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ON CHAOTIC CURVES

BY

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I. INTRODUCTION

A non-trivial topological Hausdorff space no two of whose open sets are homeomorphic is called *chaotic*. Introducing this concept in 1970, Nix [34] asked three problems (see (A), (B) and (C) in Section III), the first two of which have been answered with the help of the continuum hypothesis [4]. In this paper some sufficient conditions are stated for a space to be chaotic. These conditions enable us to verify that some well-known examples of spaces are chaotic and, therefore, they let us solve all three problems in the affirmative without any additional assumptions. Furthermore, a modification of the dendrite constructed in 1932 by Miller [33] leads to an example of a chaotic dendrite whose points are of Menger-Urysohn order at most 4. This particular space concentrates several properties in one example. There are also discussed connections between chaotic spaces and some other kinds of spaces, e.g. rigid spaces.

The author thanks Professor Bronisław Knaster who has paid the author's attention to Zarankiewicz problem [43], and Professor Eric K. van Douwen who has pointed out some early references ([21], [22], [28], [30] and [38]).

II. DEFINITIONS AND PRELIMINARY RESULTS

1. A non-trivial topological space X is said to be *chaotic* ([34], p. 975) if for any two distinct points p and q of X there exist an open neighbourhood U of p and an open neighbourhood V of q such that no open subset of U is homeomorphic to any open subset of V. We have ([34], p. 975) the following proposition which can easily be verified:

PROPOSITION 1. *A topological space X is chaotic if and only if X is a Hausdorff space no two of whose distinct open sets are homeomorphic.*

Consider a class C of topological spaces which is closed with respect to homeomorphisms (i.e., if $X \in C$, then $h(X) \in C$ for any homeomorphism

h of X onto $h(X)$) and which is closed with respect to open subspaces (i.e., if $X \in C$ and Y is an open subspace of X, then $Y \in C$). Let A be a set and let φ be a function which assigns to each pair (X, x), where $X \in C$ and $x \in X$, an element $\varphi(X, x)$ of A in a way such that

(1) φ is topologically invariant, i.e., for every homeomorphism $h: X \to h(X)$ and for every point $x \in X$ we have $\varphi(X, x) = \varphi(h(X), h(x))$;

(2) φ is unchanged by restriction to open subspaces, i.e., if Y is an open subspace of X, then for every point x of Y we have $\varphi(Y, x) = \varphi(X, x)$.

PROPOSITION 2. *Let C and φ be as above. If a topological space $X \in C$ contains a dense set K which is preserved under homeomorphisms, i.e., for every open subset $U \subset X$ and for every homeomorphism $h: U \to h(U) \subset X$ we have $h(U \cap K) \subset K$, and, furthermore, K is such that for every two distinct points x_1 and x_2 of K we have $\varphi(X, x_1) \neq \varphi(X, x_2)$, then X is chaotic.*

Indeed, suppose that a space X satisfies the conditions formulated above and that X is not chaotic. Thus, by Proposition 1, there are two distinct open sets U and V in X which are homeomorphic under a homeomorphism $h: U \to h(U) = V$. The set K being dense in X and U being open, the intersection $U \cap K$ is dense in U. Then for an arbitrary point $x \in U \cap K$ we have

$$\varphi(X, x) = \varphi(U, x) = \varphi(h(U), h(x)) = \varphi(V, h(x)) = \varphi(X, h(x)),$$

whence we conclude $x = h(x)$. Thereby $U = h(U) = V$, a contradiction.

2. A *continuum* means a compact connected Hausdorff space. A *curve* means a one-dimensional continuum.

PROPOSITION 3. *Assume that*

(i) *a continuum is such that no two distinct non-degenerate subcontinua are homeomorphic.*

Then the continuum is chaotic.

Indeed, let X be a continuum which is not chaotic. Then, by Proposition 1, there are two distinct open subsets U and V of X which are homeomorphic under a homeomorphism $h: U \to h(U) = V$. Without loss of generality we may assume that U and V are disjoint. Let $p \in U$ and $q \in V$. The continuum X being a regular space (cf. [11], Theorem 29.12, p. 263, and Theorem 18.8, p. 121), there exist (see [11], Corollary 18.7, p. 120) open sets U_1 and V_1 such that

$$p \in U_1 \subset \overline{U}_1 \subset U \quad \text{and} \quad q \in V_1 \subset \overline{V}_1 \subset V.$$

Let C_p and C_q denote the components of $\overline{U}_1$ and $\overline{V}_1$ which contain p and q, respectively. Both C_p and C_q have common points with boundaries of $\overline{U}_1$ and $\overline{V}_1$ (see [24], § 47, III, Theorem 1, p. 172) and, therefore, they are non-degenerate. Then there are non-degenerate subcontinua $K_p \subset C_p$ and $K_q \subset C_q$ such that $h(K_p) = K_q$. Thus X does not have property (i).

It will be shown that the inverse to Proposition 3 is not true even for metric locally connected continua (see Statements 10 and 13 in the sequel).

3. De Groot has studied *rigid spaces*, i.e., spaces having a trivial autohomeomorphism group ([15], see also [18], [31], [35], [39] and [40]).

