COLLOQUIUM MATHEMATICUM

VOL. XXI 1970 FASC. 1

CLOSED MAPPINGS AND LOCAL DIMENSION

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Introduction. In [3] Jung showed that if X is compact metric and $f(X) = Y$ is continuous, then

(i) $$\dim X \leqslant \dim A + \dim f,$$

where $A = \{y \in Y : \dim f^{-1}(y) \geqslant \dim X - \dim Y\}$. The author [4] has shown the inequality for X general metric with f a closed mapping. The present paper proves two related inequalities for closed mappings on separable metric spaces which involve local dimension. It is proved that if Y is compact, then

(ii) $$\dim X \leqslant \dim B + \dim f,$$

where $B = \{y \in Y : \dim f^{-1}(y) \geqslant \dim X - \dim_y Y\}$. If Y is not compact, then (ii) may fail, but in any case the following inequality holds

(iii) $$\dim X \leqslant \dim B + \dim f + 1,$$

where B is as described above. Examples are given to show that (iii) cannot be improved. It should be noted that (iii) implies Menger's Second Fundamental Theorem ([8], p. 135). In [6] Lelek has shown that the set B in (ii) has $\dim B > 0$ whenever X is an n-dimensional Cantor manifold with f non-constant.

We assume from now on that X and Y are separable metric spaces with f a continuous function from X onto Y. By $\dim_x X$ is meant the dimension of X at the point x. By $\dim f$ is meant the supremum of the dimensions of the point inverses of f. We assume the notation in [2].

Preliminaries. In this section we establish certain lemmas which will be necessary in the proof of our main results. For the most part these are improvements of lemmas appearing in [4] and [6].

LEMMA 1. *Let $\{X_i\}_{i=0}^{\infty}$ be a countable collection of F_σ's in X with $\dim X_0 = 0$. Let $\{U_a\}$ be any open cover of X. Then there is a locally finite open refinement $\{V_j\}$ such that for $x \in X$*

$$0 < \dim_x X_i \cup \{x\} < \infty$$

implies that

$$\dim_x[\mathrm{Fr}(V_j)\cap X_i]\cup\{x\}<\dim_x X_i\cup\{x\}$$

for all j *and*

$$\dim \mathrm{Fr}(V_j)\leqslant \dim X-1$$

with $\mathrm{Fr}(V_j)\cap X_0=\emptyset$ *for all* j.

Proof. First we note that we can consider $\{U_a\}$ to be a countable cover $\{U_j\}$ which is locally finite. Let $\{F_j\}$ be a closed cover of X with $F_j\subset U_j$. By the lemma in Lelek's paper [6], for each j there is an open set V_j with $F_j\subset V_j\subset \bar{V}_j\subset U_j$ having the properties of the lemma. Thus we can find an open cover $\{V_j\}$ of X having the desired properties. It will be locally finite since $V_j\subset U_j$ for each j.

LEMMA 2. *Let* $f(X)=Y$ *be a closed mapping with* $\dim X=n$ *and* $\dim Y=m$, *n and m finite, with* $\dim f<n$. *Let* $\{Y_i\}_{i=0}^{\infty}$ *be a countable collection of* F_σ*'s in* Y *with* $\dim Y_0=0$. *Then there is a closed set* $C\subset Y$ *with the following properties*:

(1) $\dim C\leqslant m-1$,

(2) $y\in Y$ *and* $0<\dim_y Y_i\cup\{y\}<\infty$ *implies that*

$$\dim_y[C\cap Y_i]\cup\{y\}\leqslant \dim_y Y_i\cup\{y\}-1,$$

(3) $C\cap Y_0=\emptyset$, *and*

(4) $\dim f^{-1}(C)\geqslant n-1$.

Proof. Let $D\subset X$ be closed with $g:D\to S^{n-1}$ continuous such that g cannot be extended continuously to all of X. Since $\dim f^{-1}(y)\leqslant n-1$ for all $y\in Y$, we may extend g to a mapping $g_y\colon \bar{U}_y\to S^{n-1}$, where U_y is an open set containing $D\cup f^{-1}(y)$. Let $O_y=Y-f(X-U_y)$. Then $\{O_y\}$ is an open cover of Y with $y\in O_y$. Let $\{V_j\}$ be an open refinement of $\{O_y\}$ satisfying the conditions of Lemma 1. Then if $B_j=f^{-1}(V_j)$ and $V_j\subset O_y$, then $B_j\subset U_y$. Thus g has an extension to $D\cup\bar{B}_j$ for all j. Since $\{B_j\}$ is locally finite, we can apply Lemma 8 in [4] to say that there must be a j with $\dim \mathrm{Fr}(B_j)\geqslant n-1$. Thus $f^{-1}(\mathrm{Fr}(V_j))\geqslant n-1$. Let $C=\mathrm{Fr}(V_j)$. Then C satisfies (1), (2), (3), and (4) of the lemma.

