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WEAK AUTOMORPHISMS OF INTEGRAL DOMAINS

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1. Introduction. In this paper we use terminology and notation of [5]. The concept of a weak isomorphism was introduced by A. Goetz and E. Marczewski (see [3]). In particular, we define a weak automorphism of an algebra $\mathfrak{A} = (A; \boldsymbol{F})$ as a bijection $\tau\colon A \to A$ such that the mapping $f \to f^*$ defined by the equality

$$f^*(x_1, \ldots, x_n) = \tau\big(f(\tau^{-1}(x_1), \ldots, \tau^{-1}(x_n))\big) \tag{1}$$

is, for each $n = 1, 2, \ldots$, a bijection from the set $\boldsymbol{A}^{(n)}(\mathfrak{A})$ of all n-ary algebraic operations of the algebra $\mathfrak{A}$ onto itself. By $\mathrm{Aut}(\mathfrak{A})$ and $\mathrm{Aut}^*(\mathfrak{A})$ we denote the sets of all automorphisms of $\mathfrak{A}$ and all weak automorphisms of $\mathfrak{A}$, respectively.

Let $a_{k+1}, \ldots, a_n$ be arbitrary (but fixed) elements of A and let $\tau \in \mathrm{Aut}^*(\mathfrak{A})$. It is easy to see that if for every $x_1, \ldots, x_k \in A$ holds the equality

$$f(x_1, \ldots, x_k, a_{k+1}, \ldots, a_n) = g(x_1, \ldots, x_k, a_{k+1}, \ldots a_n),$$

then for every $x_1, \ldots, x_k \in A$ holds also the equality

$$f^*\big(x_1, \ldots, x_k, \tau(a_{k+1}), \ldots, \tau(a_n)\big) = g^*\big(x_1, \ldots, x_k, \tau(a_{k+1}), \ldots, \tau(a_n)\big).$$

Furthermore, if $f \in \boldsymbol{A}^{(n)}$, $g_1, \ldots, g_n \in \boldsymbol{A}^{(k)}$ and $h = f(g_1, \ldots, g_n)$, then $h^* = f^*(g_1^*, \ldots, g_n^*)$ $(n, k = 0, 1, \ldots)$.

It follows that, in particular, a weak automorphism (weak isomorphism) image of an algebra belonging to a certain equational class of algebras is again an algebra belonging to the same class.

Note also (for the sake of the proof of Theorem 3) that if τ is a weak automorphism of an algebra $\mathfrak{A} = (A; \boldsymbol{F})$, then τ is an isomorphism of $\mathfrak{A}$ onto algebra $\mathfrak{A}^* = (A; \boldsymbol{F}^*)$, where $\boldsymbol{F}^*$ is the class of all operations induced by $\boldsymbol{F}$ according to formula (1). And, conversely, for any two given isomorphic algebras $\mathfrak{A}$ and $\mathfrak{A}^*$ with the same fundamental set A and such that $\boldsymbol{F}^* \subset \boldsymbol{A}(\mathfrak{A})$, if $\varphi\colon A \to A$ yields an isomorphism between $\mathfrak{A}$ and $\mathfrak{A}^*$, then φ is a weak automorphism of the algebra $\mathfrak{A}$.

Corollaries 2 and 3 from Theorem 3 are based upon remark that for every algebra $\mathfrak{A}$ the group $\mathrm{Aut}(\mathfrak{A})$ is a normal subgroup of the group $\mathrm{Aut}^*(\mathfrak{A})$ (see [2]).

Weak automorphisms of various algebras were investigated by several authors (see [3]). Recently J. Dudek investigated weak automorphisms of linear spaces and of certain related algebras (see [2]). In the present paper we shall describe (see section 3) weak automorphisms of integral domains which are either infinite or have 2, 3 or 4 elements by treating them as abstract algebras with the only fundamental constant 0 or with the two fundamental constants 0 and e (unity) — Theorem 3 and Corollary 1. Moreover, we shall also give a description of weak automorphisms of finite simple integral domains (Theorem 2), whence it will follow that Theorem 3 cannot be extended to cover all finite integral domains. Section 2 contains one simple theorem concerning rings with the unity. I owe the problem to J. Dudek whom I want also to thank for numerous discusions which greatly influenced the paper.