It is easy to verify the following

PROPOSITION 4. *A space X is rigid if and only if X contains a dense subset K such that each point of K is a fixed point with respect to any homeomorphism of X onto X.*

De Groot and Wille (see [18], 4, p. 444) considered also continua which are rigid under topological transformations into themselves. We rename such spaces admitting the following

Definition. A topological space X is called *strongly rigid* provided the only homeomorphism of X into itself is the identity of X onto X.

It is obvious that each strongly rigid space is rigid. An example of a rigid but not strongly rigid plane universal curve is described in [18], p. 442.

Similarly to Proposition 4 for rigid spaces we have the following proposition for strongly rigid ones, the proof of which is again immediate.

PROPOSITION 5. *A space X is strongly rigid if and only if X contains a dense subset K such that each point of K is a fixed point with respect to any homeomorphism of X onto a subspace of X.*

PROPOSITION 6. *Each chaotic space is rigid.*

In fact, if a space X is chaotic, then it is Hausdorff by Proposition 1. Suppose that X is not rigid, and let h be a homeomorphism of X onto itself which is different from the identity. Thus there exists a point $p \neq h(p) = q$. Let U and V be disjoint open neighbourhoods of p and q, respectively. Hence some open neighbourhood $U' \subset U$ of p is transformed homeomorphically under h onto an open neighbourhood $V' = h(U') \subset V$ of q, and, therefore, X is not chaotic.

The converse implication does not hold (see the final part of Section V). Moreover, we cannot substitute "strongly rigid" for "rigid" in Proposition 6: there are spaces which are chaotic but not strongly rigid (see Statements 7 and 8).

PROPOSITION 7. *If a continuum satisfies condition* (i) *of Proposition* 3, *then it is strongly rigid.*

Indeed, let X be a continuum which is not strongly rigid. Then there is a homeomorphism $h\colon X \to h(X) \subset X$ of X onto a subcontinuum $h(X)$ of X which is not the identity. Thus in X there exists a point $p \neq h(p) = q$. Let U and V be disjoint open neighbourhoods of p and q, respectively.

The further arguments are exactly the same as in the proof of Proposition 3.

Obviously, the inverse implication does not hold.

4. Recently, Scott has studied [37] another property stronger than rigidity. A space X is said to be *totally inhomogeneous* if for any two distinct points p and q of X the spaces $X\setminus\{p\}$ and $X\setminus\{q\}$ are not homeomorphic. It is clear that if X is totally inhomogeneous, then it is rigid. The converse implication fails, even for locally compact metric spaces ([37], Example, p. 489). But if X is a compact Hausdorff rigid space, then X is totally inhomogeneous ([37], Theorem 1, p. 490) and, therefore, for compact Hausdorff spaces rigidity coincides with total inhomogeneity. Thus it follows from Proposition 6 that each compact Hausdorff chaotic space is totally inhomogeneous. The converse implication does not hold ([37], p. 492).

5. A property which is related in some way to the notion discussed above is incompressibility. A topological space is said to be *incompressible* if it is homeomorphic to no proper subspace of itself (see [12]; some properties and examples of such spaces are discussed also in [10] and [19]). It is evident that each strongly rigid space is incompressible.

Another related concept is that of a reversible space. Recall that a topological space is said to be *reversible* if each continuous one-to-one mapping of the space onto itself is a homeomorphism (see [36], p. 129).

III. SOME KNOWN EXAMPLES

In 1970 Nix [34], introducing the concept of the chaotic space, proposed the following problems:

(A) Do chaotic spaces exist?

(B) Do chaotic spaces of cardinality of the continuum exist?

(C) Do completely normal, connected and locally connected chaotic spaces exist?

Problems (A) and (B) were solved in the affirmative in 1974 by Berney [4] who has constructed — assuming the continuum hypothesis — an example of a separable metric space X which has the cardinality of the continuum, is linearly ordered (in fact, X is a subspace of the unit interval of reals) and is chaotic. For an announcement of another answer (of all three problems) see [20], Theorem 4, p. 232. Scott [37], considering totally inhomogeneous spaces, has shown that, given any infinite cardinal $\varkappa$ and a compact Hausdorff space X of cardinality $2^\varkappa$ such that X has a basis of cardinality $\varkappa$ and that every non-empty open subset of X has the cardinality $2^\varkappa$, there are subspaces R_a (for $a < 2^\varkappa$) each being chaotic, dense in X and of cardinality $2^{\varkappa}$, which are pairwise disjoint and pairwise non-homeomorphic (see [37], the proof of Theorem 2, p. 491, and a remark on p. 492). Thus chaotic spaces exist in profusion. However, the spaces R_a are very non-constructive ([37], p. 492). It will be shown that many constructive examples of spaces are known in the literature (known before Nix asked questions (A), (B) and (C)) which can — not too hard — be verified as chaotic spaces. For example, let us note that as early as in 1926 Knaster asked if there exists an infinite subspace of a linear space that is homeomorphic to none of its proper subsets (i.e., that is incompressible; see [21], p. 201). This question has been answered by Kuratowski who has shown ([21], 3, p. 207-208) that the real line has a dense rigid subspace. It is easy to see that no two distinct open subsets of this example are homeomorphic, so it is chaotic.

