$$

Then in X there are the center U and a finite direct product V of S.B. non-abelian subgroups under σ such that X is expressed as $X = U \times V$.

Proof. We can find a sequence $X = F_0 \supset F_1 \supset \dots \supset F_{n+1} = \{e\}$ of normal subgroups that satisfy the properties of Lemma 4. Let $k \geq 2$ be the greatest integer such that X/F_{k-1} is abelian. By Lemma 6 there is a completely σ -invariant normal subgroup C such that

$$(a) \quad X/F_k = (C/F_k) \times (F_{k-1}/F_k),$$

where C/F_k is abelian and F_{k-1}/F_k is non-abelian. Apply Lemma 5 for $(F_{k-1}/F_{k+1}, F_k/F_{k+1})$; then there is a subgroup C_1 such that C_1 is normal in F_{k-1} and

$$(b) \quad F_{k-1}/F_{k+1} = (C_1/F_{k+1}) \times (F_k/F_{k+1}).$$

Since $C_1/F_{k+1} \cong F_{k-1}/F_k$, C_1/F_{k+1} is an S.B. non-abelian group under σ . Hence C_1/F_{k+1} has no center in F_{k-1}/F_{k+1} , so that C_1/F_{k+1} is normal in X/F_{k+1} . Consequently, C_1 is normal in X . From (a) and (b) it follows that

$$(c) \quad X/F_{k+1} = (C/F_{k+1}) \times (C_1/F_{k+1}).$$

If F_{k+1}/F_{k+2} is abelian, we can easily prove as above that there is a completely σ -invariant subgroup C_2 such that C_2 is normal in C_1 and

$$(d) \quad C_1/F_{k+2} = (C_2/F_{k+2}) \times (F_{k+1}/F_{k+2}),$$

where C_2/F_{k+2} has no center. It is easy to see that C_2 is normal in X . From (c) and (d) we obtain

$$(e) \quad X/F_{k+2} = (C/F_{k+2}) \times (C_2/F_{k+2}).$$

If F_{k+1}/F_{k+2} is non-abelian, X/F_{k+2} splits into a direct product group as in (e). After a finite number of steps we have a direct product splitting $X = C \times X_1$ of a completely σ -invariant normal subgroup C and an S.B. non-abelian normal subgroup X_1 under σ . Apply the above argument for (C, σ) and repeat this process inductively. Then we get the conclusion of the lemma (since the entropy of (X, σ) is finite).

LEMMA 8. *If X has no center, then X/H has no center for every completely σ -invariant normal subgroup H .*

Proof. Since X is zero-dimensional, X contains a sequence $A_1 \supset A_2 \supset \dots$ of open normal subgroups with $\bigcap A_n = \{e\}$. Put

$$H_n = \bigcap_{j=-\infty}^{\infty} \sigma^j A_n \quad \text{for } n \geq 1.$$

Obviously, the sequence $\{H_n\}$ decreases and $\bigcap H_n = \{e\}$. Since $(X/H_n, \sigma)$ is ergodic, for every $n \geq 1$ there are in X normal subgroups U_n and V_n such that U_n/H_n is the center of X/H_n , and V_n/H_n splits into a finite direct product of S.B. non-abelian subgroups under σ , and such that

$$X/H_n = (U_n/H_n) \times (V_n/H_n)$$

(by Lemma 7). Put

$$C_n = \overline{\prod_{i=n}^{\infty} U_i} \quad \text{and} \quad B_n = \overline{\prod_{i=n}^{\infty} V_i} \quad \text{for } n \geq 1$$

(\bar{E} means the closure of E). Obviously, $X = C_n B_n$ ($n \geq 1$), and hence

$$X = X_0 B, \quad \text{where } X_0 = \bigcap_{n=1}^{\infty} C_n \text{ and } B = \bigcap_{n=1}^{\infty} B_n.$$

It is easy to see that X_0 is abelian. Indeed, for every $n \geq 1$,

$$X_0 H_n / H_n \subset C_n / H_n = \overline{\prod_{i=0}^{\infty} (U_{n+i} H_n / H_n)}.$$

