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FIXED POINT THEOREMS FOR PSEUDO-MONOTONE MULTIFUNCTIONS

BY

R. E. SMITHSON (LARAMIE, WYOMING)

1. Pseudo-monotone multifunctions. In 1962. Ward [7] published some results on pseudo-monotone functions. These results were related to some of his earlier work reported in [6] and they were also extensions of results of Hamilton [1] and Kelley [2]. The purpose of the present paper is to extend the notion of a pseudo-monotone function to multifunctions and to show that Ward's results hold for continuum valued multifunctions.

By a *multifunction* $F: X \to Y$ we mean a correspondence such that $F(x)$ is a non-empty subset of Y for all $x \in X$. Further, F is said to be *continuum* (*compact*) *valued* in case $F(x)$ is a subcontinuum (compact subset) of Y for each $x \in X$, where a continuum is a compact, connected, Hausdorff space. We say that F is *upper semicontinuous* (u. s. c.) in case $F^{-1}(A) = \{x\colon F(x) \cap A \neq \emptyset\}$ is closed for each closed set $A \subset Y$. The symbol $\emptyset$ is used to denote the empty set and A^* denotes the closure of A. We shall also use the following definitions.

Definitions. Let F: $X \to Y$ be a multifunction on the space X into the space Y.

(1) F is *monotone* iff $F^{-1}(y)$ is connected for each $y \in Y$.

(2) F is *pseudo-monotone* iff whenever $A \subset X$ and $B \subset Y$ are closed connected sets with $B \subset F(A)$, there is a component C of $F^{-1}(B) \cap A$ such that $B \subset F(C) = \bigcup \{F(x)\colon x \in C\}$.

In [7] Ward noted that in general the notions of monotone and pseudo-monotone maps are independent. However, as we shall see below, they are related in certain spaces. The first lemma is well known and its proof is omitted (see Smithson [4]).

LEMMA 1. *Let F: $X \to Y$ be a u. s. c. continuum valued multifunction on X into Y. If A is a compact, connected subset of X, then $F(A)$ is a compact, connected subset of Y. Hence, if X any Y are continua, the image of any subcontinuum of X is a subcontinuum of Y.*

LEMMA 2. *Let $F: X \to Y$ be a u. s. c., compact valued, monotone multifunction on the compact, T_2-space X into the Hausdorff space Y. If B is a connected subset of $F(X)$, then $F^{-1}(B)$ is connected.*

Proof. Suppose that $F^{-1}(B) = A_1 \cup A_2$, where A_1 and A_2 are non-empty separated sets. Then set $B_i = \bigcup \{F(x): x \in A_i\} \cap B$ for $i = 1, 2$, clearly, B_1 and B_2 are non-empty. To see that they are separated suppose that $y \in B_1^* \cap B_2$. Then, since $F^{-1}(y)$ is connected, $F^{-1}(y) \subset A_2$. On the other hand, $F(A_1^*)$ is a closed set which contains B_1. Hence, $B_1^* \subset F(A_1^*)$ and therefore there is an $x \in A_1^*$ such that $y \in F(x)$. But then $x \in A_1^* \cap A_2$ which is a contradiction. Thus $A_1^* \cap A_2 = \emptyset$ and similarly $B_1 \cap B_2^* = \emptyset$. Since this contradicts the assumption that B is connected, we conclude that $F^{-1}(B)$ is connected.

Recall that a continuum is *hereditarily unicoherent* in case the intersection of any two of its subcontinua is connected.

LEMMA 3. *Let X be a hereditarily unicoherent continuum and let Y be a Hausdorff space. If $F: X \to Y$ is a u. s. c., compact valued, monotone multifunction, then F is pseudo-monotone.*

Proof. Let $A \subset X$, $B \subset Y$ be closed connected sets with $B \subset F(A)$. Then $F^{-1}(B)$ is closed and connected by Lemma 2. Thus $C = F^{-1}(B) \cap A$ is a subcontinuum of X. Finally, if $y \in B$, then there is an $x \in A$ such that $y \in F(x)$ and thus $x \in C$. Hence, $B \subset F(C)$, and F is pseudo-monotone.

The principal tool in achieving the desired extensions of Ward's results is the following lemma.