1. As an answer to a question asked by de Groot and McDowell [17], Lozier [31] has proved the following

STATEMENT 1. *For any infinite cardinal $\varkappa$ there exists a completely regular, non-compact, zero-dimensional space $X^{\varkappa}$ of cardinality $\varkappa$ which is chaotic.*

Indeed, it is shown that any two distinct points of $X^{\varkappa}$ have different values under a topologically invariant, ordinal-valued function $k_X(x)$ (where $X = X^{\varkappa}$ and $x \in X$) which is unchanged by restriction to open subspaces (see [31], Theorem 1, p. 819; cf. [37], p. 492). Taking $k_X(x)$ for $\varphi(X, x)$ in our Proposition 2, we see that $X^{\varkappa}$ is chaotic.

2. In 1955 de Groot ([14], p. 204) raised the question "Does there exist a connected set which cannot be mapped continuously and non-degenerately onto any proper subset?". In the same year Anderson [1] raised the questions "Does there exist a non-degenerate continuum which admits only the identity or a constant mapping into itself? If so, does there exist one all of whose non-degenerate subcontinua have this property?". And Moore (see [9], p. 241) has asked whether there exists a hereditarily indecomposable continuum having property (i) (see Proposition 3). Answering all these questions by one example, Cook [9] constructed the continuum M_1 whose properties can be summarized as follows:

STATEMENT 2. *There exists a metric continuum M_1 such that*

(1) *if H is a subcontinuum of M_1 and f is a mapping of H onto the non-degenerate subcontinuum K of M_1, then $H = K$ and f is the identity mapping onto itself;*

(2) *the identity is the only mapping of M_1 onto its non-degenerate subcontinuum;*

(3) *M_1 is hereditarily indecomposable;*

(4) *no non-degenerate subcontinuum of M_1 is contained in the plane;*

(5) *M_1 is one-dimensional*;

(6) *M_1 has property* (i);

(7) *M_1 is strongly rigid*;

(8) *M_1 is chaotic.*

In fact, (1), (3) and (6) are proved in [9], Theorems 8 and 9. Condition (2) is an immediate consequence of (1). For (5) and (4) see [9], Note, p. 248. And, finally, (7) and (8) follow from (6) by Propositions 3 and 7.

In connection with de Groot's paper [14] recall that in 1931 Lindenbaum claimed without proof that (see [28], p. 114; cf. [29], p. 131, and [30], p. 185; cf. also [23], § 35, I, p. 423-426, especially Theorem 5, p. 426):

(a) there is a family of $2^{\mathfrak{c}}$ subspaces of the real line none of which is a continuous image of another ($\mathfrak{c}$ denotes the cardinality of the real line).

This was proved (in an extended form) in 1947 by Sierpiński ([38], p. 30). Kuratowski has shown ([22], p. 34 and 38) that the same family can be used for (a) and for

(b) there is a family of $2^{\mathfrak{c}}$ subspaces of the real line none of which can be embedded into another.

Kuratowski's methods are the same as the later construction by de Groot [14].

3. In 1959 Anderson and Choquet [2] constructed three continua M, M' and M'' each of which has property (i) of Proposition 3 and, thereby, is a chaotic space. We recall here other properties of these continua.

It is proved in [2], Theorem III (3), that the continuum M contains no uncountable collection of disjoint non-degenerate subcontinua. Continua having this property are called *Suslinian* (see [26], p. 131; cf. [27], p. 55). The continuum M being Suslinian, it is hereditarily decomposable ([26], 1.1; cf. [2], p. 347 and 352). M is constructed in the plane in such a way that none of its subcontinua separates the plane (see [2], Theorem III (1)), thereby, being hereditarily decomposable, it is ***hereditarily unicoherent***, i.e. every two of its subcontinua have connected intersection (see [2], p. 352; cf. [7], (38), p. 82). Continua which are hereditarily decomposable and hereditarily unicoherent are called ***λ-dendroids*** (see [6], the definition on p. 15 and Theorem 1 on p. 16). Thus M is one-dimensional (cf. [6], (1.6), p. 16). Consequently, the continuum M can be described as a plane Suslinian λ-dendroid having property (i). Therefore, we have

STATEMENT 3. *There exists a metric continuum M such that*

(1) *M is contained in the plane*;

(2) *no subcontinuum of M separates the plane*;

(3) *M is hereditarily unicoherent*;

(4) *M contains no uncountable collection of disjoint non-degenerate subcontinua*;

(5) *M is hereditarily decomposable*;
(6) *M is one-dimensional*;
(7) *M has property* (i);
(8) *M is strongly rigid*;
(9) *M is chaotic.*

The second example constructed in [2] is a continuum M' which again has property (i), which lies in the plane but, in contrast to the previous example of M, every non-degenerate subcontinuum of which separates the plane. A plane continuum having the latter property has been constructed for the first time by Whyburn [42] (for another construction see [25], Example 1, p. 276-281). A plane continuum having that property must be one-dimensional. So we have (see [2], Theorem IV, p. 353)

STATEMENT 4. *There exists a metric continuum M' such that*
(1) *M' is contained in the plane*;
(2) *every subcontinuum of M' separates the plane*;
(3) *M' is one-dimensional*;
(4) *M' has property* (i);
(5) *M' is chaotic.*