Considerations of the present paper are in some fragments similar to those of L. A. Hinrichs, I. Niven and C. J. van den Eynden concerning fields (see [4]). However, since these authors did not use the concept of a weak automorphism, often did apply division, and the present paper deals with the smaller set of algebraic operations — arguments used below are essentially different from [4].

2. Weak automorphisms of rings with unity. The ring $\mathfrak{R} = (R; +, -, 0, \cdot)$ with the unity $e \neq 0$ can be considered as an abstract algebra in two ways: for the set of fundamental operations we can take either two operations of two variables, namely addition $+$ and multiplication $\cdot$, and one unary operation of inverse element $-x$ (zero can be regarded here as algebraic operation $0(x) = x+(-x) = 0$ $(x \in R)$) or these three operations together with the constant operation $f(x) = e$ (which in the first conception is not algebraic), thus obtaining the abstract algebra $\mathfrak{R}_1 = (R; +, -, 0, \cdot, e)$.

In the first case, polynomials of arbitrary many variables with integral coefficients and without a free summand are algebraic operations. Besides, two different polynomials can correspond to one algebraic operation (e.g., a finite ring). In $\mathfrak{R}_1$ all polynomials with integral coefficients (we treate free summand as a multiple of unity e) are algebraic operations.

By R_e we shall denote the subring of R generated by e.

The following theorem is analogous to Theorem 2.1 in [2]:

THEOREM 1. *Let $\mathfrak{R} = (R; +, -, 0, \cdot)$ be a ring with unity e and let $\mathfrak{R}_1 = (R; +, -, 0, \cdot, e)$ be an algebra obtained from $\mathfrak{R}$ by adjoining the constant e to fundamental operations. Then a mapping $\varphi: R \to R$ is a weak automorphism of the algebra $\mathfrak{R}_1$ if and only if for every $x \in R$ there is*

$$\varphi(x) = \tau(x) + a, \tag{2}$$

where $\tau \in \mathrm{Aut}^*(\mathfrak{R})$, $a \in R_e$.

Proof. First we show that if $\tau \in \mathrm{Aut}^*(\mathfrak{R})$, then the mapping φ defined by (2) is a weak automorphism of the algebra $\mathfrak{R}_1$. Taking use of group operation $+$, it is easy to verify that φ is a bijection. It remains to show that φ induces a one-to-one mapping of $A^{(n)}(\mathfrak{R}_1)$ onto itself. For this purpose it suffices to verify that for every algebraic operation $f \in A^{(n)}(\mathfrak{R}_1)$ the operation induced by φ is an algebraic operation of the algebra $\mathfrak{R}_1$. Any n-ary algebraic operation g in $\mathfrak{R}_1$ is a polynomial of n variables with integral coefficients. It has the form

$$g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) + b,$$

where $f \in A^{(n)}(\mathfrak{R})$ is a polynomial with integral coefficients, without a free summand, and $b \in R_e$ (R_e is a ring generated by e). Hence we have

$$\varphi\Big(g\big(\varphi^{-1}(x_1), \ldots, \varphi^{-1}(x_n)\big)\Big) = \tau\Big(f\big(\tau^{-1}(x_1 - a), \ldots, \tau^{-1}(x_n - a)\big)\Big) + b. \tag{3}$$

Since $f \in A^{(n)}(\mathfrak{R})$, there is $f^* \in A^{(n)}(\mathfrak{R})$ and so operation (3) induced by φ is algebraic in $\mathfrak{R}_1$.

Now let $\varphi \in \mathrm{Aut}^*(\mathfrak{R}_1)$. Let us define a mapping $\tau: R \to R$ by

$$\tau(x) = \varphi(x) - \varphi(0). \tag{4}$$

Obviously, $\varphi(0) \in R_e$. We shall show that τ is a weak automorphism of the algebra $\mathfrak{R}$. Similarly as before, it is sufficient to verify that for every $f \in A^{(n)}(\mathfrak{R})$ the operation f^* induced by τ is an algebraic operation in $\mathfrak{R}$. Since f^* is a polynomial with integral coefficients, it remains to show that $f^*(0) = 0$. From the definition (4) of τ we have

$$\tau\Big(f\big(\tau^{-1}(x_1), \ldots, \tau^{-1}(x_n)\big) \\ = \varphi\Big(f\big(\varphi^{-1}(x_1 + \varphi(0)), \ldots, \varphi^{-1}(x_n + \varphi(0))\big)\Big) - \varphi(0).$$

Hence

$$\tau\Big(f\big(\tau^{-1}(0), \ldots, \tau^{-1}(0)\big)\Big) = \varphi\big(f(0, \ldots, 0)\big) - \varphi(0) = \varphi(0) - \varphi(0) = 0,$$

which completes the proof of Theorem 1.

