COLLOQUIUM MATHEMATICUM

VOL. LVI 1988 FASC. 2

STRICT-2-ASSOCIATEDNESS FOR THIN SETS

BY

KATHRYN E. HARE (EDMONTON)

1. Introduction. Let G denote a compact abelian group and $\hat{G} = \Gamma$ its necessarily discrete, abelian, dual group. When E is a subset of Γ, an integrable function f on G will be called an *E-function* provided its Fourier transform $\hat{f}$ vanishes on the complement of E. Similarly, an E-function f will be called an *E-polynomial* if the support of its Fourier transform, denoted by $\operatorname{supp}\hat{f}$, is finite.

A subset E of Γ is said to be a *$\Lambda(p)$ set*, $p > 0$, if for some $0 < r < p$ there is a constant $c(p, r, E)$ such that

$$\|f\|_p \leqslant c(p, r, E)\|f\|_r$$

for all E-polynomials f.

For standard results on $\Lambda(p)$ sets see [11] and [8].

DEFINITION ([8], 9.3). $E \subset \Gamma$ is said to be *strictly-2-associated* with a measurable subset S of G if there is a constant $c = c(S, E) > 0$ such that

$$\|1_S f\|_2^2 \geqslant c\|f\|_2^2$$

for all E-polynomials f.

Any such c will be called a *constant of strict-2-associatedness for E and S.*

In [9] Miheev proves that any $\Lambda(p)$ set in $\boldsymbol{Z}$, with $p > 2$, is strictly-2-associated with all subsets of the circle of positive measure. This improves upon Zygmund's result ([14], V.6) for lacunary series in $\boldsymbol{Z}$.

We extend Miheev's result to a formally larger class of subsets of $\boldsymbol{Z}$ and obtain the same conclusion for certain $\Lambda(p)$ sets in arbitrary discrete abelian groups. In the process we develop a new arithmetic property of $\Lambda(p)$ sets.

Following Blei [1], call E a *uniformizable $\Lambda(2)$ set* if for each $\varepsilon > 0$ there exists a constant $c(E, \varepsilon)$ such that for all $v \in l^2(E)$ there is a continuous function g satisfying

(i) $\|g\|_\infty \leqslant c(E, \varepsilon)\|v\|_2$;

(ii) $\hat{g}(\chi) = v(\chi)$ for all $\chi \in E$;

(iii) $\|\hat{g}|_{\Gamma\setminus E}\|_2 \leqslant \varepsilon\|v\|_2$.

The smallest such constant $c(E, \varepsilon)$ will be referred to as the *unformizable $\Lambda(2)$ constant for E and ε.*

All uniformizable $\Lambda(2)$ sets are $\Lambda(2)$ sets, and all $\Lambda(p)$ sets with $p > 2$ are uniformizable $\Lambda(2)$ sets [1].

In Section 2 we will prove

THEOREM A. *If E is a uniformizable $\Lambda(2)$ subset of Z, then E is strictly-2-associated with each subset S of the circle group T having positive Lebesgue measure.*

Although we follow the basic outline used in [9], we present new proofs of the intermediate steps. Our proofs of these steps are simpler and they can be generalized to groups other than Z and T.

The arithmetic structure of $\Lambda(2)$ sets plays an important role in the proof.

As an application of Theorem A we mention two corollaries. The proof of the first is immediate from Theorem A.

COROLLARY 1. *If E is a uniformizable $\Lambda(2)$ subset of Z, then the only E-function in $L^2(T)$ which may vanish on any subset of T of positive measure is the identically zero function.*

COROLLARY 2. *Suppose E is a uniformizable $\Lambda(2)$ subset of Z and $\{a_n\}_{n=1}^{\infty}$ is a sequence of complex numbers such that*

$$f = \sum_{n \in E} a_n e^{int}$$

converges pointwise on some subset of T of positive measure. Then $f \in L^2(T)$, i.e.,

$$\sum_{n \in E} |a_n|^2 < \infty .$$

Proof. Let S_N be the N-th partial sum of f. By an application of Egoroff's theorem we may conclude that $\{S_N\}_{N=1}^{\infty}$ converges uniformly on some set S of positive measure. Thus

$$\sup_N \|1_S S_N\|_2 < \infty .$$

Since E and S are strictly-2-associated and S_N is an E-polynomial, $\sup_N \|S_N\|_2 < \infty$, and thus $f \in L^2(T)$.

If G is not a connected group, then one can construct polynomials which vanish on certain subsets of G having positive measure; thus in the general setting a further hypothesis is required to obtain the conclusion of Theorem A

This hypothesis is related to the following definition given in [8], 8.2:

$E \subset \Gamma$ is said to be *X_0-subtransversal* for a subset X_0 of Γ if, whenever χ and ψ are distinct elements of E, then $\chi\psi^{-1} \notin X_0$.

With this terminology we can state the general theorem, whose proof is the content of Section 3.

THEOREM B. *Let G be a compact abelian group and let E, a subset of Γ, be a uniformizable $\Lambda(2)$ set which is X_0-subtransversal for all finite subgroups X_0 of Γ. Then E is strictly-2-associated with all subsets of G having positive Haar measure.*

In proving this we are led to consider a new arithmetic property of $\Lambda(2)$ sets in discrete abelian groups other than $\boldsymbol{Z}$.

