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A REMARK ON HYPOELLIPTIC DIFFERENTIAL AND CONVOLUTION OPERATORS

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Gorin and Grushin [3] have proved the following theorem:

Let $P(D)$ be a differential polynomial in R^n with constant coefficients. If for some $m \geqslant 0$ all C^m-solutions in R^n of the partial differential equation

$$P(D)u = 0 \tag{1}$$

have continuous derivatives of order $m+1$ in a neighbourhood of the origin, then the operator $P(D)$ is hypoelliptic, i. e. all solutions of equation (1) in the space of distributions $\mathcal{D}'$ are C^∞-functions.

In this note we give a simple proof of a similar statement, which involves solutions of greater smoothness. In fact, we assume that all C^∞-solutions of equation (1) belong in a domain $V \subset R^n$ to a Gevrey class. In that case the operator $P(D)$ appears to be hypoelliptic of a type determined by the Gevrey class in question. The result holds also for invertible convolution operators.

We use the usual notation of the theory of distributions in R^n (see e. g. [5]).

Let Ω be an open set of R^n and a a positive number. We denote by $\Gamma_a(\Omega)$ the class of C^∞-functions in Ω with the following property: If $u \in \Gamma_a(\Omega)$, then for every compact set $K \subset \Omega$ there are constants A_K and $B_K > 0$ (which may depend on u) such that

$$\sup_{x \in K} |D^p u(x)| \leqslant A_K B_K^{|p|} p^{ap} = A_K B_K^{p_1+\ldots+p_n} p_1^{ap_1} \ldots p_n^{ap_n} \tag{2}$$

for all multi-indices $p = (p_1, \ldots, p_n)$. In particular, $\Gamma_1(\Omega)$ consists of real analytic functions in Ω.

The operator $P(D)$ is said to be *of type* a, if all solutions $u \in \mathcal{D}'$ of equation (1) are in $\Gamma_a(R^n)$.

THEOREM. *If $P(D)$ is a differential polynomial with constant coefficients and if all C^∞ solutions in R^n of equation* (1) *belong in a domain V to $\Gamma_a(V)$, $a \geqslant 1$, then $P(D)$ is of type a.*

Proof. We may assume without loss of generality that V contains the n-sphere K_r: $|x| \leqslant r$.

If for a function $u \in \Gamma_a(V)$ and for the compact set $K = K_r$ the constant B_K in (2) is less than e^{-1}, then

$$\sum_p \frac{1}{p!}\big(\sup_{|x|\leqslant r} |D^p u(x)|\big)^{1/a} < \infty. \tag{3}$$

We denote by H the subset of $\Gamma_a(V)$ consisting of functions u such that $B_{K_r} \leqslant \frac{1}{3}$. Then series (3) converges for all $u \in H$. Moreover, it is easy to verify that

$$\|u\|_{1/a} = \sum_p \frac{1}{p!}\big(\sup_{|x|\leqslant r} |D^p u(x)|\big)^{1/a}$$

is an F-norm in H; in fact it is a $1/a$-norm (see e. g. [4], p. 164). With this F-norm H is a linear topological space, which is metrisable and complete.

On the other hand, the space $\mathcal{E}$ of C^∞-functions in R^n with the topology defined by the system of seminorms

$$N_{k,l}(u) = \sup_{\substack{|x|\leqslant k\\ |p|\leqslant l}} |D^p u(x)|$$

is a Fréchet space.

All functions $u \in H$ and $u \in \mathcal{E}$, which are solutions of equation (1), form subspaces H_E and $\mathcal{E}_E$ of the spaces H and $\mathcal{E}$ respectively. H_E and $\mathcal{E}_E$ are themselves metrisable and complete.

Now, by assumption, the restriction u_V to the domain V of every function $u \in \mathcal{E}_E$ is in $\Gamma_a(V)$. Consider the set $\mathcal{E}_{EH}$ of all $u \in \mathcal{E}_E$, whose restrictions u_{K_r} to K_r are in H_E. Provided with the topology induced by $\mathcal{E}$, $\mathcal{E}_{EH}$ is again metrisable and complete. Furthermore, the map $u \to u_{K_r}$ of $\mathcal{E}_{EH}$ into H_E is continuous. Thus, by an argument similar to that used in [1] (see chapter II, § 6, proposition 8), there is a semi-norm $N_{k,l}$ and a constant C such that

$$\|u_{K_r}\|_{1/a} \leqslant C[N_{k,l}(u)]^{1/a} \tag{4}$$

for all $u \in \mathcal{E}_{EH}$.

But $\mathcal{E}_{EH}$ contains all exponential functions $e^{i\zeta x}$ with $P(i\zeta) = 0$, $\zeta = \xi + i\eta$, $\xi, \eta \in R^n$. Therefore inequality (4) implies

$$e^{|\zeta|^{1/a} + r|\eta|} \leqslant C(1+|\zeta|)^l e^{k|\eta|}$$

and, consequently,

$$|\zeta|^{1/a} \leqslant C' + C'' \log(1+|\zeta|) + M|\eta|, \tag{5}$$

where C', C'' and M are constants. Since M must be positive, we infer from (5) that the zeros of the polynomial $P(i\zeta)$ satisfy the condition

$$0 < M^{-1} \leqslant \liminf_{|\zeta|\to\infty} \frac{|\eta|}{|\zeta|^{1/a}}, \tag{6}$$

which is sufficient for the operator $P(D)$ to be of type a (see e. g. [6]). The proof is thus complete.

Let S be a distribution with a compact support, which is invertible in $\mathscr{D}'$, i. e. for any $T \in \mathscr{D}'$ the convolution equation

$$S * u = T$$

has a solution $u \in \mathscr{D}'$.

We say that S is *of type* $a > 0$, if all solutions $u \in \mathscr{D}'$ of the homogeneous equation

$$S * u = 0$$

are in $\Gamma_a(R^n)$.

The analogue of our theorem for convolution operators is now clear. It can be proved in the same way. The last sufficient condition follows by the methods of Ehrenpreis [2].

REFERENCES

[1] N. Bourbaki, *Espaces vectoriels topologiques I*, Paris 1953.

[2] L. Ehrenpreis, *Solution of some problems of division IV*, American Journal of Mathematics 82 (1960), p. 522-587.

[3] Е. А. Горин и В. В. Грушин, *Об определении гипоэллиптических уравнений*, Успехи Математических Наук 16 (1961), p. 163-166.

[4] G. Köthe, *Topologische lineare Räume I*, Berlin-Göttingen-Heidelberg 1960.

[5] L. Schwartz, *Théorie des distributions I/II*, Paris 1957/59.

[6] Г. Е. Шилов, *Локальные свойства решений дифференциальных уравнений в частных производных с постоянными коэффициентами*, Успехи Математических Наук 14 (1959), p. 3-44.

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