The properties of the third continuum, M'', are — apart from property (i) — again property (4) formulated in Statement 3 for M (thus M'' is Suslinian, and hence hereditarily decomposable; this implies, by the Mazurkiewicz theorem (see [24], § 48, V, Remark 2, p. 206), that M'' is one-dimensional). But an essentially new property of M'' is its hereditary non-planability. So we have ([2], Theorem V, p. 353)

STATEMENT 5. *There exists a metric continuum M'' such that*
(1) *no non-degenerate subcontinuum of M'' is contained in the plane*;
(2) *M'' contains no uncountable collection of disjoint non-degenerate subcontinua*;
(3) *M'' is hereditarily decomposable*;
(4) *M'' is one-dimensional*;
(5) *M'' has property* (i);
(6) *M'' is chaotic.*

Another example of a plane continuum which has property (i) (and which thereby is chaotic) was published in 1961 by Andrews [3]. An additional property of Andrews' example M''' is the chainability.

Recall that a metric continuum X is said to be *tree-like* (arc-like = snake-like = chainable) if it is degenerate or if, for every positive number ε, there is an ε-map throwing X into a finite tree (an arc). The following statement is known for curves:

(*) *Every tree-like curve is hereditarily unicoherent.*

In fact, let C be any subcontinuum of the tree-like curve and let g be a map from C onto the unit circle. Since C is also tree-like and the unit circle is a linear graph, g is homotopic to a constant (see [5], Theorem 1, p. 74). Consequently, C is unicoherent (see [24], § 57, II, Theorem 2, p. 437).

It can easily be proved, in the same manner as in [2], p. 352, that the Andrews example M''' contains no uncountable collection of disjoint non-degenerate subcontinua. Thus M''' is Suslinian and, as previously, it is hereditarily decomposable ([26], 1.1, p. 131), whence it is one-dimensional ([24], § 48, V, Remark 2, p. 206). Further, M''' being chainable, it is tree-like, thus, by (*), is hereditarily unicoherent. So the continuum M''' can be described as a plane chainable Suslinian λ-dendroid having property (i). Therefore, we have

STATEMENT 6. *There exists a metric continuum M''' such that it is chainable and has all properties listed in Statement 3 for the continuum M.*

4. In 1958 de Groot and Wille ([18], 2, p. 442) constructed a plane locally connected continuum P of positive measure, universal with respect to the class of all plane curves and rigid with respect to those transformations of P onto itself which are locally topological. We prove that P is chaotic. To this end we recall here the construction of P.

Let D be a disc in the plane and let $\{a_i\}$ be a dense sequence of points in the interior of D. We define a sequence of *propellers* in D. The first is bounded by a two-bladed curve having a_1 as its centre, which does not meet the boundary of D. We proceed by induction. Suppose that the first $n-1$ propellers have already been defined. Let a_n' be the first member of the sequence a_i which is in no previously constructed propeller. Then the n-th propeller is $n+1$ bladed, with centre at a_n', and lies inside a circle which misses all previously constructed propellers and the boundary of D. Moreover, we take care that the diameters of the propellers tend to zero. The space P is the disc D with the interiors of all propellers removed.

To see that P is chaotic, consider the class C of topological spaces generated by P, i.e. the class consisting of P, all open subspaces of P and all their homeomorphic images. If we take a suitable small neighbourhood U of an a_i' in P, then $U \setminus \{a_i'\}$ has $i+1$ components which have a_i' as their limit point. For every other point $p \in P \setminus \bigcup \{a_i' : i = 1, 2, \ldots\}$ and for a suitable small neighbourhood U of p in P the set $U \setminus \{p\}$ is connected. Define $\varphi(P, x)$ as the number of components of $U \setminus \{x\}$ for a suitable small neighbourhood U of x. The set K of all centres a_i' of the propellers is dense in P (see [18], p. 443). It is easy to verify that all conditions mentioned in Proposition 2 are fulfilled. Hence P is chaotic. So we have

STATEMENT 7. *There exists a metric continuum P such that*

(1) *P is contained in the plane*;

(2) *P is locally connected*;

(3) *P is one-dimensional*;

(4) *P is universal with respect to the class of all plane curves*;

(5) *P is of positive measure;*

(6) *P is rigid with respect to local homeomorphisms;*

(7) *P is not strongly rigid;*

(8) *P is chaotic.*

A similar construction may be carried out in the three-space, giving a chaotic universal locally connected curve (see [18], p. 443):

STATEMENT 8. *There exists a metric continuum P' such that*

(1) *P' is locally connected;*

(2) *P' is one-dimensional;*

(3) *P' is universal with respect to the class of all curves;*

(4) *P' is of positive measure;*

(5) *P' is rigid with respect to local homeomorphisms;*

(6) *P' is not strongly rigid;*

(7) *P' is chaotic.*

Starting with a $(k+1)$-dimensional ball, where k is an arbitrary natural, and carrying out a very similar construction we get

STATEMENT 9. *For every natural k there exists a metric continuum P'' such that*

(1) *P'' is locally connected;*

(2) *P'' is k-dimensional;*

(3) *P'' is of positive measure;*

(4) *P'' is chaotic.*

5. Another example of a locally connected rigid continuum whose construction is described in [18] is a rigid dendrite. For a very similar example see [37], p. 492. Like previously we show that this continuum is not only rigid but also chaotic. In order to attain this we recall here the construction of that continuum.