Since $U_{n+i} H_n / H_n$ is a factor of U_{n+i} / H_{n+i} that is the center of X / H_{n+i} ($i \geq 0$), it is central in X / H_n . Therefore, $X_0 H_n / H_n \cong X_0 / (X_0 \cap H_n)$ is abelian, and so is X_0 . As above we get

$$B H_n / H_n \subset B_n / H_n = \overline{\prod_{i=0}^{\infty} (V_{n+i} H_n / H_n)} \quad (n \geq 1).$$

Fix $n \geq 1$. For every $i \geq 0$, V_{n+i} / H_{n+i} is expressed as a finite direct product of S.B. non-abelian normal subgroups under σ . Since $V_{n+i} H_n / H_n$ is a factor of V_{n+i} / H_{n+i} for $i \geq 0$, $V_{n+i} H_n / H_n$ is a finite direct product of S.B. non-abelian normal subgroups under σ (by (I)). Since

$$V_{n+i} H_n / H_n \subset X / H_n = (U_n / H_n) \times (V_n / H_n) \quad \text{for } i \geq 0,$$

we have $V_{n+i} H_n / H_n \subset V_n / H_n$ ($i \geq 0$), so that $B_n / H_n = V_n / H_n$. Remark that $B H_n / H_n$ is expressed as a finite direct product of S.B. non-abelian normal subgroups under σ (i.e., $B H_n / H_n$ has no center). Since n is arbitrary, B has no center. Hence $X_0 \cap B = \{e\}$, i.e., $X = X_0 \times B$. By assumption of the lemma, $X = B$ (i.e., $X_0 = \{e\}$). Let H be the subgroup in the statement of the lemma. For every $n \geq 1$, $X / H H_n = B / H H_n$ is a factor of B / H_n . Therefore, by (I), $X / H H_n$ has no center, and so does X / H since $H_n \downarrow \{e\}$.

LEMMA 9. Let X_0 be the center of X . Then there exists a completely σ -invariant normal subgroup B such that X splits into the direct product $X = X_0 \times B$.

This follows from the proof of Lemma 8.

LEMMA 10. Assume that X has no center. Then there exists a sequence $\{X_j: j \geq 1\}$ of normal subgroups such that, for every $j \geq 1$, X_j is an S.B. non-abelian subgroup under σ and such that

$$X = \bigtimes_{j \geq 1} X_j.$$

Proof. Let $\{F_n: n \geq 0\}$ be a sequence of normal subgroups chosen as in (VI). It follows from Lemma 8 that X / F_2 has no center. By Lemma 5, X contains a normal subgroup D_1 such that D_1 / F_2 is an S.B. non-abelian group under σ and $X / F_2 = (D_1 / F_2) \times (F_1 / F_2)$. Hence $X = D_1 F_1$. Since D_1 / F_2 and

F_1/F_2 are non-abelian, in the same way we have direct product splittings

$$D_1/F_3 = (D_2/F_3) \times (F_2/F_3) \quad \text{and} \quad F_2/F_3 = (D'_2/F_3) \times (F_2/F_3),$$

where D_2 and D'_2 are normal subgroups of X . Obviously,

$$X/F_3 = (D_2/F_3) \times (D'_2/F_3) \times (F_2/F_3) \quad \text{and} \quad D_1 = D_2 F_2.$$

Notice that each term of the above right-hand side is non-abelian and that $X = D_2 F_1$. Applying the above argument for D_2/F_4 , we see that D_2/F_4 splits into a direct product $D_2/F_4 = (D_3/F_4) \times (F_3/F_4)$ of S.B. non-abelian normal subgroups D_3/F_4 and F_3/F_4 . It is easily checked that D_3 is normal in X . Since $D_2 = D_3 F_3$, we have $X = D_3 F_1$.

Repeating this process, we see that there is a sequence $D_1 \supset D_2 \supset \dots$ of normal subgroups such that, for every $n \geq 1$, D_n/F_{n+1} is an S.B. non-abelian group under σ and $X = D_n F_1$. Put $X_1 = \bigcap D_n$. Then $X = X_1 F_1$. Since $X = X_1 F_1 \not\cong F_1$, we obtain

