COLLOQUIUM MATHEMATICUM

VOL. XX 1969 FASC. 2

A NOTE ON THE LEAST ELEMENT MAP*

BY

A. R. BEDNAREK (GAINESVILLE, FLORIDA)

In their paper [2] Franklin and Wallace consider extensions of an earlier result of Capel and Strother [1] which asserts that *if a compact Hausdorff space is provided with a closed partial order, then the function which maps each closed set with least element into its least element is continuous* (=continuous with respect to the Vietoris topology). In particular, Franklin and Wallace relax the assumptions that the space be compact and that the relation be a partial order. The purpose of this note is to demonstrate that there is no loss of generality in the assumption that the space be compact.

If R is a *relation on* X, that is, $R \subset X \times X$, a set $A \subset X$ is said to have a *least element* x (when clarity demands, we shall say *R-least element*) if and only if

(i) $$x \in A,$$

(ii) $$x \times A \subset R,$$

(iii) $$(A \times x) \cap R = \{(x, x)\}.$$

The result of Franklin and Wallace may be stated as follows:

THEOREM A (Franklin-Wallace). *If a topological space X is provided with a closed relation, the function which maps each closed compact set with least element into its least element is continuous.*

Actually the statement of this result in [2] includes the assumption that the space is Hausdorff and attention is restricted to compact sets with least element. However, the assumption that the space be Hausdorff is not necessary in the Franklin-Wallace proof (it is used for the other results generated in [2]), so the Theorem A follows directly from [2].

As noted above, our purpose here is to prove that Theorem A follows from, and thus is equivalent to, the following:

* The author is grateful to the National Science Foundation for its support of this research under GP 6505.

THEOREM B. *If X is a compact space provided with a closed relation, the function which maps each closed set with least element into its least element is continuous.*

Proof. Let X be a topological space, R a closed relation on X and Σ the family of closed compact subsets of X having an R-least element. We let X_∞ be the one-point compactification on X, and define the relation R_∞ on X_∞ by:

$$R_\infty = R \cup (\infty \times X_\infty) \cup (X_\infty \times \infty).$$

Furthermore, we let Σ_∞ denote the *closed* subsets of X_∞ having R_∞-least element.

(i) R_∞ is closed.

This follows directly from the fact that $(X_\infty \times X_\infty)\setminus R_\infty = (X \times X)\setminus R$.

(ii) $\Sigma_\infty = \Sigma \cup \{\{\infty\}\}$.

If $K \in \Sigma$, then K is a closed compact subset of X having an R-least element. Since $R \subset R_\infty$ and $\infty \notin K$, K has an R_∞-least element. Moreover, K is a closed subset of X_∞, thus $K \in \Sigma_\infty$. Since $(\infty, \infty) \in R_\infty$, $\{\infty\}$ has an R_∞-least element, and being closed, $\{\infty\} \in \Sigma_\infty$.

If $K \in \Sigma_\infty$ and $\infty \notin K$, then K has an R-least element. Since K is closed in X_∞, $X_\infty \setminus K$ is open; that is, $K = X_\infty \setminus (X_\infty \setminus K)$ is a closed compact subset of X, therefore $K \in \Sigma$.

If $\infty \in K \in \Sigma_\infty$, then ∞ must be the R_∞-least element of K. But $X_\infty \times \infty \subset R_\infty$ implies that $K = \{\infty\}$, consequently $\Sigma_\infty = \Sigma \cup \{\{\infty\}\}$.

Now applying Theorem B, we get that the R_∞-least element map $f_\infty \colon \Sigma_\infty \to X_\infty$ is continuous. If $f \colon \Sigma \to X$ is the R-least element map, and if U is open in X, then since $f^{-1}(U) = f_\infty^{-1}(U) \cap \Sigma$ and $f_\infty^{-1}(U)$ is open in Σ_∞, we see that f is continuous.

Remark. It would be interesting to determine whether or not there is any loss in generality in assuming the relation is a *partial order* (**P 671**). Since, on the sets having R-least element the relation enjoys properties close to those of an order it seems plausible that similar to the above, one might define a partial order P related to the original relation so that $\Sigma_P \supset \Sigma_R$ and $f_P | \Sigma_R = f_R$.

REFERENCES

[1] C. E. Capel and W. L. Strother, ***Multi-valued functions and partial order***, Portugaliae Mathematica 17 (1958), p. 41-47.

[2] S. P. Franklin and A. D. Wallace, ***The least element map***, Colloquium Mathematicum 15 (1966), p. 217-221.

Recu par la Rédaction le 29. 5. 1968