Taking $a = 0$ in Theorem we have

COROLLARY. *Every weak automorphism of $\mathfrak{R}$ is simultaneously a weak automorphism of $\mathfrak{R}_1$.*

3. Weak automorphisms of integral domains. In the sequel $\mathfrak{R} = (R; +, -, 0, \cdot)$ will be an integral domain, i.e. an associative-commutative ring without divisors of zero, with the unity e, and containing at least 2 elements. As is well known, if integral domain R has a finite number q of elements, then it is a field, and so $q = p^m$, where p is a prime and m is a certain positive integral (the number p is a rank of the additive group and it is called *characteristic* of the integral domain or of the field, resp.). Thus all integral domains with the fixed finite number q of elements are isomorphic and their multiplicative group is a cyclic group of rank $q-1$. Hence in integral domain R with the finite number q of elements equality $x^q = x$ holds for every $x \in R$.

If integral domain R has infinitely many elements, then the set of algebraic operations of the algebra $\mathfrak{R}_1$ (or of the algebra $\mathfrak{R}$) will be isomorphic to the set of polynomials over the ring R_e generated by unity e with (or without) the free summand and so it is isomorphic to the set of polynomials with either integral coefficients if the characteristic of the ring is 0, or with the coefficients in Z_p (the ring of integral numbers modulo p) if the characteristic of the ring is p (see e.g. [1], Proposition 9, p. 27).

THEOREM 2 (cf. also [4], p. 539). *Let $\mathfrak{R} = (R;\ +,\ -,\ 0,\ \cdot)$ be a finite simple field and let $\mathfrak{R}_1 = (R;\ +,\ -,\ 0,\ \cdot,\ e)$ be an abstract algebra obtained from $\mathfrak{R}$ by adjoining the unity e as a constant to fundamental operations. Then every permutation of the set R is a weak automorphism of $\mathfrak{R}_1$, and permutation τ of R is a weak automorphism of $\mathfrak{R}$ if and only if $\tau(0) = 0$.*

Proof. Consider an arbitrary permutation τ of the set R which preserves the element zero. It follows from the interpolation formula of Lagrange that there exists a polynomial with integral coefficients, of a degree $\leqslant p-1$, and without a free summand, which for every $x \in R$ takes the value $\tau(x)$. So the operations $x \oplus y = \tau\big(\tau^{-1}(x) + \tau^{-1}(y)\big)$, $x \odot y = \tau\big(\tau^{-1}(x) \cdot \tau^{-1}(y)\big)$ are algebraic, and the mapping $f \mapsto f^*$ defined by (1) will be a bijection of the set of algebraic operations of the algebra $\mathfrak{R}$ onto itself. Therefore every permutation of the set R preserving 0 is a weak automorphism of $\mathfrak{R}$.

It follows from Theorem 1 that a weak automorphism of $\mathfrak{R}_1$ can take an arbitrary value for element 0, and since weak automorphism of $\mathfrak{R}$ is simultaneously a weak automorphism of the algebra $\mathfrak{R}_1$ (see Corollary to Theorem 1), every permutation of R is a weak automorphism of the algebra $\mathfrak{R}_1$.

From Theorem 2 we get at once

COROLLARY. *If R is a p element field, where p is a prime, then $\mathfrak{R} = (R;\ +,\ -,\ 0,\ \cdot)$ has $(p-1)!$ weak automorphism and $\mathfrak{R}_1 = (R;\ +,\ -,\ 0,\ \cdot,\ e)$ has $p!$ weak automorphisms.*

Now we shall prove the main result of this paper.

