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ON REGULAR CONGRUENCES

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In [5], Suszko investigated the difference between equational theories in a nominal language (the usual equational theories) and equational theories in a sentential language having an identity connective. Any congruence θ induced by a sentential equational theory in an algebra $\boldsymbol{A}$ is *regular*, i.e.,

(1) $$\text{if } a\circ b \equiv c\circ c(\theta) \text{ for some } c, \text{ then } a \equiv b(\theta).$$

(The operation $\circ$ corresponds to the identity connective.)

Słomiński in [4] studied congruences θ determined by a polynomial term (which we denote by $a\circ b$) in the following way: (1) holds, and

(2) $$a\circ a \equiv b\circ b(\theta).$$

Słomiński's congruences are thus a subclass of the regular congruences. All congruences in a group, or ring, or Boolean algebra have properties (1) and (2) (for an appropriate term $a\circ b$). In [4], Słomiński gave necessary and sufficient conditions for every congruence on an algebra $\boldsymbol{A}$ to have properties (1) and (2). In section 2, we give necessary and sufficient conditions for all congruences on $\boldsymbol{A}$ to be regular. We also show how all of Słomiński's theorems may be generalized to apply to regular congruences. Sections 1 and 3 contain some related theorems. In section 1, we relate nominal equational theories to sentential ones. In section 3, conditions are given on an algebra in order that it contain at least one proper regular congruence.

1. Nominal and sentential equational theories. Let K be the class of all algebras of a fixed type. Let N be the set of terms in the corresponding open language (i.e., N is the free algebra in K on a countably infinite set of generators). We will identify an equational theory with a subset of $N\times N$ (e.g., instead of saying $\alpha = \beta$ is true in $\boldsymbol{A}$, we say $\langle \alpha, \beta\rangle$ is true in $\boldsymbol{A}$, where $\alpha, \beta \in N$).

Suppose $\circ$ is a binary operational symbol not in N. Let K^+ be the class of all algebras $A = \langle A; \circ, \ldots \rangle$ such that $\langle A; \ldots \rangle \in K$. Let N^+ be the sentential language with *connectives* $\circ, \ldots$ (as an algebra, N^+ is the free algebra in K^+ on $\aleph_0$-generators).

The consequence operations E_0 and E were defined in [5]. E_0 is the standard equational consequence operation (applicable to both $N \times N$ and $N^+ \times N^+$. E is defined on $N^+ \times N^+$ by the rules of E_0 as well as the rule

$$\frac{\langle a \circ \beta, \gamma \circ \gamma \rangle}{\langle a, \beta \rangle}.$$

An E (E_0) *theory* is a subset T of $N^+ \times N^+$ ($N \times N$) such that $E(T) = T$. A theory T is *invariant* if, whenever $\langle a, \beta \rangle \in T$, and k is an endomorphism of N (or N^+), then $\langle k(a), k(\beta) \rangle \in T$; i.e. T is invariant iff T is "closed under substitution".

Definition. Suppose $A \in K$ and φ: $N \to A$ is a homomorphism. Then

$$E_0(\varphi; A) = \{\langle a, \beta \rangle \in N \times N \colon \varphi(a) = \varphi(\beta)\},$$

and

$$E_0(A) = \bigcap_{\varphi} E_0(\varphi; A).$$

For $A \in K^+$, $E(\varphi, A)$ and $E(A)$ are defined similarly.

THEOREM 1. *T is an invariant E_0-theory iff there is some algebra $A \in K$ such that $T = E_0(A)$.*

Notation. ω_A denotes the *identity congruence in the algebra A*. (Regular congruences were defined in the introduction.)

THEOREM 2. *T is an invariant E-theory iff there is some algebra A in K^+ such that ω_A is regular and $T = E(A)$.*

For any algebra A in K with at least two elements $0 \neq 1$, define the operation $\circ$ on A as follows:

$$a \circ b = \begin{cases} 1, & a = b, \\ 0, & a \neq b. \end{cases}$$

Let A^+ be A with this additional operation. Note that ω_{A^+} is regular: $a \circ b = c \circ c \Rightarrow a = b$.

