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WEAK LIMIT OF MEASURES THAT IS NOT A POINTWISE LIMIT ON COMPACT SETS

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In this note we construct a sequence (μ_n) of finite Borel measures on the set Q of rational numbers, weakly convergent to a finite measure μ and such that, for any non-empty compact set $K \subset Q$, the sequence $\mu_n(K)$ does not converge to $\mu(K)$. This solves the problem P 1005 posed by Fleming Tøpsoe ([1]).

Before presenting the solution it will be useful to notice that *every compact set in Q has an isolated point.* Indeed, otherwise the set would be countable and perfect in the space of real numbers, which contradicts the well-known fact that each perfect set of a complete metric space is uncountable.

Let $\{q^i: i \in N\}$ be a fixed denumeration of Q. Consider an infinite matrix $(q_j^i) \subset Q$ such that

(i) $|q_j^i - q^i| < 1/j$ for all $i, j \in N$;

(ii) each rational number appears in the matrix at most once.

Such a matrix exists, since Q is dense in itself.

Now, we construct the desired sequence of measures as follows:

$$\mu_n(\{q_n^i\}) = 2^{-i} \quad \text{and} \quad \mu_n(\{q\}) = 0 \text{ for } q \notin \{q_n^i: i \in N\}.$$

The sequence $\{\mu_n\}$ converges weakly to the measure μ, where $\mu(\{q^i\}) = 2^{-i}$, $i \in N$.

In fact, if f is a continuous function on Q such that $|f| < B$, then

$$\Big|\int_Q f\,d\mu_n - \int_Q f\,d\mu\Big| = \Big|\sum_{i=1}^{\infty} 2^{-i}\big(f(q_n^i) - f(q^i)\big)\Big|.$$

For any $\varepsilon > 0$ we can find an integer I such that

$$\sum_{i=I+1}^{\infty} 2^{-i} < \frac{\varepsilon}{4B}$$

([1]) See Colloquium Mathematicum 37 (1977), p. 177.

and an integer N such that for every $n \geqslant N$ and $i \leqslant I$

$$|f(q_n^i)-f(q^i)| < \frac{\varepsilon}{2},$$

so that

$$\sum_{i=1}^{\infty} 2^{-i}|f(q_n^i)-f(q^i)| \leqslant \frac{\varepsilon}{2}\sum_{i=1}^{I} 2^{-i}+2B\sum_{i=I+1}^{\infty} 2^{-i} < \varepsilon.$$

To conclude this it is sufficient to check that, for a compact set $K \subset Q$, $\mu_n(K) \to \mu(K)$ if and only if K is empty. We split the proof in two cases.

Case I. *K is finite.*

Then there exists a positive integer N such that no element of K appears in rows of the matrix lying beneath the N-th row. Therefore, $\mu_n(K) = 0$ for $N \leqslant n$ and, on the other hand, $\mu(K) = 0$ if and only if $K = \emptyset$.

Case II. *K is infinite.*

Then, as we noticed at the beginning, K has an isolated point. Let q^m be the first (in our denumeration of Q) isolated point of K. There are only finitely many elements of K in the m-th column, since no sequence of elements of K converges to q^m. Let

$$I = \{i : i < m,\ q^i \notin K\}.$$

Then the i-th column for $i \in I$ contains only finitely many elements of K as well, since K is closed. Hence, we can find an integer N such that for every $n \geqslant N$ and $i \in I \cup \{m\}$ we have $q_n^i \notin K$. Thus

$$\mu(K) = \sum_{\substack{i \notin I \\ i<m}} 2^{-i}+2^{-m}+r, \quad \text{where } 0 < r \leqslant \sum_{i>m} 2^{-i} = 2^{-m},$$

and

$$\mu_n(K) \leqslant \sum_{\substack{i \notin I \\ i<m}} 2^{-i}+2^{-m},$$

so that, eventually,

$$\mu(K)-\mu_n(K) \geqslant r \quad \text{for } n \geqslant N.$$

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