$$X_1 F_n \not\cong (X_1 \cap F_1) F_n \quad \text{for all } n \geq 1,$$

so that

$$X_1 F_{n+1}/F_{n+1} \not\cong (X_1 \cap F_1) F_{n+1}/F_{n+1}.$$

Since $X_1 F_{n+1}/F_{n+1} \subset D_n/F_{n+1}$, $X_1 F_{n+1}/F_{n+1}$ is an S.B. non-abelian group under σ . Consequently, $(X_1 \cap F_1) F_{n+1}/F_{n+1}$ is trivial by (I), i.e., $(X_1 \cap F_1) F_{n+1} = F_{n+1}$ ($n \geq 1$). Since $F_n \downarrow \{e\}$, we have $X_1 \cap F_1 = \{e\}$, and so $X = X_1 \times F_1$. Since F_1 has no center and (F_1, σ) is ergodic, we can apply the above argument for F_1 and repeat this process inductively. Then we can easily find a sequence $\{X_j; j \geq 1\}$ of S.B. non-abelian normal subgroups under σ such that $X = X_1 \times \dots \times X_n \times F_n$ for every $n \geq 1$. Now, put

$$X^{(1)} = F_2 \quad \text{and} \quad X^{(j)} = \overline{\prod_{i \neq j} X_i} \quad \text{for } j \geq 2.$$

Then we have $X^{(j)} = X_1 \times \dots \times X_{j-1} \times X_{j+1} \times F_{j+2}$ ($j \geq 2$). Hence

$$\bigcap_{j=1}^n X^{(j)} = F_{n+1} \quad \text{for all } n > 0,$$

and so

$$\bigcap_{j=1}^{\infty} X^{(j)} = \{e\}$$

since $F_n \downarrow \{e\}$. It is clear that $\overline{\prod X_j} = X$. Therefore, X is expressed as a direct product

$$X = \bigtimes_{j \geq 1} X_j$$

of the normal subgroups X_j .

Proof of the Theorem. As before, let X_0 be the center of X . By Lemma 9 there is a completely σ -invariant normal subgroup B such that $X = X_0 \times B$. Obviously, B contains no center and both (X_0, σ) and (B, σ) are ergodic under the normalized Haar measure. By Lemma 10 we get the conclusion of the Theorem.

2. Applications of the Theorem. In this section we shall prove the statements mentioned in Section 0. As before, X will be a zero-dimensional compact metric group and σ will be an automorphism of X . (X, σ) is said to satisfy *specification* if for every $\varepsilon > 0$ there is an $M(\varepsilon) > 0$ such that, for every $k \geq 1$ and k points $x_1, \dots, x_k \in X$ and for every set of integers $a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M(\varepsilon)$ ($2 \leq i \leq k$) and for every integer p with $p \geq b_k - a_1 + M(\varepsilon)$, there is a point $x \in X$ such that $d(\sigma^n x, \sigma^n x_i) < \varepsilon$ for $a_1 \leq n \leq b_i$ ($1 \leq i \leq k$) and $\sigma^p x = x$.

APPLICATION 1. Let X_0 be the center of X . Then (X, σ) satisfies *specification* iff (X, σ) is ergodic and (X_0, σ) obeys *specification*.

This follows from the Theorem proved in Section 1. The necessary and sufficient condition for (X_0, σ) to satisfy *specification* is given in Theorem 2 of [2] as follows:

(A) There are a zero-dimensional compact metric abelian group \bar{X}_0 and an automorphism $\bar{\sigma}$ of \bar{X}_0 such that \bar{X}_0 is a B. group under $\bar{\sigma}$ and (X_0, σ) is an algebraic factor of $(\bar{X}_0, \bar{\sigma})$.

Equivalently,

(B) X_0 contains a sequence $X_0 = F_0 \supset F_1 \supset \dots$ of completely σ -invariant normal subgroups such that $\bigcap F_n = \{e\}$ and, for every $n \geq 0$, F_n/F_{n+1} is an S.B. group under σ (compare with (VI)).

It is proved in [1], Theorem 1, that there is an ergodic zero-dimensional abelian automorphism without the *specification* property.

A sequence $\{x_i: a < i < b\}$ ($a = -\infty$ or $b = \infty$ is permitted) of points in X is a δ -pseudo-orbit if $d(\sigma x_i, x_{i+1}) < \delta$ ($a < i < b-1$). A point $x \in X$ ε -traces $\{x_i: a < i < b\}$ if $d(\sigma^i x, x_i) < \varepsilon$ ($a < i < b$). We say that (X, σ) has the *pseudo-orbit tracing property* (in abbreviation, P.O.T.P.) if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that every δ -pseudo-orbit $\{x_i: a < i < b\}$ in X is ε -traced by a point $x \in X$.