LEMMA 4. *Let $F: X \to X$ be a u. s. c., pseudo-monotone, continuum valued multifunction on the continuum X. If X contains a cut point p, then there is a proper subcontinuum X_0 of X such that $F(x) \cap X_0 \neq \emptyset$ for all $x \in X_0$.*

Proof. Suppose $X \setminus p = A \cup B$, where A and B are non-empty separated sets. Define a function $r: X \to A^*$ by: $r(x) = p$ if $x \in B$ and $r(x) = x$ if $x \in A$. Then r is a continuous function. Now define a multifunction $G: A^* \to A^*$ by $G(x) = r(F(x))$. Then G is u.s.c. and continuum valued. Now let $K = \bigcap_{n=1}^{\infty} G^n(A^*)$.

Since A^* is compact and connected, K is a subcontinuum of A^*. Furthermore, $K \subset F(K)$. For let $x \in K$; then for each $n > 1$ there exists an $x_n \in G^{n-1}(A^*)$ such that $x \in G(x_n)$. Further, if x_0 is a cluster point of the sequence $\{x_n; n > 1\}$, $x \in F(x_0)$ since F is u. s. c. But $G^{n+1}(A^*)$ is compact and contained in $G^n(A^*)$ and thus there is a cluster point of the sequence in K.

Now, since $K \subset F(K)$ and since F is pseudo-monotone, there exists a component K_1 of $K \cap F^{-1}(K)$ such that $K \subset F(K_1)$ and thus we have $F(x) \cap K \neq \emptyset$ for all $x \in K_1$. Since $K_1 \subset K$, $K_1 \subset F(K_1)$, and we proceed inductively. Suppose a chain of subcontinua $K_0, \ldots, K_n$ of K have been defined such that $F(x) \cap K_i \neq \emptyset$ for $x \in K_{i+1}$, $i = 0, \ldots, n-1$, and such that $K_{i-1} \subset F(K_i)$ $(i = 1, \ldots, n)$, where $K_0 = K$. Then $K_n \subset F(K_n)$ and another application of the pseudo-monotonicity of F gives a component K_{n+1} of $K_n \cap F^{-1}(K_n)$ such that $K_n \subset F(K_{n+1})$ and we have $F(x) \cap K_n \neq \emptyset$ for all $x \in K_{n+1}$. Then set $X_0 = \bigcap_{i=0}^{\infty} K_n$. Clearly, X_0 is a subcontinuum of X and if $x \in X_0$, then $F(x) \cap K_n \neq \emptyset$ for all n. For $x \in X_0$, let $\mathscr{F} = \{F(x) \cap K_n: n \geqslant 0\}$. Then $\mathscr{F}$ has the finite intersection property and therefore $\bigcap \mathscr{F} \neq \emptyset$. Thus $F(x) \cap X_0 \neq \emptyset$ and the lemma is proved.

Remark. If $F: X \to X$ is u. s. c. and if $X_0 \subset X$ such that $F(x) \cap X_0 \neq \emptyset$ for all $x \in X_0$, then $G: X_0 \to X$, defined by $G(x) = F(x) \cap X_0$, is u. s. c. Furthermore, if X is a hereditarily unicoherent continuum, if X_0 is a subcontinuum of X and if F is pseudo-monotone, then G is pseudo-monotone.

THEOREM 1. *Let X be a hereditarily unicoherent continuum and let $F: X \to X$ be a u. s. c., continuum valued, pseudo-monotone multifunction on X. Then there exists a subcontinuum X_0 of X which is minimal with respect to $F(x) \cap X_0 \neq \emptyset$ for all $x \in X_0$ and X_0 contains no cut points.*

Proof. In view of Lemma 4 and the remark following it all we need to show is that there is a subcontinuum X_0 which is minimal with respect to $F(x) \cap X_0 \neq \emptyset$ for all $x \in X_0$. For this let S be the set of all subcontinua of X with the given property. Partial order S by inclusion and let S_0 be a chain in S. Then $K = \bigcap S_0$ is a subcontinuum of X which is contained in each member of S_0. Let $x \in K$ and set $\mathscr{F} = \{F(x) \cap Y: Y \in S_0\}$. Since S_0 is a chain, $\mathscr{F}$ has the finite intersection property and so $\bigcap \mathscr{F} \neq \emptyset$. Hence, $F(x) \cap K \neq \emptyset$. Therefore, by Zorn's Lemma, S has a minimal element and the theorem follows.