Of course, similar corollaries to those given above follow from Theorem B for uniformizable $\Lambda(2)$ sets which are X_0-subtransversal for all finite subgroups of Γ.

As $\boldsymbol{Z}$ contains no finite subgroups other than the trivial one, the subtransversality condition in Theorem B is automatically satisfied. Thus Theorem B reduces to Theorem A when $\Gamma = \boldsymbol{Z}$. Indeed, whenever G is a connected group, there are no non-trivial finite subgroups in Γ.

If there are distinct characters χ and ψ in E with $\chi\psi^{-1}$ an element of a finite subgroup X_0, then the E-polynomial $\chi - \psi$ vanishes on the annihilator of X_0, an open set since X_0 is finite, and hence a subset of G with positive measure. Obviously, E fails to be strictly-2-associated with this set, so the subtransversality condition is necessary.

A set E is said to *tend to infinity* if for each finite set $\Delta \subset \Gamma$ there is a finite set $F \subset E$ such that if χ and ψ are distinct elements of $E \setminus F$, then $\chi\psi^{-1} \notin \Delta$. For subsets of $\boldsymbol{Z}$ this means that, for each positive integer N, only finitely many points of E differ in absolute value by at most N.

It is known ([8], Ch. 9) that the conclusion of Theorem B holds for $\Lambda(4)$ sets tending to infinity and satisfying the subtransversality condition. As there are many known examples of $\Lambda(4)$ sets not tending to infinity, our work improves upon this result. The presentation given in [8] is a synthesis of the work of Bonami [2] and López [7].

2. Strict-2-associatedness in $\boldsymbol{Z}$. Throughout this section, E will denote a uniformizable $\Lambda(2)$ set in $\boldsymbol{Z}$ and S will denote a subset of the circle T with positive Lebesgue measure.

Roughly, the idea of the proof of Theorem A is to show first that whenever subsets of a uniformizable $\Lambda(2)$ set are all strictly-2-associated with S with a common constant of strict-2-associatedness, and the subsets have sufficiently large gaps between them, then their union is again strictly-2-associated with S; and then to show that any uniformizable $\Lambda(2)$ set is the union of such a collection of subsets. This outline, which will also be followed in the proof of Theorem B, is the general scheme used in [9].

The first step is precisely the statement of Lemma 2.1.

For sets F_1 and F_2 in Z let $d(F_1, F_2) = \min\{|n_1 - n_2|: n_i \in F_i\}$.

LEMMA 2.1. *Given $c > 0$ there is an integer $N = N(E, S, c)$ such that if the subsets $\{E_i\}_{i\in I}$ of E are such that for all $i \in I$*

$$\|1_S f\|_2^2 \geqslant c\|f\|_2^2$$

whenever f is an E_i-polynomial, and

$$d(E_i, E_j) > N$$

for all $i \neq j$, then $\bigcup_{i\in I} E_i$ is also strictly-2-associated with S.

Before proceeding to the proof we make two remarks.

Remarks. 1. This lemma replaces Theorem 5 of [9] whose proof does not seem to adapt easily to uniformizable $\Lambda(2)$ sets or to the general setting.

2. The proof of Lemma 2.1 makes use of a property of uniformizable $\Lambda(2)$ sets discussed in [4], namely that there is a Young function $\Phi(x) = \varphi(x^2)$ with φ a "strongly convex" function, and a constant K such that

$$\|f\|_\Phi^2 \leqslant K\|f\|_2^2$$

whenever f is an E-polynomial. Without loss of generality, assume φ is itself a Young function, and thus has a conjugate ψ. Set $\beta(x) = \psi(x^2)$. It follows that for all measurable functions f and g

$$\|fg\|_2 \leqslant \sqrt{2}\|f\|_\Phi\|g\|_\beta.$$

We will be using this notation in the proof below.

Proof of Lemma 2.1. Let $\varepsilon = c/24K$ and choose a trigonometric polynomial P satisfying $\|P - 1_S\|_\beta^2 < \varepsilon$.

Let $N = N(E, S, c) = \max\{|n-m|: n, m \in \operatorname{supp}\hat{P}\}$.

Let $\{E_i\}_{i\in I}$ be subsets of E which satisfy the hypothesis of the lemma with this choice of N. We estimate $\|1_S f\|_2^2$ for any $\bigcup_{i\in I} E_i$-polynomial. Write f as $\sum_{i\in I} f_i$ with f_i an E_i-polynomial for each $i \in I$. Observe that if, for some $m \in Z$,

$$\widehat{f_i P}(m) \neq 0 \quad \text{and} \quad \widehat{f_j P}(m) \neq 0,$$

then

$$d(\operatorname{supp}\hat{f}_i, \operatorname{supp}\hat{f}_j) \leqslant N.$$

But, clearly,

$$d(E_i, E_j) \leqslant d(\operatorname{supp}\hat{f}_i, \operatorname{supp}\hat{f}_j),$$