THEOREM 3. *Let $\mathfrak{R}_1 = (R;\ +,\ -,\ 0,\ \cdot,\ e)$ be an integral domain which is either infinite or has $2, 3$ or 4 elements, treated as an abstract algebra with the unity e as a fundamental constant. Then a bijection $\tau\colon R \to R$ is a weak automorphism of $\mathfrak{R}_1$ if and only if the element $\tau(e) - \tau(0)$ has* *an inverse and for every* $x, y \in R$ *the conditions*

$$\tau(x+y) = \tau(x)+\tau(y)-\tau(0), \tag{5}$$

$$\tau(x\cdot y) = \big[\tau(x)\cdot\tau(y)-\tau(0)\big(\tau(x)+\tau(y)\big)+\tau(e)\tau(0)\big]\big(\tau(e)-\tau(0)\big)^{-1}, \tag{6}$$

where $\tau(0), \tau(e) \in R_e$, *are satisfied.*

Moreover, we have

$$\big(\tau(e)-\tau(0)\big)^{-1} = \tau^{-1}(e)-\tau^{-1}(0), \tag{7}$$

$$\tau(0)\big(\tau^{-1}(e)-\tau^{-1}(0)\big) = -\tau^{-1}(0). \tag{8}$$

Proof. First we prove the necessity. From the remarks of Section 1 it follows that to operations $+$ and $\cdot$ correspond, by a fixed weak automorphism τ, associative and commutative operations $\oplus$ and $\odot$ such that the operation $\odot$ will be distributive with regard te $\oplus$, and will have the neutral element $\tau(e)$, the operation $\oplus$ will be a group operation with the neutral element $\tau(0)$, where $\tau(0)$ and $\tau(e)$, being algebraic constants in $\mathfrak{R}_1$, belong to the subring R_e generated by unity e.

In the case of $|R| = 2, 3$ or 4 the hypothesis can be verified directly. Let thus R contain infinitely many elements. We shall use the isomorphism of the set of algebraic operations of $\mathfrak{R}_1$ with the ring of polynomials over R_e.

As follows from commutativity of multiplication, $x\odot y$ is a symmetric polynomial of two variables and so we can write it in the form

$$\tau\big(\tau^{-1}(x)\cdot\tau^{-1}(y)\big) = x\odot y = f_1(x)+f_2(y)+g(x, y), \tag{9}$$

where f_1 and f_2 are polynomials of one variable and g is a polynomial of two variables over R_e. Taking into consideration associativity

$$(x\odot y)\odot z = x\odot(y\odot z), \tag{10}$$

we see that the polynomial (9) must be of degree 1 with respect to each variable, for if it is, e.g., of degree $n > 2$ with respect to variable x, then in the left-hand side of equality (10) we have a polynomial of degree n^2 with respect to x, and in the right-hand side — of degree n with respect to the same variable. Therefore $n^2 = n$ (modulo the characteristic of the ring R, if it is finite).

In view of the commutativity of $\odot$ we get the general form

$$x\odot y = a_{1\tau}(x+y)+a_{2\tau}xy+a_{3\tau}e, \tag{11a}$$

where $a_{1\tau}, a_{2\tau}, a_{3\tau}$ belong either to Z or to Z_p (this depends on the characteristic of the ring R).

Operation $\oplus$ has similar form:

$$x\oplus y = b_{1\tau}(x+y)+b_{2\tau}xy+b_{3\tau}e. \tag{11b}$$

However, from distributivity of $\odot$ with regard to addition $\oplus$ we have

$$(12)\qquad \begin{aligned} & a_{1\tau}b_{1\tau}(x+y)+a_{1\tau}b_{2\tau}xy+a_{1\tau}b_{3\tau}e+a_{1\tau}z+ \\ & +a_{2\tau}b_{1\tau}(x+y)z+a_{2\tau}b_{2\tau}xyz+a_{2\tau}b_{3\tau}z+a_{3\tau}e \\ = {} & b_{1\tau}[a_{1\tau}(x+y+2z)+a_{2\tau}(x+y)z+2a_{3\tau}e]+ \\ & +b_{2\tau}[a_{1\tau}(x+z)+a_{2\tau}xz+a_{3\tau}e][a_{1\tau}(y+z)+a_{2\tau}yz+a_{3\tau}e]+b_{3\tau}e. \end{aligned}$$

Thus it follows that $b_{2\tau}=0$.