Let us start with the sequence $\{T_i\}$, $i = 1, 2, \ldots$, where each T_i is the union of i straight-line unit segments emanating from one point, called the *origin* of T_i. We proceed by induction. Define Y_0 as the unit straight-line segment. Let x be the mid point of Y_0, and define Y_1 as the union of Y_0 and a diminished copy of $T_{j_1} = T_1$ in such a way that the diameter of this copy is less than $1/2$ and that the point x is the only common point of Y_0 and T_1. Assume now that a dendrite Y_n has been defined in such a way that it is the union of finitely many straight-line segments and such that Y_n contains (properly diminished) copies of $T_1, T_2, \ldots, T_{j_n}$, i.e., of the first j_n terms of the sequence $\{T_i\}$. To define Y_{n+1} consider all arcs contained in Y_n having either branch points or end points of Y_n as their ends, and such that each interior point of every of them is a point of (Menger-Urysohn) order 2 in Y_n. Observe that we have only finitely many, say $m(n)$, such arcs; furthermore, each such arc is a straight-line segment. Let x denote its mid point. With each point x so defined we associate, in a one-to-one way, a set T_i taken from the $m(n)$ consecutive terms of the sequence $\{T_i\}$, i.e., we use in this step of the construction the sets $T_{j_n+1}, T_{j_n+2}, \ldots, T_{j_{n+1}}$, where $j_{n+1} = j_n + m(n)$. We take each mid point x as the origin of the diminished copy of the associated set T_i (where i is one of the indices $j_n+1, j_n+2, \ldots, j_{n+1}$) in such a way that the diameter of the copy of T_i is less than $1/2^{n+1}$ and that Y_n has only the point x in common with the added copy of T_i. All this can clearly be done so carefully that the resulting set Y_{n+1} is a finite dendrite. Furthermore, the whole construction can be realized so that the closure of the union of dendrites Y_n successively obtained is itself a dendrite (cf., e.g., the end-point compactification in [13]). We may suppose then that

$$Y = \overline{\bigcup_{n=0}^{\infty} Y_n}$$

is a dendrite.

To see that Y is chaotic let us observe that the set K of all branch points of Y is a dense set in Y which must be mapped into itself under any homeomorphism of Y into Y. If we define $\varphi(Y, y)$ as the Menger-Urysohn order of Y at the point y, we see that all conditions assumed in Proposition 2 are satisfied and, therefore, Y is chaotic.

We prove now that Y is not strongly rigid. To see this we observe that for every point p of Y and for every component C_p of $Y \setminus \{p\}$ the set $C_p \cup \{p\}$ contains a homeomorphic copy of the whole Y. In fact, if p is an end point of Y, then p does not separate Y ([24], § 51, V, Theorem 4, p. 293). Thus $C_p \cup \{p\} = Y$ and the conclusion holds trivially. If p is a point of order $n > 1$, then each component C_p of $Y \setminus \{p\}$ is an open set ([24], § 49, II, Theorem 4, p. 230) and, by construction, each arc which is contained in C_p contains branch points of Y which are of arbitrarily great order. Note that the set $C = C_p \cup \{p\}$ is again a dendrite. One can map homeomorphically the segment Y_0 of Y onto the maximal straight-line segment L in C which ends at p, taking care of the fact that the mid point x of Y_0 is mapped on a branch point b of this arc. Next, choose a straight-line segment emanating from b and not contained in L and map T_1 onto it homeomorphically such that the mid point of T_1 is put on a branch point of C. If necessary, we change a little the previous homeomorphism of Y_0 onto L so that

(a) mid points of the straight-line segments emanating from x are mapped onto branch points of C,

(b) the homeomorphism is not-changed at x.

Continuing this process in a routine way (see, e.g., the proof of the universality of the universal dendrite in [32], p. 321 and 322; cf. [41], III, p. 57 and 58) we can prove that the limit transformation is a homeomorphism of Y onto a subcontinuum of C. Consequently, Y is not a strongly rigid space.

So we have the following

STATEMENT 10. *There exists a metric continuum Y such that*

(1) *Y is a dendrite;*

(2) *the set of all end points of Y is dense in Y;*

(3) *for every natural n the set of all points of Y which are of order at least n is dense in Y;*

(4) *for every natural $n \geqslant 3$ the dendrite Y contains exactly one point of order n;*

(5) *for each point p of Y and for each component C_p of $Y \setminus \{p\}$ the continuum $C_p \cup \{p\}$ contains a homeomorphic image of Y;*

(6) *Y is not strongly rigid;*

(7) *Y is chaotic.*

If we start with a k-dimensional cube (for an arbitrary natural k) or with the Hilbert cube, and if we repeat the above-given construction, being careful of the fact that the set K of the origins of all added T_i's must be dense in the whole resulting space (thus — in particular — in the cube Y_0), i.e., that $\overline{K \cap Y_0} = Y_0$, then we can get rigid locally connected continua of arbitrarily high or infinite dimension. So we have (cf. [18], p. 443)