so we must have $i=j$. Thus

$$\|fP\|_2^2=\sum_{i\in I}\|f_iP\|_2^2.$$

Since the sets $\{E_i\}_{i\in I}$ are disjoint and strictly-2-associated with S, with constant of strict-2-associatedness c, we have

$$c\|f\|_2^2=\sum_{i\in I}c\|f_i\|_2^2\leqslant\sum_{i\in I}\|f_i 1_S\|_2^2.$$

Hence

$$\begin{aligned}c\|f\|_2^2&\leqslant 2\sum_{i\in I}(\|(1_S-P)f_i\|_2^2+\|Pf_i\|_2^2)\\&\leqslant 2\sum_{i\in I}2\|1_S-P\|_\beta^2\|f_i\|_\Phi^2+2\|Pf\|_2^2\leqslant 2\sum_{i\in I}2\varepsilon\|f_i\|_\Phi^2+2\|Pf\|_2^2,\end{aligned}$$

with the last inequality following from the choice of P.

Now the functions f_i are E-polynomials, so after using the property of uniformizable $\Lambda(2)$ sets described in Remark 2, and simplifying, we see that

$$c\|f\|_2^2\leqslant 4\varepsilon K\|f\|_2^2+2\|Pf\|_2^2.$$

Similarly,

$$\|Pf\|_2^2\leqslant 2\|(1_S-P)f\|_2^2+2\|1_Sf\|_2^2\leqslant 4\varepsilon K\|f\|_2^2+2\|1_Sf\|_2^2.$$

Thus

$$c\|f\|_2^2\leqslant 12\varepsilon K\|f\|_2^2+4\|1_Sf\|_2^2.$$

Our choice of ε implies

$$\|1_Sf\|_2^2\geqslant\frac{c}{8}\|f\|_2^2$$

for all $\bigcup_{i\in I}E_i$-polynomials f.

Establishing the ideas of the second step will require several lemmas. We first assert that if E is strictly-2-associated with S, then so are the sets $E\cup\{n\}$, $n\in \mathbf{Z}$, with constant of strict-2-associatedness independent of n. Without the last requirement this can essentially be found in [8], 9.8, and is due to Bonami. Our result will be used to show that the subsets referred to in the outline of the proof do indeed have a common constant of strict-2-associatedness for the given set S.

LEMMA 2.2. *Suppose that E is strictly-2-associated with S. Then $E\cup\{n\}$ is also strictly-2-associated with S, with constant of strict-2-associatedness independent of n. Indeed, if c is a constant of strict-2-associatedness for E and*

S, then $E \cup \{n\}$ *and* S *have as a constant of strict-2-associatedness*

$$\left(\frac{c\varepsilon}{2m(V)}+m(S)+2\right)^{-1}\frac{c\varepsilon m(S)}{4m(V)},$$

where V *is an open subset of the circle which depends only on* E, c *and* S, *and* ε *is a constant determined by* V.

The proof is postponed until the end of this section.

Remark. In [9], Theorem 6, Miheev proves a similar result, without obtaining a specific constant of strict-2-associatedness. Our proof, though shorter, is still lengthy. For this reason we have postponed its presentation.

A consequence of Lemmas 2.1 and 2.2 is

COROLLARY 2.3. *If* E, *in addition to being uniformizable* $\Lambda(2)$, *tends to infinity, then* E *is strictly-2-associated with all subsets of the circle with positive measure.*

Proof. For a given set S choose the integer $N = N(E, S, m(S))$ from Lemma 2.1. Since E is assumed to tend to infinity, except for a finite set of integers, say F, distinct members of E differ by at least N.

Now apply Lemma 2.1 taking as the sets E_i the singleton sets whose union is $E \setminus F$. Each of these sets is strictly-2-associated with S, with constant of strict-2-associatedness equal to $m(S)$, and hence the choice of F ensures that $E \setminus F$ is strictly-2-associated with S. Applying Lemma 2.2 finitely many times, we conclude that E is strictly-2-associated with S.

We now examine the arithmetic structure of $\Lambda(p)$ sets and observe first that uniformizable $\Lambda(2)$ sets have "uniformly large gaps".

LEMMA 2.4. *If* $c(E, \varepsilon)$ *is the uniformizable* $\Lambda(2)$ *constant of* E *and* ε, *then*

$$|\{a+b, a+2b, \ldots, a+Nb\} \cap E| \leqslant 8\left(c(E, \varepsilon)^2+\varepsilon^2 N\right)$$

for any $a, b \in \mathbf{Z}$ *and positive integer* N ($|\cdot|$ *denotes cardinality*).

Proof. This is a straightforward modification of [11], 1.3.5.

COROLLARY 2.5. *Given any positive integer* N *there is an integer* $M = M(E, N)$ *such that any interval of length* M *contains a subinterval of length* N *free of points of* E.

Proof. If $c(E, \varepsilon)$ is the uniformizable $\Lambda(2)$ constant for E and ε, then we may take

$$M(E, N) = c(E, \varepsilon)^2\, 16N$$

with $\varepsilon = (16N)^{-1/2}$. By the lemma, the interval $(a, a+M]$ must contain a subinterval of length

$$\frac{M}{8\left(c(E, \varepsilon)^2+\varepsilon^2 M\right)} = N$$

free of points of E.