APPLICATION 2. Every zero-dimensional automorphism has P.O.T.P.

The proof is as follows. Since X is zero-dimensional, X contains a σ -invariant normal subgroup H such that (H, σ) is ergodic and $h(\sigma_{X/H}) = 0$ (see Theorem 11.1 in [10]).

First we shall prove that (H, σ) has P.O.T.P. By the Theorem proved in Section 1, H splits into a direct product

$$H = \bigtimes_{j \geq 0} H_j$$

of completely σ -invariant normal subgroups H_j , where H_0 is the center of H and, for every $j \geq 1$, H_j is an S.B. non-abelian group under σ . Obviously, (H_j, σ) ($j \geq 1$) has P.O.T.P., and so has $(\bigtimes_{j \geq 1} H_j, \sigma)$. From (VI) (ii) and the following Lemmas 11 and 12 we infer that (H_0, σ) has P.O.T.P. Hence (H, σ) has P.O.T.P.

LEMMA 11. *Let σ be an automorphism of a compact metric group Y and let K be a completely σ -invariant normal subgroup of Y . If both $(Y/K, \sigma)$ and (K, σ) have P.O.T.P., then so has (Y, σ) .*

Proof. By assumption, for every $\varepsilon > 0$ there is a $\delta > 0$ with $\delta < \varepsilon$ such that, for every δ -pseudo-orbit in K , a point in K $(\varepsilon/2)$ -traces the orbit. Choose η with $0 < \eta < \delta/3$ such that the following conditions hold:

(1) $d(\sigma x, \sigma y) < \eta$ if $d(x, y) < \eta$;

(2) for an arbitrary $(\delta/3)$ -pseudo-orbit $\{x_i: a < i < b\}$ of Y , Y/K contains a point $xK \in Y/K$ with $d(\sigma^i(xK), x_iK) < \delta/3$ ($a < i < b$) (the existence of such a point xK is ensured by assumption).

By (2), for $a < i < b$ there is $k_i \in K$ such that $d(\sigma^i x k_i, x_i) < \eta$. By (1), $d(\sigma^{i+1} x \sigma k_i, \sigma x_i) < \delta/3$. For $a < i < b-1$ we obtain

$$\begin{aligned} d(\sigma k_i, k_{i+1}) &= d(\sigma^{i+1} x \sigma k_i, \sigma^{i+1} x k_{i+1}) \\ &\leq d(\sigma^{i+1} x \sigma k_i, \sigma x_i) + d(\sigma x_i, x_{i+1}) + d(x_{i+1}, \sigma^{i+1} x k_{i+1}) < \delta, \end{aligned}$$

which implies that there is a point $k \in K$ $(\varepsilon/2)$ -tracing the orbit $\{k_i: a < i < b\}$. Since

$$d(\sigma^i(xk), x_i) \leq d(\sigma^i(xk), \sigma^i x k_i) + d(\sigma^i x k_i, x_i) < \varepsilon,$$

the point xk ε -traces the orbit $\{x_i: a < i < b\}$ in Y and the proof is completed.

LEMMA 12. *Let Y and σ be as in Lemma 11. If Y contains a sequence $Y \supset K_1 \supset \dots$ of completely σ -invariant normal subgroups such that $\bigcap K_n = \{e\}$ and, for every $n \geq 1$, $(Y/K_n, \sigma)$ has P.O.T.P., then (Y, σ) has also P.O.T.P.*

Proof. Let $\varepsilon > 0$ be given. Choose k so large that $\text{diam}(K_k) < \varepsilon/2$. Since $(Y/K_k, \sigma)$ has P.O.T.P., for every $(\varepsilon/2)$ -pseudo-orbit $\{x_i: a < i < b\}$ in Y there is a point $xK_k \in Y/K_k$ with $d(\sigma^i(xK_k), x_iK_k) < \varepsilon/2$ ($a < i < b$). Since $\text{diam}(K_k) < \varepsilon/2$, we have $d(\sigma^i x, x_i) < \varepsilon$ for $a < i < b$.

It is clear by the following Lemmas 13 and 14 that $(X/H, \sigma)$ has P.O.T.P.