In fact, by a comparison of coefficients of xyz^2 in (12) we have either $b_{2\tau}a_{2\tau}^2=0$ or $b_{2\tau}a_{2\tau}^2e=(b_{2\tau}e)(a_{2\tau}e)^2=0$, whence $b_{2\tau}=0$ or $a_{2\tau}=0$. If $b_{2\tau}\neq 0$, then in virtue of the equality $a_{2\tau}=0$ and (11a) we should have $x\odot y=a_{1\tau}(x+y)+a_{3\tau}e$. Since $\tau(e)$ is a neutral element of the operation $\odot$, we have $a_{1\tau}=1$. But using in (12) the equalities $a_{2\tau}=0$ and $a_{1\tau}=1$ we get

$$\begin{aligned} & b_{1\tau}(x+y)+b_{2\tau}xy+b_{3\tau}e+z+a_{3\tau}e \\ & \qquad = b_{1\tau}(x+y+2z+2a_{3\tau}e)+b_{2\tau}(x+z+a_{3\tau}e)(y+z+a_{3\tau}e)+b_{3\tau}e, \end{aligned}$$

whence, by a comparison of coefficients of xz, we come to a contradiction.

Taking into account $b_{2\tau}=0$ and the fact that $\tau(0)$ is a neutral element of operation $\oplus$ we get immediately from (11b) the equalities $b_{1\tau}=1$ and $b_{3\tau}e=-\tau(0)$. Therefore we get the formula

$$(*)\qquad x\oplus y=\tau\big(\tau^{-1}(x)+\tau^{-1}(y)\big)=x+y-\tau(0)$$

equivalent to (5).

Now putting into (12) the values of $b_{1\tau}$, $b_{2\tau}$, $b_{3\tau}$ we get

$$\begin{aligned} & a_{1\tau}(x+y)-a_{1\tau}\tau(0)+a_{1\tau}z+a_{2\tau}(x+y)z-a_{2\tau}\tau(0)z+a_{3\tau}e \\ & \qquad = a_{1\tau}(x+y+2z)+a_{2\tau}(x+y)z+2a_{3\tau}e-\tau(0), \end{aligned}$$

whence, by a comparison of coefficients of these poynomials, we have the equalities

$$(13)\qquad -a_{2\tau}\tau(0)=a_{1\tau}e,$$

$$(14)\qquad \tau(0)-a_{1\tau}\tau(0)=a_{3\tau}e.$$

However, putting $y=\tau(e)$ in (11a) we have

$$x=a_{1\tau}x+a_{2\tau}\tau(e)x+a_{1\tau}\tau(e)+a_{3\tau}e,$$

whence

$$(15)\qquad a_{1\tau}e+a_{2\tau}\tau(e)=e,$$

$$(16)\qquad a_{1\tau}\tau(e)+a_{3\tau}e=0.$$

From (13) and (15) we get

$$(17)\qquad a_{2\tau}\big(\tau(e)-\tau(0)\big)=e.$$

(Hence the element $\tau(e)-\tau(0)$ has the inverse.)

From formulas (14) and (16) we infer the equalities

$$a_{1\tau}\big(\tau(e)-\tau(0)\big) = -\tau(0), \tag{18}$$

$$a_{3\tau}\big(\tau(e)-\tau(0)\big) = \tau(0)\tau(e). \tag{19}$$

From equalities (17), (18) and (19) in an application to (11a) we obtain the equality

$$(**)\quad x\odot y = \tau\big(\tau^{-1}(x)\cdot\tau^{-1}(y)\big) = [xy-\tau(0)(x+y)+\tau(0)\tau(e)]\cdot\big(\tau(e)-\tau(0)\big)^{-1}$$

equivalent to (6). Therefore we proved necessity of conditions (5) and (6).

Now let $\tau: R\to R$ be a bijection, let the element $\tau(e)-\tau(0)$ has an inverse, and let equalities (5) and (6) be satisfied. For the proof of sufficiency it is sufficient, in view of remarks of Section 1, to verify that the mapping

$$\varphi(x) = \big(x-\tau(0)\big)\big(\tau(e)-\tau(0)\big)^{-1}$$

is an isomorphism of the ring $(R;\oplus,\odot)$ onto the ring $(R;+,\cdot)$. Obviously, this mapping is a bijection. Further, the equalities

$$\begin{aligned}
\varphi(x\oplus y) &= \big[\big(x+y-\tau(0)\big)-\tau(0)\big]\big(\tau(e)-\tau(0)\big)^{-1} = \varphi(x)+\varphi(y),\\
\varphi(x\odot y) &= \big[\big(xy-\tau(0)(x+y)+\tau(e)\tau(0)\big)\big(\tau(e)-\tau(0)\big)^{-1}-\tau(0)\big]\big(\tau(e)-\tau(0)\big)^{-1}\\
&= \big[\big(x-\tau(0)\big)y-\tau(0)x+\tau(e)\tau(0)-\tau(0)\big(\tau(e)-\tau(0)\big)\big]\big(\tau(e)-\tau(0)\big)^{-2}\\
&= \big[\big(x-\tau(0)\big)y-\tau(0)\big(x-\tau(0)\big)\big]\big(\tau(e)-\tau(0)\big)^{-2} = \varphi(x)\cdot\varphi(y)
\end{aligned}$$

hold. Therefore the first part of Theorem 3 is proved.