The uniform gap property enables us to show that if a uniformizable $\Lambda(2)$ set with large enough gaps between members is adjoined to a uniformizable $\Lambda(2)$ set which is strictly-2-associated with some subset S of the circle with positive measure, then the larger set is also strictly-2-associated with S.

LEMMA 2.6. *If $E' \subset E$ is strictly-2-associated with S, then there is an integer $M = M(E, E', S)$ such that if $E'' = \{n_k\} \subset E$ satisfies $|n_k - n_j| \geqslant M$ for all $j \neq k$, then $E' \cup E''$ is also strictly-2-associated with S.*

Proof. By Lemma 2.2 the sets $E' \cup \{n\}$ are strictly-2-associated with S, with some constant of strict-2-associatedness $c > 0$ independent of n. Choose the integer $N = N(E, S, c)$ as in Lemma 2.1 so that if the sets $E_i \subset E$, $i \in I$, are strictly-2-associated with S with constant c and $d(E_i, E_j) \geqslant N$ for $i \neq j$, then $\bigcup_{i \in I} E_i$ is also strictly-2-associated with S. Finally, choose $M = M(E, N)$ as in the previous corollary.

Since $|n_k - n_j| \geqslant M$ for all $j \neq k$, there is an interval of length at least N free of points of E between each pair $\{n_j, n_k\}$. Thus $E' \cup E''$ naturally partitions into sets $\{E_i\}$, with $d(E_i, E_j) \geqslant N$ for $i \neq j$, and each set E_i containing at most one integer from E''. By Lemma 2.2 these subsets of E are each strictly-2-associated with S with constant c; thus by Lemma 2.1 their union $E' \cup E''$ is also strictly-2-associated with S.

A further arithmetic property is necessary to complete the proof of Theorem A.

DEFINITION. A subset P of $\boldsymbol{Z}$ is called a *parallelepiped of dimension N* if P is the sum of N two-element sets and P has 2^N elements.

In [9] and [10] Miheev proves that $\Lambda(p)$ sets in $\boldsymbol{Z}$ for $p > 0$ do not contain parallelepipeds of arbitrarily large dimension. (He refers to these as reflexive segments.) Independently, Fournier and Pigno in [5] obtain the same conclusion for $p \geqslant 1$. For the case of uniformizable $\Lambda(2)$ sets their proof is straightforward.

The notion of parallelepiped can be generalized in the obvious way to discrete abelian groups. With the appropriate change in wording Fournier and Pigno's proof gives the same conclusion for $\Lambda(1)$ sets in the general setting. Miheev's proofs can only be adapted to the connected case. An easy proof, using the notion of probabilistic independence, shows that in discrete abelian groups, all of whose elements have order 2, no $\Lambda(p)$ sets, for any $p > 0$, can contain parallelepipeds of arbitrarily large dimension. By arguing that it is always possible to essentially reduce to one of these two cases, we have been able to obtain the same result for all discrete abelian groups [6].

Following [9] define sets of class M_n inductively as follows:

M_0 is the class of subsets of $\boldsymbol{Z}$ which tend to infinity.

M_n is the class of those subsets of $\boldsymbol{Z}$ which for each integer N are the union of two sets, one consisting of integers which are at least N units apart in absolute value, and the other a finite union of sets in class M_{n-1}.

In [9] Miheev shows that each class M_n contains a $\Lambda(4)$ set which is not a finite union of sets in class M_{n-1}. He also shows that any sequence of integers which does not contain any parallelepipeds of dimension n, $n \geqslant 2$, belongs to class M_{n-2}. In particular, any $\Lambda(p)$ set belongs to some class M_n.

Thus, any uniformizable $\Lambda(2)$ set is the union of a set with large gaps between members and a union of a finite number of sets in some class M_{n-1}. An induction argument, together with Lemma 2.6, will now complete the proof of Theorem A.

Proof of Theorem A. It suffices to show that any subset of E belonging to M_n is strictly-2-associated with S for each $n \geqslant 0$. Actually, more than this will be proved. We proceed inductively on n.

Let E' be any subset of E which is strictly-2-associated with S, and suppose $E'' \subset E$ belongs to class M_0, i.e., E'' tends to infinity. Let $M = M(E, E', S)$ as in Lemma 2.6. Since E'' tends to infinity, only finitely many points of E'', say those in the set F, differ in absolute value by less than M. Hence $E' \cup (E'' \setminus F)$ is strictly-2-associated with S. By Lemma 2.2, $E' \cup E''$ is strictly-2-associated with S.

Now suppose we have established that whenever $E' \subset E$ is strictly-2-associated with S and $E''_n \subset E$ belongs to class M_n, then $E' \cup E''_n$ is also strictly-2-associated with S.