STATEMENT 11. *For every $k = 1, 2, \ldots, n, \aleph_0$, there exists a metric continuum Y' such that*

(1) *Y' is k-dimensional;*

(2) *Y' is locally connected;*

(3) *Y' is acyclic;*

(4) *Y' is chaotic.*

Let us notice that all examples of chaotic locally connected curves constructed here (see Statements 7, 8 and 10) contain points of arbitrarily great Menger-Urysohn order. However, in [18], p. 445-446, the authors described a construction of a rigid locally connected one-dimensional continuum P_1 all points of which have order less than or equal to 6. Using the same methods as for the previous example, in particular Proposition 2 in which we take the dense set of points of order 6 in P_1 as K and the number of links of the chain which is affixed to every point of K as the invariant φ, we can prove that P_1 is chaotic. Thus we have

STATEMENT 12. *There exists a metric continuum P_1 such that*

(1) *P_1 is contained in the plane;*

(2) *P_1 is locally connected;*

(3) *P_1 is one-dimensional;*

(4) *all points of P_1 are of order less than or equal to 6;*

(5) *the set of points of order* 6 *is dense in* P_1;

(6) *each open subset of* P_1 *contains a simple closed curve*;

(7) P_1 *is strongly rigid*;

(8) P_1 *is chaotic.*

Let us come back now to the Nix problems (A), (B) and (C) recalled at the beginning of this section. Since every metric space is completely normal ([11], Theorem 29.19, p. 265), every metric locally connected continuum which is chaotic constitutes an example that gives the affirmative answer to problem (C). Furthermore, since every metric continuum is a separable space, it has the cardinality of the continuum. Therefore, each of the examples mentioned in Statements 7-12 answers all three problems (A), (B) and (C) affirmatively, without any additional assumptions, in particular without the continuum hypothesis.

IV. A CHAOTIC DENDRITE

The example of an incompressible space is a simple closed curve, while a segment is not. As another example, in particular an incompressible local dendrite, one can take the union of a finite dendrite and of finitely many disjoint circles such that every end point of the dendrite belongs to exactly one circle which is disjoint with the dendrite out of this end point.

Zarankiewicz [43] proposed a problem which can be reformulated as follows:

(D) Does there exist a dendrite which is incompressible?

The problem has been solved in the affirmative by Miller [33]. The idea of the proof is based on the following proposition (see [33], Theorem, p. 831).

PROPOSITION 8. *A dendrite X is incompressible if it contains a set K such that*

(1) *each point of K is a fixed point with respect to any homeomorphism of X onto a subcontinuum of X;*

(2) *each point of X which is not its end point lies on an arc contained in X and having its end points in K.*

Each strongly rigid space is incompressible by the definition, but not conversely, as the example of a simple closed curve shows. Note that the dendrite Y constructed above is chaotic but very far from being incompressible (this is because it has property (5) of Statement 10) while Miller's example of the dendrite S is incompressible but not chaotic; it is not rigid even since it contains open arcs as open subsets.

We show however that a modification of Miller's example leads to the construction of a chaotic dendrite D. This dendrite D justifies in its turn an affirmative answer to problems (A)-(D), but also represents a slightly stronger result than that of de Groot and Wille in [18], p. 445.

The following statement will be proved:

STATEMENT 13. *There exists a metric continuum D such that*

(1) *D is a dendrite;*

(2) *each point of D is of order not greater than* 4;

(3) *for every $n = 1, 2, 3$ or 4 the set of all points of D which are of order n is dense in D;*

(4) *D is strongly rigid;*

(5) *D is chaotic.*

1. Definition of the dendrites $E_{x_1x_2\ldots x_k}$. For the reader's convenience, we repeat here the basic constructions described in [33], § 2.

The definition of the sets $E_{x_1x_2\ldots x_k}$, where x_i $(i \leqslant k)$ can have either of the values 1 and 2, will be by induction. Within a linear interval ab ordered from a to b by $<$ choose points a_n so that

$$a_{n+1} < a_n \quad \text{and} \quad \lim_{n\to\infty} a_n = a.$$

Within each interval $a_{n+1}a_n$ choose points $a_{n,m}$ so that

$$a_{n,m} < a_{n,m+1} \quad \text{and} \quad \lim_{m\to\infty} a_{n,m} = a_n.$$

At each point a_n and $a_{n,m}$ erect a perpendicular to ab. Take these perpendiculars so that for any positive number ε only finitely many of them have their lengths greater than ε. The set of points obtained in this way is called E_1. The point a is called the *origin* of E_1, and the perpendiculars which we have erected are all referred to as perpendiculars of rank 1. It is clear that E_1 is a dendrite.

Everything is the same in the definition of E_2 except for the one change: the points $a_{n,m}$ are taken within the interval $a_{n+1}a_n$ so that

$$a_{n,m+1} < a_{n,m} \quad \text{and} \quad \lim_{m\to\infty} a_{n,m} = a_{n+1}.$$

So E_2 is also a dendrite.