Now we shall prove equalities (7) and (8) for every $\tau\in\mathrm{Aut}^*(\mathfrak{R}_1)$. By applying formula (5) we get easily the equality

$$\tau(ax) = a\tau(x)-(a-1)\tau(0), \tag{20}$$

where a is an element either of Z or of Z_p (this depends on the characteristic of the ring $\mathfrak{R}_1$). Now, considering the mapping τ^{-1} which must also be a weak automorphism and using formulas (5), (6), (20), (*), (**), and the analogous equality for τ^{-1}, we obtain

$$\begin{aligned}
&xy\big(\tau(e)-\tau(0)\big) = \tau^{-1}\big(\tau(x)\odot\tau(y)\big)\big(\tau(e)-\tau(0)\big)\\
&= \big(\tau(e)-\tau(0)\big)\tau^{-1}\big[\big(\tau(x)\tau(y)-\tau(0)\big(\tau(x)+\tau(y)\big)+\tau(0)\tau(e)\big)\big(\tau(e)-\tau(0)\big)^{-1}\big]\\
&= \tau^{-1}\big[\tau(x)\tau(y)-\tau(0)\big(\tau(x)+\tau(y)\big)+\tau(0)\tau(e)\big]-\\
&\quad-\big(\tau(e)-\tau(0)\big)\big[\big(\tau(e)-\tau(0)\big)^{-1}-e\big]\tau^{-1}(0)\\
&= \tau^{-1}\big(\tau(x)\tau(y)\big)+\tau^{-1}\big[-\tau(0)\big(\tau(x)+\tau(y)\big)\big]+\tau(0)\tau(e)\tau^{-1}(e)-\\
&\quad-\big(\tau(0)\tau(e)-e\big)\tau^{-1}(0)-3\tau^{-1}(0)+\big(\tau(e)-\tau(0)\big)\tau^{-1}(0)\\
&= \tau^{-1}\big(\tau(x)\tau(y)\big)-\tau(0)\tau^{-1}\big(\tau(x)+\tau(y)\big)+\\
&\quad+\tau(0)\tau(e)\big(\tau^{-1}(e)-\tau^{-1}(0)\big)+\tau(e)\tau^{-1}(0)-\tau^{-1}(0)\\
&= [xy-\tau^{-1}(0)(x+y)+\tau^{-1}(0)\tau^{-1}(e)]\big(\tau^{-1}(e)-\tau^{-1}(0)\big)^{-1}-\\
&\quad-\tau(0)[x+y-\tau^{-1}(0)]+\tau(0)\tau(e)\big(\tau^{-1}(e)-\tau^{-1}(0)\big)+\tau(e)\tau^{-1}(0)-\tau^{-1}(0).
\end{aligned}$$

Therefore we gained the equality

$$xy\big(\tau(e)-\tau(0)\big) = xy\big(\tau^{-1}(e)-\tau^{-1}(0)\big)^{-1}-\tau^{-1}(0)\big(\tau^{-1}(e)-\tau^{-1}(0)\big)^{-1}(x+y)+ \\ +\tau^{-1}(0)\tau^{-1}(e)\big(\tau^{-1}(e)-\tau^{-1}(0)\big)^{-1}-\tau(0)(x+y)+\tau(0)\tau^{-1}(0)+ \\ +\tau(0)\tau(e)\big(\tau^{-1}(e)-\tau^{-1}(0)\big)+\tau(e)\tau^{-1}(0)-\tau^{-1}(0),$$

whence by a comparison of coefficients of xy and $x+y$ we get formulas (7) and (8). This completes the proof of Theorem 3.