Let $E' \subset E$ be any set strictly-2-associated with S and assume that $E''_{n+1} \subset E$ belongs to class M_{n+1}. Again choose $M = M(E, E', S)$ as in Lemma 2.6. Since E''_{n+1} belongs to class M_{n+1}, it is the union of two sets E_1 and E_2, where any two distinct integers in E_1 differ by at least M, and E_2 is a finite union of sets in class M_n, say

$$E_2 = \bigcup_{i=1}^{N} F_i, \quad F_i \in M_n.$$

Because the gaps in E_1 are sufficiently large, Lemma 2.6 implies that $E' \cup E_1$ is strictly-2-associated with S, and by the induction hypothesis so is $(E' \cup E_1) \cup F_1$. By applying the induction hypothesis $N-1$ more times, we see that

$$E' \cup E_1 \cup \Big(\bigcup_{i=1}^{N} F_i \Big) = E' \cup E''_{n+1}$$

is strictly-2-associated with S. This completes the induction step, and hence the proof.

The converse question remains open: If E is strictly-2-associated with all subsets of T of positive measure, then is E a uniformizable $\Lambda(2)$ set? **(P 1361)** This would have an affirmative answer if it could be shown that all $\Lambda(2)$ sets are actually uniformizable $\Lambda(2)$, since it is known that a subset of $\boldsymbol{Z}$ is $\Lambda(2)$ if and only if it is strictly-2-associated with all subsets of T of sufficiently large measure [3]. Indeed, the same comments apply if T is replaced by any compact abelian group G and Z by its dual.

To prove Lemma 2.2 we need the following result:

LEMMA 2.7. *Let g be a real-valued, non-negative, integrable function which is not identically zero. Then there is an $\varepsilon(g) > 0$ such that if $n \neq k$, then*

$$\int_T g(x)|e^{inx} - e^{ikx}|^2\, dm \geqslant 2\varepsilon(g).$$

Here m is normalized Lebesgue measure on the circle.

Remark. More generally, it is known [13] that if N is any positive integer, then there is an $\varepsilon(g, N)$ such that if f is a trigonometric polynomial with N non-zero Fourier coefficients, then

$$\int_T g|f|^2\, dm \geqslant \varepsilon(g, N)\|f\|_2^2.$$

In Lemma 3.5 we present a more general version of Lemma 2.7 and provide a proof.

Proof of Lemma 2.2. Since E is a uniformizable $\Lambda(2)$ set, we can find $\delta > 0$ so that whenever $m(A) < \delta$

$$\|1_A f\|_2^2 \leqslant \frac{c}{2}\|f\|_2^2.$$

This is a basic property of uniformizable $\Lambda(2)$ sets, established in [4].

The function

$$v \mapsto m(S) - 1_S * 1_{-S}(v) = m\big(S\backslash(S-v)\big)$$

is continuous, so there is a neighbourhood V of 0 on the circle with

$$m\big(S\backslash(S-v)\big) < \delta$$

whenever $v \in V$. Hence if $S_v = (S-v) \cap S$ and f is an E-polynomial, then

$$\|1_{S_v} f\|_2^2 \geqslant \|1_S f\|_2^2 - \|1_{S\backslash S_v} f\|_2^2 \geqslant \frac{c}{2}\|f\|_2^2 \tag{1}$$

whenever $v \in V$.

Given any $(E \cup \{n\})$-polynomial f and $v \in V$, let

$$f_v(x) = f(x+v) - e^{inv} f(x).$$

Observe that

$$\hat{f}_v(k) = \hat{f}(k)(e^{ikv} - e^{inv}) \tag{2}$$

so that f_v is an E-polynomial.

Now the choice of S_v ensures that

$$\|1_S f\|_2^2 \geqslant \frac{1}{m(V)} \int_V \int_{S_v} \frac{|f(x+v)|^2 + |f(x)|^2}{2} \, dm(x)\, dm(v),$$

so that by applying the basic inequality

$$2(|a|^2 + |b|^2) \geqslant |a+b|^2$$

to the line above and then using (1) and (2) we obtain

$$\begin{aligned} \|1_S f\|_2^2 &\geqslant \frac{1}{4m(V)} \int_V \int_{S_v} |f_v(x)|^2 \, dm(x)\, dm(v) \\ &\geqslant \frac{1}{4m(V)} \int_V \frac{c}{2} \|f_v\|_2^2 \, dm(v) \\ &= \frac{1}{4m(V)} \frac{c}{2} \sum_{k \neq n} |\hat{f}(k)|^2 \int_V 1_V |e^{ikv} - e^{inv}|^2 \, dm(v). \end{aligned}$$

From Lemma 2.7 we see that

$$\|1_S f\|_2^2 \geqslant \frac{1}{4m(V)} \frac{c}{2} \sum_{k \neq n} |\hat{f}(k)|^2 \, 2\varepsilon(1_V). \tag{3}$$

The basic inequality used before also shows that

$$\begin{aligned} \|1_S f\|_2^2 &\geqslant \frac{1}{2} \int_S |\hat{f}(n) e^{inx}|^2 \, dm(x) - \int_S \Big| \sum_{k \neq n} \hat{f}(k) e^{ikx} \Big|^2 dm(x) \\ &\geqslant \frac{1}{2} |\hat{f}(n)|^2 m(S) - \sum_{k \neq n} |\hat{f}(k)|^2. \end{aligned} \tag{4}$$

Thus by considering the two cases:

(i) $$\sum_{k \neq n} |\hat{f}(k)|^2 \geqslant \tau \|f\|_2^2$$

or

(ii) $$\sum_{k \neq n} |\hat{f}(k)|^2 \leqslant \tau \|f\|_2^2$$

for

$$\tau = m(S) \left(\frac{c\varepsilon(1_V)}{2m(V)} + m(S) + 2 \right)^{-1}$$

and substituting into (3) or (4), respectively, we obtain the conclusion of the lemma with $\varepsilon = \varepsilon(1_V)$.