Let us now suppose that we have defined dendrites $E_{x_1x_2\ldots x_k}$, where $x_i = 1$ or 2 for $i \leqslant k$. Let us suppose furthermore that we have defined the expressions origin of $E_{x_1x_2\ldots x_k}$ and perpendiculars of rank k of $E_{x_1x_2\ldots x_k}$. To define the set $E_{x_1x_2\ldots x_k1}$ we proceed as follows. We replace each perpendicular of rank k of $E_{x_1x_2\ldots x_k}$ by a set E_1 whose origin is the foot of the perpendicular. Furthermore, we do this, as we clearly can, so that the resulting set $E_{x_1x_2\ldots x_k1}$ is a dendrite. By the origin of $E_{x_1x_2\ldots x_k1}$ we mean merely the origin of $E_{x_1x_2\ldots x_k}$, and by perpendiculars of rank $k+1$ of $E_{x_1x_2\ldots x_k1}$ — the perpendiculars of rank 1 of the sets E_1 used in obtaining $E_{x_1x_2\ldots x_k1}$ from $E_{x_1x_2\ldots x_k}$.

Everything is the same in the definition of $E_{x_1x_2\ldots x_k2}$ except for the one change: in obtaining $E_{x_1x_2\ldots x_k2}$ from $E_{x_1x_2\ldots x_k}$ we use sets E_2 instead of E_1.

2. Construction of the dendrite D. This construction, after Miller's construction in [33], § 3, is achieved by the use of the following sequence of sets: $E_1, E_{21}, E_{221}, \ldots, E_{22\ldots 21}, \ldots$

First re-label these sets in the following order: $W_1, W_2, W_3, \ldots, W_n, \ldots$

We begin with a set W_1 whose origin is a point a and adjoin to it three line segments ac, ad and ae so that the only point which any two of the sets W_1, ac, ad and ae have in common is the point a. Let us denote the resulting dendrite by D_1. Observe that just one of the four distinct arcs $ab \subset W_1$, ac, ad and ae of D_1 which meet at a (namely the arc ab) has the property that there is a sequence of branch points on it of Menger-Urysohn's order 3 which converges to a.

So far our construction and Miller's one in [33] are exactly the same. But now we proceed in another way. We consider the arc in D_1 having either two branch points of D_1 or a branch point and an end point of D_1 as its end points and such that all other points of the arc are not branch points of D_1. In other words, we consider the maximal arc in D_1 having the property that all its points of order 2 are of order 2 in D_1. It is evident from the construction that every such arc is a line segment. Denote the mid point of this segment by x. We obtain, of course, a countable infinity of points x. With this countable infinity of points we associate, in a one-to-one way, the sets W_n of odd indices n, i.e.,

$$W_3, W_5, W_7, \ldots, W_{2m+1}, \ldots,$$

and take x as the origin of the associated set $W_{2m+1} = W(x)$ in such a way that D_1 and $W(x)$ have only the point x in common. Moreover, we attach to the point x a straight-line segment having x as one end point and having only x in common with $W(x) \cup D_1$. All this can be clearly done so that the resulting set D_2 is a dendrite. Observe that, for every point y of D_2 of Menger-Urysohn's order 4, just one of the four essentially distinct arcs of D_2 which meet at y (namely the arc contained in $W(x)$ if $y = x$ or in W_1 if $y = a$) has the property that there is a sequence of branch points on it of order 3 which converges to y. Now, D_3 is related to D_2 in the same way as D_2 is related to D_1, except that we make use of sets $W_{2(2m+1)}$ instead of sets W_{2m+1}. In general, D_{n+1} is related to D_n in the same way as D_n is related to D_{n-1} except that we make use of sets $W_{2^{n-1}(2m+1)}$ instead of sets $W_{2^{n-2}(2m+1)}$. It can be observed easily that, for every point y of D_n of order 4, just one of the four essentially disjoint arcs in D_n which meet at y has the property mentioned previously. It is well known that such a construction can be carried through so that the closure of the union of the dendrites D_n successively obtained is itself a dendrite. We may suppose then that $D = \overline{\bigcup_{n=1}^{\infty} D_n}$ is a dendrite.

3. The proof of the properties of D. We notice first that any branch point of D is either of order 3 or of order 4. The points of order 4 are the point a of D_1 and the points x which arise at successive stages of our process of construction. We denote the set of all points of order 4 of D by K. Since we take in the construction the mid points x of all maximal arcs whose interior points are of order 2 in D, the set K is dense in D. Furthermore, notice that the above-mentioned property of points of order 4 in each D_n is preserved in D. Precisely, if y is in K, then there is just one arc of four arcs in D, ending at y and disjoint out of it, such that it contains a sequence of branch points of order 3 converging to y. Thus an open neighbourhood about a point $y \in K$ contains points of order 3 in D and, henceforth, the density of the set of all points of order 3 in D follows from the density of the set K of all points of order 4 in D. The set of all points of order 2 is always dense in any dendrite ([32], p. 309; cf. [24], § 51, VI, Theorem 8, p. 302). Finally, it can be observed simply by the construction of D that the set of all end points of D is dense in D.