In [6] Ward stated that if X is a continuum each of whose non-degenerate subcontinua has a cutpoint, then X is hereditarily unicoherent. Thus the following corollaries are immediate from Theorem 1.

COROLLARY 1.1. *Let X be a continuum such that each of its non-degenerate subcontinua has a cut point. If $F: X \to X$ is a u. s. c., continuum valued, pseudo-monotone multifunction, then there exists an $x_0 \in X$ such that $x_0 \in F(x_0)$.*

COROLLARY 1.2. *Let X be a continuum such that each of its non-degenerate subcontinua has a cut point. If $F: X \to X$ is a u. s. c., continuum valued, monotone multifunction, then there exists $x_0 \in X$* such that $x_0 \in F(x_0)$.

2. Partially ordered topological spaces. In [7] Ward gave a generalization of the single valued version of Theorem 1 to partially ordered topological spaces. In this section we extend this result as well as another related result (see [5]) to multifunctions. The results of this section are also related to results of Smithson [3].

We shall follow the conventions in [5]. Thus a *partially ordered topological space* (POTS) is partially ordered set $(X, \leqslant)$ together with a topology in which the sets

$$L(x) = \{a:\ a \leqslant x\} \quad \text{and} \quad M(x) = \{a:\ x \leqslant a\}$$

are closed for each $x \in X$. Further, two elements x and y of X are *comparable* if $x \leqslant y$ or $y \leqslant x$.

By a unit for X, we mean an element $e \in X$ such that $L(e) = X$. We say that a subset $A \subset X$ is bounded away from the unit e in case there exists a $y \in X$, $e \neq y$, such that $A \subset L(y)$. Note that X can have at most one unit. We also need the following conditions:

I. *Let* $F\colon X \to X$ *be a multifunction on* $(X, \leqslant)$. *If* $x_1 \leqslant x_2$ *and if* $y_1 \in F(x_1)$, *then there is a* $y_2 \in F(x_2)$ *such that* $y_1 \leqslant y_2$.

II. *Let* $F\colon X \to X$ *be a multifunction on* $(X, \leqslant)$. *If* $x_1 \leqslant x_2$ *and if* $y_2 \in F(x_2)$, *then there is a* $y_1 \in F(x_1)$ *such that* $y_1 \leqslant y_2$.

We also need the following lemma which is immediate from the definition of upper semi-continuity.

LEMMA 5. *Let* $F\colon X \to X$ *be a u. s. c., compact valued multifunction on the Hausdorff topological space* X, *and let* $\{x_n;\ n \geqslant 1\}$ *be a sequence in* X *such that* $x_n \in F(x_{n+1})$. *If* x_0 *is a cluster point of* $\{x_n;\ n \geqslant 1\}$, *then some element of* $F(x_0)$ *is a cluster point of the sequence.*

Theorem 2 is an extension of Theorem 8 of [5].

THEOREM 2. *Let* X *be a compact Hausdorff POTS which contains a unit* e. *Let* $F\colon X \to X$ *be a u. s. c., compact valued multifunction on* X. *If there exists an* $x \in X$, $x \neq e$, *such that* x *is comparable to some point in* $F(x)$ *and if for each such* x *either* (i) *there is a monotone sequence* $\{x_n\colon x_n \in F(x_{n-1}),\ x_0 = x\}$ *which is bounded away from* e *or* (ii) $F^{-1}(x) \cap L(x) \neq \emptyset$, *then there exists an* $x_0 \in X$ *such that* $x_0 \in F(x_0)$.

Proof. First suppose that x is comparable to a member of $F(x)$ and let $\{x_n\colon x_n \in F(x_{n-1})\}$ be a monotone sequence which is bounded away from e. Then by results of Ward [5], the sequence converges. If $x_n \to z_0$, then $z_0 \neq e$, and since $x_n \in F(x_{n-1})$ and F is u. s. c., it follows that $z_0 \in F(z_0)$ and we are done.