3. Strict-2-associatedness in the general setting. We turn now to the proof of Theorem B. The main difficulty in adapting Theorem A is to establish an appropriate interpretation of the notion of "uniformly large gaps".

In the remainder of the paper we will be using multiplicative notation.

DEFINITION. Let F be a finite subset of Γ. For $\chi, \psi \in \Gamma$ we say that χ is *F-equivalent to ψ* if for some positive integer m there is a sequence

$$\chi = \chi_1, \chi_2, \ldots, \chi_m = \psi$$

with $\chi_{i+1}\chi_i^{-1} \in F$ for $i = 1, \ldots, m-1$. Such a sequence is called an *F-chain joining χ to ψ.*

If $\chi_i \in E \subset \Gamma$ for $i = 1, \ldots, m$, then $\chi_1, \ldots, \chi_m$ is said to be an *F-chain in E joining χ to ψ* and in this case χ is said to be *(E, F)-equivalent to ψ.*

When F is a symmetric subset of Γ containing the identity 1, this relation is an equivalence relation.

This terminology is suggested by [8], 8.9.

The notion of "uniformly large gaps" will be replaced by

THEOREM 3.1. *Suppose $E \subset \Gamma$ does not contain parallelepipeds of dimension n, and F is a finite symmetric subset of Γ containing* 1. *Then there is a constant $s = s(n, F)$ such that whenever $\{\chi_i\} \subset E$ satisfies $\chi_i\chi_j^{-1} \notin F^s$ for $i \neq j$, then χ_i and χ_j belong to distinct (E, F)-equivalence classes if $i \neq j$.*

Before proceeding with the proof we make some remarks and establish a lemma.

Remarks. 1. As we are now using multiplicative notation, a parallelepiped of dimension N is a set of cardinality 2^N which is the product of N two-element sets.

2. Taking $\Gamma = \boldsymbol{Z}$ and $F = [-N, N]$, this theorem implies that if n and m belong to E and $|n-m| \geqslant sN$, then between n and m there is an interval of length N free of points of E. Thus this theorem is a generalization of the uniform gap property for $\boldsymbol{Z}$ as outlined in Corollary 2.5.

The proof of the next lemma is motivated in part by [12].

LEMMA 3.2. *Let F be any finite subset of Γ. For each positive integer n there is a constant $k = k(n, F)$ such that if $\{\chi_i\}_{i=1}^r$ is an F-chain joining χ_1 and χ_r, with $\chi_i \neq \chi_j$ if $i \neq j$, and $r \geqslant k$, then $\{\chi_i\}_{i=1}^r$ contains a parallelepiped of dimension n.*

Proof. $k(1, F) = 2$ works since any 2-element set is a parallelepiped of dimension one.

Now proceed inductively assuming the result for n. We consider the F-chain of distinct terms $\{\chi_i\}_{i=1}^r$ with $r \geqslant 2k(n)|F|^{k(n)-1} = k(n+1)$. (We write $k(n)$ and $k(n+1)$ instead of $k(n, F)$ and $k(n+1, F)$ for ease of notation.)

Notice that each of the sets

$$B_1 = \{\chi_i\}_{i=1}^{k(n)}, \ B_2 = \{\chi_i\}_{i=k(n)+1}^{2k(n)}, \ \ldots, \ B_N = \{\chi_i\}_{i=(N-1)k(n)+1}^{Nk(n)},$$

where $N = k(n+1)/k(n)$, forms an F-chain of $k(n)$ distinct terms; so by the induction hypothesis each contains a parallelepiped of dimension n.

Observe that any two subsets of Γ, say $A = \{\alpha_i\}_{i=1}^m$ and $B = \{\beta_i\}_{i=1}^m$, are translates of one another, i.e., $A = B\chi$ for some $\chi \in \Gamma$, if $\alpha_{i+1}\alpha_i^{-1} = \beta_{i+1}\beta_i^{-1}$ for all $i = 1, \dots, m-1$. If in addition $A \cap B = \emptyset$ and A contains the parallelepiped $P_1 \dots P_N$ of dimension N, then $A \cup B$ contains $P_1 \dots P_N\{1, \chi\}$, a parallelepiped of dimension $N+1$.