Now, we show that D is chaotic. Let p and q be two distinct points of D. Let $r \in pq \setminus \{p, q\}$, where pq means the (only) arc having p and q as its ends. Let U and V be defined as components of the set $D \setminus \{r\}$ that contain p and q, respectively. Take arbitrary open sets $U' \subset U$ and $V' \subset V$ and suppose, on the contrary, that there is a homeomorphism h of U' onto V'. First, we notice that $K \cap U' \neq \emptyset \neq K \cap V'$, the set K being dense in D. Further, observe that h must carry each point of $K \cap U'$ into a point of $K \cap V'$, since no point of D is of order greater than 4 and K contains all points of order 4 of D. Take $u \in K \cap U'$ and put $v = h(u) \in K \cap V'$. Let us assume for definiteness (the argument is similar in the opposite case) that the set W_n which has the point u as its origin is of lower index than the set W_n which has v as its origin. Consider now the arc $ub_u \subset U'$ which is the only arc of four arcs ending at u and disjoint out of u that contains a sequence of branch points of order 3 converging to u. Assume that $vb_v \subset V'$ has a similar meaning. It is clear that there are a subarc ub'_u of ub_u and a subarc vb'_v of vb_v such that $h(ub'_u) = vb'_v$. We can take b'_u so close to u and, similarly, b'_v so close to v that ub'_u and vb'_v are straight-line segments. Any branch point of D on ub'_u is mapped under h into a branch point of D on vb'_v. If $W(u) = W_1$, we see that we have already reached a contradiction. For $W_1 = E_1$ and $W(v) = W_n = E_{22\ldots21}$, which means that ub'_u contains branch points being limit points of branch points from the left, while vb'_v contains no such points. If $W(u) = W_2$, we fix our attention upon some one branch point of D interior to ub'_u. Let us denote this point by r_u and put $r_v = h(r_u) \in vb'_v$. Consider the perpendiculars to ub'_u and vb'_v at r_u and r_v, respectively. Denote these perpendiculars by $r_u s_u$ and $r_v s_v$. Now, since $W_2 = E_{21}$, r_u is a limit point along $r_u s_u$ of branch points of D which are, in turn, limit points of branch points of D from below along $r_u s_u$, while $r_v s_v$ contains no such points since $W(v) = W_n = E_{22\ldots21}$. It is obvious that the argument exemplified above can be extended to apply to the general case where $W(u) = W_n$ and $W(v) = W_m$ for $n < m$ and $m < n$, respectively. It follows that $u = v$ if $u, v \in K$ and $h(u) = v$. Thus D is chaotic.

Exactly the same arguments can be applied to show that if h is any homeomorphism of D onto a subcontinuum of D, then each point of K must be fixed under h. Since K is dense in D, we conclude from Proposition 5 that D is strongly rigid. In particular, it is incompressible.

An outline of the above-given construction has been published in [8].

V. FINAL REMARKS

1. The existence of a subset K in a dendrite X such that K has properties (1) and (2) of Proposition 8 is sufficient but not necessary for the dendrite X to be incompressible. Indeed, to construct a dendrite which is incompressible but which does not contain any subset K satisfying (1) and (2) of Proposition 8 we proceed as follows. Locate the dendrite Y (see Statement 10) in the plane such that there exists a straight line L in the plane which has exactly one point, namely an end point e of Y, in common with Y. Let Y^s denote the image of Y under the symmetry s with respect to L. Thus $Y \cap Y^s = \{e\}$ and we see that $Y \cup Y^s$ is a dendrite. It is easy to verify that the only non-trivial homeomorphism of $Y \cup Y^s$ onto a subcontinuum of it is the symmetry s. Thus $Y \cup Y^s$ is incompressible. But s leaves only the point e fixed, whence we conclude that if a set K satisfies (1), then $K = \{e\}$; but then (2) is not true.

2. To show that the converse of Proposition 6 does not hold we give an example of a strongly rigid but not chaotic locally connected curve. To this end take two copies D' and D'' of the dendrite D described in Statement 13, fix an end point e' of D', and let e'' be the copy of e' in D''. Next identify e' and e'', respectively, with two different points of the locally connected and strongly rigid curve P_1 of Statement 12. The union $X = D' \cup P_1 \cup D''$ obtained in this way is obviously a locally connected curve. The two components of $X \setminus P_1$, i.e., the open sets $D' \setminus \{e'\}$ and $D'' \setminus \{e''\}$, are clearly homeomorphic and, therefore, X is not chaotic. Consider an arbitrary homeomorphism h of X onto a subcontinuum of itself. Since P_1 contains a dense set of points of order 6 and there is no such point in D' and D'', we conclude that $h(P_1) \subset P_1$, whence — P_1 being a strongly rigid space — h must be the identity on P_1. Thus $h(e') = e'$, $h(e'') = e''$, and, therefore, $h(D') \subset D'$ and $h(D'') \subset D''$. But D is strongly rigid and h must be the identity on both D' and D''. So h is the identity on the whole X, i.e., X is strongly rigid.

Added in proof. The following four papers are essentially related to the topic of this article:

A. S. Besicovitch, *Totally heterogeneous continua*, Proceedings of the Cambridge Philosophical Society 41 (1945), p. 96-103.

V. Kannan and M. Rajagopalan, *Constructions and applications of rigid spaces*, Advances in Mathematics 29 (1978), p. 89-130.

— *Constructions and applications of rigid spaces II*, American Journal of Mathematics 100 (1978), p. 1139-1172.

— *Constructions and applications of rigid spaces III*, Canadian Journal of Mathematics 30 (1978), p. 926-932.

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