It is worth to remark that taking into consideration formulas (7) and (8) one can write equality (6) in a simpler form:

$$\tau(xy) = \big(\tau^{-1}(e)-\tau^{-1}(0)\big)\tau(x)\tau(y)+\tau^{-1}(0)\big(\tau(x)+\tau(y)\big)-\tau(e)\tau^{-1}(0).$$

Remark 1. It is well known that if $\mathfrak{R}_1$ is a field, then equalities (*) and (**), where $\tau(0)$ and $\tau(e)$ are certain (arbitrary) elements in R, define new operations $\oplus$ and $\odot$ with respect to which R is again a field isomorphic to the original one (see [7], p. 11). It was proved in [4] that this form is also necessary for the set R with new operations $\oplus$ and $\odot$ defined by symmetric polynomials to be a field isomorphic with the original field (Theorem 1, p. 537).

In view of Theorems 1 and 3 and of the observation that a weak automorphism of $\mathfrak{R} = (R; +, -, 0, \cdot)$ must carry the only algebraic constant 0 onto itself we get

COROLLARY 1. *If $\mathfrak{R} = (R; +, -, 0, \cdot)$ is an integral domain which is either infinite or has $2, 3$ or 4 elements, then a bijection $\tau\colon R \to R$ is a weak automorphism of $\mathfrak{R}$ if and only if the element $\tau(e)$ has an inverse and for every $x, y \in R$ the equalities*

$$\tau(x+y) = \tau(x)+\tau(y), \tag{5'}$$

$$\tau(xy) = [\tau(e)]^{-1}\tau(x)\tau(y), \tag{6'}$$

where $\tau(e)\in R_e$, are satisfied.

Moreover, equality

$$[\tau(e)]^{-1} = \tau^{-1}(e) \tag{7'}$$

also holds true.

Remark 2. Note that neither the hypothesis of Theorem 3 nor that of Corollary 1 can be extended to all finite integral domains. For let $p > 3$ be a prime and let R be a finite simple field with p elements. Then every permutation τ of R preserving 0 and e is, in virtue of Theorems 1 and 2, a weak automorphism of both $\mathfrak{R}$ and $\mathfrak{R}_1$. If formulas (5), (6) and (5'), (6') were satisfied for every weak automorphism of $\mathfrak{R}_1$ or $\mathfrak{R}$, respectively, then for the considered permutations we should have

$$x\oplus y = \tau\big(\tau^{-1}(x)+\tau^{-1}(y)\big) = x+y,$$
$$x\odot y = \tau\big(\tau^{-1}(x)\tau^{-1}(y)\big) = xy.$$

So the field R would have at least $(p-2)!$ automorphisms which is impossible.

From Corollary 1 we obtain two following corollaries:

COROLLARY 2. *Let* $\mathfrak{R} = (R; +, -, 0, \cdot)$ *be an integral domain. Then the factor group* $\mathrm{Aut}^*(\mathfrak{R})/\mathrm{Aut}(\mathfrak{R})$ *is isomorphic to the multiplicative group* G_e *of all divisors of unity which are elements of the subring* R_e *of* R.

Proof. It follows from Corollary 1 that the mapping $h\colon \mathrm{Aut}^*(\mathfrak{R}) \to G_e$ defined by $h(\tau) = [\tau(e)]^{-1}$ is correct. We shall prove that this mapping is a homomorphism. Let $\tau_1, \tau_2 \in \mathrm{Aut}^*(\mathfrak{R})$. Then, in view of Corollary 1 and of formula (20), the equalities

$$\tau_2\tau_1(xy) = [\tau_1(e)]^{-1}\tau_2\big(\tau_1(x)\tau_1(y)\big) = [\tau_1(e)]^{-1}[\tau_2(e)]^{-1}\tau_2\big(\tau_1(x)\big)\tau_2\big(\tau_1(y)\big)$$

hold. Therefore $\big[\tau_2\big(\tau_1(e)\big)\big]^{-1} = [\tau_2(e)]^{-1}[\tau_1(e)]^{-1}$.

It is easy to see that kernel of h is the group of automorphisms of $\mathfrak{R}$, which completes the proof of Corollary 2.

From Corollaries 1 and 2 it follows immediately that

COROLLARY 3. *If* $\mathfrak{R}$ *is an integral domain of characteristic* 0, *then for every weak automorphism either* $[\tau(e)]^{-1} = e$ *or* $= -e$, *and the factor group* $\mathrm{Aut}^*(\mathfrak{R})/\mathrm{Aut}(\mathfrak{R})$ *is the two-element group.*

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