Since the set $\{\chi_i\}_{i=1}^r$ is an F-chain, there are only $|F|$ choices for each of the characters $\chi_{i+1}\chi_i^{-1}$. Thus there can be at most $|F|^{k(n)-1}$ different sets of the form

$$\{\chi_{i+1}\chi_i^{-1}\}_{i=(j-1)k(n)+1}^{jk(n)-1}, \quad j = 1, \dots, N.$$

(Here we count different orderings as different sets.) But N was chosen to be twice this number, so at least two of the sets $B_1, \dots, B_N$ must be translates. Their union, and hence $\{\chi_i\}_{i=1}^r$, must contain a parallelepiped of dimension $n+1$. This completes the induction step.

We now prove Theorem 3.1. Recall that by assumption E does not contain parallelepipeds of dimension n.

Proof of Theorem 3.1. We will show that if $\chi_i\chi_j^{-1} \notin F^s$ for all $i \neq j$ with $s = k(n, F)$, then χ_i and χ_j belong to distinct (E, F)-equivalence classes when $i \neq j$.

Suppose not. Then there is an F-chain in E, say $\psi_1, \dots, \psi_m$, joining some pair χ_i, χ_j. If two of the characters, ψ_k and ψ_l, were identical, then the sequence $\psi_1, \dots, \psi_k, \psi_{l+1}, \dots, \psi_m$, upon renumbering, would still be an F-chain joining χ_i and χ_j, so we may assume $\psi_2, \dots, \psi_{m-1}$ are distinct.

Because $\psi_{i+1}\psi_i^{-1} \in F$ for $i = 1, \dots, m-1$, it follows that $\chi_j \in F^{m-1}\chi_i$, and since $F^{m-1} \subset F^s$ if $m-1 \leqslant s$, we must have $m > s = k(n, F)$.

Thus the F-chain $\{\psi_i\}_{i=1}^m$ in E consists of at least $k(n, F)$ distinct terms, and hence by the lemma must contain a parallelepiped of dimension n. This contradiction establishes the theorem.

The remaining steps in the proof of Theorem B are similar to those of Theorem A. We will state the necessary lemmas in complete generality and briefly indicate how their proofs differ from those of Theorem A.

Throughout, E will be assumed to be a uniformizable $\Lambda(2)$ subset of Γ, and S a subset of G with positive Haar measure.

By making the appropriate changes in the wording of the proof of Lemma 2.1 we obtain

LEMMA 3.3. *Given $c > 0$ there is a finite symmetric set $F = F(E, S, c)$, containing 1, so that if the subsets $\{E_i\}_{i\in I}$ of E are such that for all $i \in I$*

$$\|1_S f\|_2^2 \geqslant c\|f\|_2^2$$

whenever f is an E_i-polynomial, and

$$E_i E_j^{-1} \cap F = \emptyset \quad \text{for } i \neq j,$$

then $\bigcup_{i \in I} E_i$ is also strictly-2-associated with S.

Instead of Lemma 2.2 we have

LEMMA 3.4. *Suppose E is strictly-2-associated with S. There is a finite subgroup X of Γ depending on E and S such that whenever the set $E \cup \{\chi\}$, $\chi \in \Gamma$, is X-subtransversal, then $E \cup \{\chi\}$ is strictly-2-associated with S, with constant of strict-2-associatedness independent of χ.*

Again a constant of strict-2-associatedness can be specified.

Instead of using Lemma 2.7 in the proof of this lemma we need the following generalization:

LEMMA 3.5. *Let g be a real-valued, non-negative, integrable function which is not identically zero. There are a finite subgroup X_0 of Γ and a constant $\varepsilon(g) > 0$ such that if χ, $\psi \in \Gamma$ and $\chi\psi^{-1} \notin X_0$, then*

$$\int_G g\,|\chi-\psi|^2 \geqslant 2\varepsilon(g).$$

Remark. As with Lemma 2.7 this is a special case of a result applying to all polynomials P with supp $\hat{P}$ X_0-subtransversal ([8], 8.14). We present here a proof of the special case above for completeness.

Proof. Choose $A \subset G$ of positive measure and $\delta > 0$ so that $g \geqslant \delta$ on A.

Observe that whenever χ, $\psi \in \Gamma$, $\operatorname{Re} \hat{1_A}(\chi\psi^{-1}) \leqslant m(A)$ with equality if and only if $\chi\psi^{-1} = 1$ on A, and hence on the smallest open subgroup containing A. Let X_0 be the annihilator of this subgroup.

If G is connected, the only open subgroup of G is G itself, and thus X_0 would be trivial. In general, X_0 is a finite subgroup of Γ and $\operatorname{Re} \hat{1_A}(\chi\psi^{-1}) = m(A)$ if and only if $\chi\psi^{-1} \in X_0$. An application of the Riemann–Lebesgue Lemma yields an $\varepsilon > 0$ so that

$$\operatorname{Re} \hat{1_A}(\chi\psi^{-1}) < (1-\varepsilon)\,m(A) \quad \text{whenever } \chi\psi^{-1} \notin X_0.$$

Thus, if $\chi\psi^{-1} \notin X_0$, we have

$$\int g\,|\chi-\psi|^2 \geqslant \delta \int 1_A\,|\chi-\psi|^2 = \delta\big(2m(A) - 2\operatorname{Re} \hat{1_A}(\chi\psi^{-1})\big) \geqslant 2\varepsilon\delta m(A).$$

Setting $\varepsilon(g) = \varepsilon\delta m(A)$, the lemma is established.