THEOREM 3. *$E_0(A) = E(A^+) \cap (N \times N)$. Thus, every invariant E_0-theory is the restriction of an E-theory to the appropriate set of terms.*

2. Regular algebras. All algebras in this section contain a binary operation denoted by $a \circ b$. An algebra A will be called *regular* ([1]) if every congruence on A is regular.

([1]) The reader is warned that in [5], algebras having not less than 2 regular congruences were called *regular*.

Let A be an algebra, $a, b \in A$.

Definition. $\theta(a, b)$ is the smallest congruence θ in A such that $a \equiv b\ (\theta)$ (see [3], p. 52).

The main theorem in [4] is theorem 2 in section 4. Its counterpart for regular congruences is the following

THEOREM 1. *A is regular iff, for all a, b in A,*

$$a \equiv b\Big(\bigcap_{c \in A} \theta(a \circ b, c \circ c)\Big).$$

Proof. Suppose A is regular. Then each congruence $\theta(a \circ b, c \circ c)$ is regular. Hence $a \equiv b\ \theta(a \circ b, c \circ c)$ for all c. Conversely, if θ is any congruence and $a \circ b \equiv c \circ c\ (\theta)$ for some c, then $\theta(a \circ b, c \circ c) \leqslant \theta$. By the assumption, $a \equiv b\ \theta(a \circ b,\ c \circ c)$, so $a \equiv b\ (\theta)$. Thus all congruences are regular.

We now state theorems 2, 3, 4 and 5 which are analogues of Słomiński's theorems 3, 4, 9 and 11, respectively (see [4], section 4). The proofs of these theorems follow from theorem 1 in the way Słomiński's theorems follow from theorem 2 in section 4 of [4]. We use the terminology of [4].

THEOREM 2. *A is regular iff, for any a, b, c in A, there is an $\langle a \circ b, c \circ c\rangle$--chain from a to b.*

THEOREM 3. *A is regular iff every 3-generated subalgebra of A is regular.*

Let (S) be Słomiński's condition S_2 (see [4], p. 336) modified as follows: "$\langle \Psi(x, y), \Psi(x, x)\rangle$" should be changed to "$\langle x \circ y, z \circ z\rangle$".

THEOREM 4. *If a class K of algebras satisfies condition* (S), *then every algebra in K is regular.*

THEOREM 5. *Let K be an equational class. Then every algebra in K is regular iff K satisfies* (S).

Remark. There are no familiar examples of algebras which are regular but do not satisfy $a \circ a = b \circ b$. Such algebras occur only in the study of sentential equational theories (as was pointed out in [5]).

Example. Let $A = \langle B, \circ\rangle$, where B is the four-element Boolean algebra $\{0, 1, a, -a\}$ and where the $\circ$-operation is defined by the following table:

	0	1	a	$-a$
0	1	0	0	0
1	0	1	0	0
a	0	0	a	0
$-a$	0	0	0	$-a$

Then the only congruences on A are the trivial congruence and ω_A. Both are regular.

3. The existence of regular congruences. All algebras in this section are assumed to have the $\circ$-operation and *contain at least two elements.* Let K be the class of all algebras which satisfy the conditional equality

$$x\circ y = z\circ z \rightarrow x = y.$$

THEOREM 1. *An algebra $\boldsymbol{A}$ is in K iff $\omega_{\boldsymbol{A}}$ is regular.*

For any algebra $\boldsymbol{A}$, $H(\boldsymbol{A})$ is the class of all homomorphic images of $\boldsymbol{A}$ (containing not less than 2 elements).

THEOREM 2. *An algebra $\boldsymbol{A}$ has at least one proper regular congruence iff $H(\boldsymbol{A})\cap K \neq \emptyset$.*

Proof. Suppose $\boldsymbol{A}$ has a proper regular congruence θ. Then $\boldsymbol{A}/\theta \in H(A)\cap K$. Conversely, if $\boldsymbol{B}\in H(A)\cap K$, where $h\colon \boldsymbol{A}\rightarrow\boldsymbol{B}$ is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{B}$, then $\theta = \ker h$ $(a \equiv b\ (\theta) \Leftrightarrow h(a) = h(b))$ is a proper regular congruence. Indeed, θ is proper, since $\boldsymbol{B}$ has at least two elements. θ is regular, since if $a\circ b \equiv c\circ c\ (\theta)$, then

$$h(a)\circ h(b) = h(c)\circ h(c).$$

Since $\boldsymbol{B}\in K$, $h(a) = h(b)$, we have $a \equiv b\ (\theta)$.