Next suppose that x is comparable to a point in $F(x)$ and that $F^{-1}(x) \cap L(x) \neq \emptyset$. Thus let $y_1 \in F^{-1}(x)$ and suppose that $y_1 \leqslant x$. Then y_1 is comparable to a point in $F(y_1)$ and select $y_2 \in F^{-1}(y_1)$ such that $y_2 \leqslant y_1$. (If case (i) ever applies to a member y_n of the sequence we are constructing, we would be done.) Thus assume that $y_1, \ldots, y_n$ have been chosen so that $y_i \in F(y_{i+1})$ and $y_{i+1} \leqslant y_i$ for all $i = 1, \ldots, n-1$, and select $y_{n+1} \in F^{-1}(y_n)$ such that $y_{n+1} \leqslant y_n$. Then the sequence $\{y_n;\ n \geqslant 1\}$ must have a limit point y_0, and $y_0 \in F(y_0)$ by Lemma 5. Thus in either case the theorem holds.

The order theoretic analogue of pseudo-monotone for a function $F: X \to X$ on a partially ordered set is the condition

III. *If $x \leqslant y$ for some $y \in F(x)$, then $F^{-1}(x) \cap L(x) \neq \emptyset$.*

We also need

IV. *If $x_1 \leqslant z \leqslant x_2$, if $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$ with $y_1 \leqslant y_2$, then there exists a $y_3 \in F(z)$ such that $y_1 \leqslant y_3 \leqslant y_2$.*

Further, we shall use the following two properties on a partially ordered set X with unit e.

(α) *There exist elements a, b and p* of X *such that $L(a) \cap L(b) = p$.*

(β) *If $x \in X \setminus (L(a) \cup L(b))$, then $p \leqslant x$ and each of the sets $L(x) \cap L(a)$ and $L(x) \cap L(b)$ has a supremum.*

THEOREM 3. *Let X be a non-degenerate compact, Hausdorff POTS with unit e which has properties* (α) *and* (β). *Let F be a u. s. c., point closed multifunction on X which satisfies Conditions* I, II, III *and* IV. *If for each minimal element q, $F(q)$ contains a minimal element, then there is an $x_0 \in X$, $x_0 \neq e$, such that $x_0 \in F(x_0)$.*

Proof. If there is an $x \in X$, $x \neq e$ such that there is a $y \in F(x)$ with y less than or equal to x, then the desired fixed point is obtained from Theorem 2. (By using Condition II we satisfy (i) of Theorem 2.)

Thus suppose that this does not occur and let a, b and p be the elements given in (α). By (β) p is a minimal element and hence, $f(p)$ contains a minimal element q. Further, if $q \notin L(a) \cup L(b)$, then $p \leqslant q$ by (β) and, since q is minimal, q and p would be equal. In this case, p is a fixed point with $p \neq e$ (since X is non-degenerate). Thus either $q \leqslant a$ or $q \leqslant b$. Suppose that $q \leqslant a$; by Condition I, there is a $y_1 \in F(a)$ such that $q \leqslant y_1$. Note that $y_1 \notin L(b) \cup L(a)$, and so by ($\beta$) there exists a $t_1 = \sup(L(y_1) \cap L(a))$ with $p \leqslant t_1$. Then, by Condition IV, there is a $y_2 \in F(t_1)$ such that $q \leqslant y_2 \leqslant y_1$. Also $y_2 \in X \setminus (L(a) \cup L(b))$. Hence, set $t_2 = \sup(L(y_2) \cap L(a))$ as before. Then $t_2 \leqslant a$ and so $p \leqslant t_2 \leqslant t_1$. Inductively, construct a sequence t_n such that $p \leqslant t_n \leqslant t_{n-1}$ and $t_n = \sup(L(y_n) \cap L(a))$, where $y_n \in F(t_{n-1})$, with $q \leqslant y_n \leqslant y_{n-1} \leqslant y_1$.

Now, the sequence $t_n \to t_0$ since it is monotone decreasing and $t_0 \leqslant t_n$ for all n. Also $t_0 \leqslant y_n$ for each n and so $t_0 \leqslant y_0$, where $y_n \to y_0$. But $y_n \in F(t_{n-1})$. Therefore, since F is u. s. c., $y_0 \in F(t_0)$ and hence, t_0 is less than or equal to some element of $F(t_0)$. This implies that Condition (i) of Theorem 2 is satisfied. That Condition (ii) is satisfied follows from Condition III and the above discussion. Consequently, Theorem 2 applies and the result follows.

Each single valued order preserving function satisfies Conditions I, II and IV. Hence, Theorem 3 is an extension of Ward's result.

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