The proof of Lemma 3.4 is carried out in the same manner as that of Lemma 2.2, taking for X the annihilator of the smallest open subgroup containing V, where V is chosen in the same way as in Lemma 2.2.

Finally, we replace Lemma 2.6 by

LEMMA 3.6. *Suppose E is X_0-subtransversal for all finite subgroups X_0 of Γ and suppose $E' \subset E$ is strictly-2-associated with S. Then there is a finite set F_1 depending on E, E' and S, so that whenever $E'' = \{\chi_i\} \subset E$ satisfies $\chi_i \chi_j^{-1} \notin F_1$ if $i \neq j$, then $E' \cup E''$ is also strictly-2-associated with S.*

Proof. Let $c > 0$ be a constant of strict-2-associatedness for all of the sets $E' \cup \{\chi\}$, $\chi \in E$, and then choose the finite set $F = F(E, S, c)$ as outlined in Lemma 3.3.

Now use the generalization of the notion of "uniformly large gaps". Since E is a uniformizable $\Lambda(2)$ set, it does not contain parallelepipeds of arbitrarily large dimension, and thus Theorem 3.1 may be applied. Choose the constant s so that if $\chi_i \chi_j^{-1} \notin F^s$ for $i \neq j$, and $E'' = \{\chi_i\}_{i \in I}$, then the (E, F)-equivalence class containing χ_i does not contain any other $\chi_j \in E''$. Denote by E_i the elements of this class which belong to $E' \cup E''$. Set

$$E_0 = E' \cup E'' \setminus \bigcup_{i \in I} E_i.$$

We take for F_1 the finite set F^s.

The proof is concluded as before by applying Lemma 3.3 since the sets $\{E_i\}_{i \in I \cup \{0\}}$ are strictly-2-associated with S, with constant of strict-2-associatedness c, and by construction of the equivalence relation, $E_i E_j^{-1} \cap F = \emptyset$ for all $i \neq j$, $i, j \in I \cup \{0\}$.

The definition of the classes M_n is easily transferred to the general setting by replacing "integer N" with "finite set Δ". Thus a set belongs to class M_0 if it tends to infinity as defined in Section 1, and a set belongs to class M_n if, for each finite set Δ, the original set can be expressed as the union of finitely many sets belonging to class M_{n-1} and a set $\{\chi_i\}$ with $\chi_i \chi_j^{-1} \notin \Delta$ for $i \neq j$.

The proof given in [9] can be reworded to show that any set not containing parallelepipeds of dimension n belongs to class M_{n-2}. Thus again each $\Lambda(p)$ set, $p > 0$, belongs to some class M_n.

The proof of Theorem B now follows with the same induction argument as Theorem A, replacing Lemmas 2.2 and 2.6 by Lemmas 3.4 and 3.6, respectively.

In conclusion, the author thanks J. Fournier for directing this work.

REFERENCES

[1] R. C. Blei, *Multidimensional extensions of the Grothendieck inequality*, Arkiv Mat. 17 (1979), pp. 51–68.

[2] A. Bonami, *Étude des coefficients de Fourier des fonctions de $L^p(G)$*, Ann. Inst. Fourier (Grenoble) 20 (1970), pp. 335–402.

[3] J. J. F. Fournier, *Two observations about 2-associatedness*, preprint, 1982.

[4] — *Uniformizable $\Lambda(2)$ sets and uniform integrability*, Colloq. Math. 51 (1987), pp. 119–129.

[5] – and L. Pigno, *Analytic and arithmetic properties of thin sets*, Pacific J. Math. 105 (1983), pp. 115–141.
[6] K. E. Hare, *Arithmetic properties of thin sets*, ibidem 131 (1988), pp. 143–155.
[7] J. López, *Fatou–Zygmund properties on groups*, Ph. D. Dissertation, University of Oregon, 1975.
[8] – and K. Ross, *Sidon Sets*, Lecture Notes in Pure and Appl. Math. 13, Marcel Dekker, Inc., New York 1975.
[9] I. M. Miheev, *On lacunary series*, Math. USSR-Sb. 27 (1975), pp. 481–502; translated from Mat. Sb. 98 (140) (1975), pp. 538–563.
[10] – *Trigonometric series with gaps*, Anal. Math. 9 (1983), pp. 43–55.
[11] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. 9 (1960), pp. 203–227.
[12] B. L. van der Waerden, *How the proof of Baudet's conjecture was found*, Studies in Pure Math., presented to R. Rado and L. Mirsky (eds.), London, 1971, pp. 251–260.
[13] A. Zygmund, *On a theorem of Hadamard*, Ann. Soc. Polon. Math. 21 (1948), pp. 52–69; *Errata*, ibidem, pp. 357–358.
[14] – *Trigonometric Series*, Vol. I, Cambridge University Press, Cambridge 1959.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA
CANADA T6G 2E1

Reçu par la Rédaction le 25.4.1986;
en version modifiée le 23.12.1986