We now consider the class of "Boolean $\circ$-algebras" $\boldsymbol{A} = \langle \boldsymbol{B}, \circ\rangle$, where $\boldsymbol{B}$ is a Boolean algebra (with $0, 1, \cap, \rightarrow, -$) and $\circ$ is some binary operation on $\boldsymbol{B}$. The example of the last section is one such algebra. That example shows that it is possible that there be congruences θ on such algebras having no Boolean filter $F(\theta)$ with the property

$$a \equiv b\ (\theta) \Leftrightarrow a\circ b\in F(\theta).$$

Definition A. Let $\boldsymbol{A}$ be a Boolean $\circ$-algebra, and θ a congruence on $\boldsymbol{A}$. θ is *Boolean regular* if there is a Boolean filter $F = F(\theta)$ such that

(a) $a \equiv b\ (\theta) \Leftrightarrow a\circ b\in F(\theta)$, and

(b) $a, a\circ b\in F(\theta) \Rightarrow b\in F(\theta)$.

The reader familiar with [1] or [2] will see that if $\langle \boldsymbol{A}, P\rangle$ is an SCI-matrix and A is a Boolean $\circ$-algebra, then A contains a proper Boolean regular congruence — that defined by the SCI-filter P.

Definition B. Let K^+ be the class of all structures $\langle \boldsymbol{A}, F\rangle$ such that $\boldsymbol{A}$ is a Boolean $\circ$-algebra, $F \subset A$ and the following are true:

$$F(1),$$

$$F(x)\ \&\ F(x\rightarrow y) \Rightarrow F(y),$$

$$x = y \Rightarrow F(x\circ y),$$

$$F(x\circ y) \Rightarrow x = y.$$

THEOREM 3. *Let $\boldsymbol{A}$ be a Boolean $\circ$-algebra. ω_A is Boolean regular iff there is some $F \subset A$ such that $\langle \boldsymbol{A}, F\rangle \in K^+$.*

THEOREM 4. *Let $\boldsymbol{A}$ be a Boolean $\circ$-algebra. $\boldsymbol{A}$ has some proper Boolean regular congruence iff there is some $\boldsymbol{B} \in H(A)$ and some $F \subset \boldsymbol{B}$ such that $\langle \boldsymbol{B}, F\rangle \in K^+$.*

Proof. Suppose

$$h\colon \boldsymbol{A} \xrightarrow{\text{onto}} \boldsymbol{B}$$

is a homomorphism and $\langle \boldsymbol{B}, F\rangle \in K^+$. Then ω_B is Boolean regular. Then it is easy to see that $\theta = \ker h$ is a Boolean regular congruence on $\boldsymbol{A}$, since the inverse image of F is a Boolean filter in $\boldsymbol{A}$.

Conversely, suppose θ is a proper Boolean regular congruence on $\boldsymbol{A}$, and G is a Boolean filter such that $a \equiv b\ (\theta)$ iff $a \circ b \in G$, as in definition A. Let

$$h\colon A \to A/\theta, \qquad h(a) = |a| = \{b\colon a \equiv b\ (\theta)\},$$

be the canonical map. We claim $F = h(G)$ is a Boolean filter in $\boldsymbol{B}$ and $|a| = |b| \Leftrightarrow |a| \circ |b| \in F$.

First, we show $\check{h}F = G$ ($\check{h}$ is the inverse image of h). Indeed, if $h(a) = h(b)$, where $a \in G$, then $|a| = |b|$, so $a \equiv b\ (\theta)$. Thus $a \circ b \in G$ and, by definition A, $b \in G$. Hence $h(b) \in F$.

Now, suppose $|a|$, $|a \to b| \in F$. By the above, $a, a \to b \in G$, so $b \in G$, and $|b| \in F$. This shows that F is a Boolean filter. Similarly, if $|a|$, $|a \circ b| \in F$, then $|b| \in F$. Lastly,

$$|a| = |b| \Leftrightarrow a \circ b \in G \Leftrightarrow |a| \circ |b| \in F.$$

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