LEMMA 13. *If $h(\sigma) = 0$, then X contains a sequence $X = K_0 \supset K_1 \supset \dots$ of completely σ -invariant normal subgroups such that $\bigcap K_n = \{e\}$ and, for every $n \geq 0$, K_n/K_{n+1} is finite.*

Proof. This is obtained using (III) inductively. Since X is zero-dimensional, X contains a sequence $X = A_0 \supset A_1 \supset \dots$ of open normal subgroups

such that $\bigcap A_n = \{e\}$. Since X/A_1 is finite, there is a normal subgroup $F_1 \supset A_1$ such that X/F_1 is simple. Put

$$K_1 = \bigcap_{-\infty}^{\infty} \sigma^i F_1$$

and apply (III). Then X/K_1 is finite since $h(\sigma) = 0$. Since $K_1 \cap A_1$ is open in K_1 , we can find a normal subgroup F_2 of X such that $F_2 \supset K_1 \cap A_1$ and K_1/F_2 is simple. Put

$$K_2 = \bigcap_{-\infty}^{\infty} \sigma^i F_2$$

and apply (III). Then K_1/K_2 is finite since $h(\sigma_{K_1/K_2}) = 0$. Repeat the same process to get K_3, K_4, \dots . After a finite number (r , say) of steps, we have $K_r \subset A_1$ since X/A_1 is finite. Then, replace A_1 with A_2 and produce K_{r+1}, K_{r+2}, \dots . In this way our requirement is obtained.

LEMMA 14. *If $h(\sigma) = 0$, then (X, σ) has P.O.T.P.*

Proof. Since $h(\sigma) = 0$, there is a sequence $\{K_n\}$ of normal subgroups as in Lemma 13. For every $\varepsilon > 0$ there are K_n and δ with $0 < \delta < \varepsilon$ such that

$$\{x \in X: d(x, e) < \delta\} \subset K_n \subset \{x \in X: d(x, e) < \varepsilon\}.$$

Let $\{x_i: a < i < b\}$ be an arbitrary δ -pseudo-orbit in X , i.e., $d(\sigma x_i, x_{i+1}) < \delta$, $a < i < b-1$ (without loss of generality we may assume $a+1 \leq 0$). Then $\sigma x_i x_{i+1}^{-1} \in K_n$ ($a < i < b-1$). Hence $\sigma^i x_0 x_i^{-1} \in K_n$ since $\sigma K_n = K_n$, i.e., $d(\sigma^i x, x_i) < \varepsilon$ ($a < i < b$). This shows that (X, σ) has P.O.T.P.

We have proved that both (H, σ) and $(X/H, \sigma)$ have P.O.T.P. Using Lemma 12 again, we see that (X, σ) has P.O.T.P. The proof of Application 2 is completed.

REFERENCES

- [1] N. Aoki, *Zero-dimensional automorphisms having a dense orbit*, Journal of the Mathematical Society of Japan 33 (1981), p. 693-700.
- [2] — *Zero-dimensional automorphisms with specification*, Monatshefte für Mathematik 95 (1983), p. 1-17.
- [3] — and M. Dateyama, *Relationship between algebraic numbers and expansiveness of automorphisms on compact abelian groups*, Fundamenta Mathematicae 117 (1983), p. 21-35.
- [4] R. Bowen, *Some systems with unique equilibrium state*, Mathematical Systems Theory 8 (1974), p. 193-202.
- [5] M. Denker, C. Grillenberger and K. Sigmund, *Ergodic theory on compact spaces*, Lecture Notes in Mathematics 527, Berlin - Heidelberg - New York 1976.
- [6] P. R. Halmos, *Lectures on ergodic theory*, The Mathematical Society of Japan, 1956.
- [7] I. Kaplansky, *Groups with representations of bounded degree*, Canadian Journal of Mathematics 1 (1949), p. 105-112.

- [8] A. Morimoto, *Stochastic stability of group automorphisms* (preprint).
[9] L. Pontrjagin, *Topological groups*, Godon and Breach Science Publ. Inc., 1966.
[10] S. A. Yuzvinskiĭ, *Metric properties of endomorphisms of compact groups*, American Mathematical Society Translations 66 (1968), p. 63-98.

DEPARTMENT OF MATHEMATICS
TOKYO METROPOLITAN UNIVERSITY
TOKYO

*Reçu par la Rédaction le 15. 1. 1981;
en version modifiée le 10. 12. 1981*