LEMMA 3. *Let* $B=\{y\in Y\colon \dim X-\dim_y Y\leqslant \dim f^{-1}(y)\}$, *where* $f(X)=Y$ *is a closed mapping with* $\dim X=n$ *and* $\dim Y=m$. *Then* B *is an* F_σ *in* Y *whenever n and m are both finite.*

Proof. Let $B_k=\{y\in Y\colon \dim f^{-1}(y)\geqslant k\}$ and let $C_k=\{y\in Y\colon \dim_y Y\geqslant \dim X-k=n-k\}$. Then B_k is an F_σ ([4], Theorem 1) and C_k is an F_σ ([8], p. 127). Now suppose that $y\in B_k\cap C_k$. Then $\dim f^{-1}(y)\geqslant k$ and $\dim X-\dim_y Y\leqslant k$. Thus $y\in B$. Now if $y\in B$, then let $k=\dim X-\dim_y Y$. Then $y\in C_k\cap B_k$. Thus $B=(B_0\cap C_0)\cup\ldots\cup(B_n\cap C_n)$, and B is an F_σ.

Main theorems. We proceed immediately to the first main result.

THEOREM 1. *Let* $f(X) = Y$ *be a closed mapping with* $\dim X = n \geqslant \dim Y = m$ *and* m *finite. Let* $B = \{y \in Y: n - \dim_y Y \leqslant \dim f^{-1}(y)\}$. *Then*

$$\dim X \leqslant \dim B + \dim f + 1.$$

Proof. We may suppose that $n < \infty$ for otherwise $\dim f$ is infinite by the Hurewicz theorem ([2], Theorem VI. 7, p. 91). If $m = 0$, then the result again holds by the Hurewicz theorem. Suppose that the theorem is true for all lesser values of m and let $m > 0$. We have three cases to consider.

Case (i). $\dim f = n$.

This case is trivial since $\dim f = \dim X$.

In the next two cases let $D = \{y \in Y: \dim f^{-1}(y) = n-1\}$.

Case (ii). $\dim f < n$ and $\dim D > 0$.

In this case there is a $y \in D$ with $\dim_y D > 0$. Thus $\dim_y Y > 0$. Therefore $y \in B$ and $B \neq \emptyset$. Thus the theorem holds in this case also.

Case (iii). $\dim f < n$ and $\dim D \leqslant 0$.

In this case let C be closed in Y with the following properties:

(1) $0 < \dim_y Y$ implies that $\dim_y C \cup \{y\} \leqslant \dim_y Y - 1$;

(2) $0 < \dim_y B \cup \{y\}$ implies the inequality $\dim_y [C \cap B] \cup \{y\} \leqslant \dim_y B \cup \{y\} - 1$;

(3) $C \cap D = \emptyset$; and

(4) $\dim f^{-1}(C) \geqslant n-1$.

Such a set C exists by Lemmas 2 and 3.

Now if $\dim B = 0$, we may replace (3) by (3′) $C \cap (D \cup B) = \emptyset$ since then $D \cup B$ will be an F_σ of dimension at most zero. So we will assume C satisfies (1) through (4) and in addition if $\dim B = 0$,

(3′) $C \cap (D \cup B) = \emptyset$.

Now consider $f|f^{-1}(C): f^{-1}(C) \to C$. If we let

$$B^* = \{y \in C: \dim f^{-1}(C) - \dim_y C \leqslant \dim f^{-1}(y)\},$$

then by our induction assumption we have

$$\text{(iv)} \qquad \dim f^{-1}(C) \leqslant \dim B^* + \dim f|f^{-1}(C) + 1,$$

since $\dim C < \dim Y$. Now suppose that B^* is not contained in B. Then let $y \in B^* - B$. Then we have

$$\text{(v)} \qquad \dim f^{-1}(y) \geqslant n - 1 - \dim_y C = n - (\dim_y C + 1),$$

by (4). The only way for (v) to hold and for $y \notin B$ is for $n - \dim_y Y > n - (\dim_y C + 1)$. But then this implies that $\dim_y Y < \dim_y C + 1$. Thus $\dim_y Y = \dim_y C$. But this only holds when $\dim_y Y = 0$ by property (1) of C. But then $\dim_y C = 0$ also and $\dim f^{-1}(y) \geqslant n-1$. Thus $y \in D$ and we have a contradiction, by (3). Thus we conclude that $B^* \subset B$.

Since $B^* \subset B$, we have either, by (2) or (3′), that $\dim B^* \leqslant \dim B - 1$ or $B^* = \emptyset$. We now show that $B^* \neq \emptyset$ because if $B^* = \emptyset$, then

$$n-1 \leqslant -1+\dim f|f^{-1}(C)+1,$$

by (4) and (iv). Thus $n-1 \leqslant \dim f|f^{-1}(C)$. This is impossible since $C \cap D = \emptyset$. Therefore $B^* \neq \emptyset$ and $\dim B^* \leqslant \dim B - 1$. This implies that

$$n-1 \leqslant \dim B - 1 + \dim f|f^{-1}(C)+1$$

and thus that

$$n \leqslant \dim B + \dim f + 1.$$

Now the theorem is completely proved.

COROLLARY. *If* $\dim X = n < \infty$ *and* $X_n = \{x \in X: \dim_x X = n\}$, *then* $\dim X_n \geqslant \dim X - 1$.

Proof. Let f be the identity mapping in Theorem 1 and let $Y = X$. Then $B = X_n$ and $\dim f = 0$. Thus $\dim X \leqslant \dim X_n + 1$. This is Menger's famous theorem ([8], p. 135).

Example 1. A space X is said to be *weakly n-dimensional* provided $\dim X = n$ and $\dim X_n = n-1$ (see [5], p. 298, or [8], p. 138). There are such spaces for every positive integer n [7]. Let X be weakly n-dimensional and f the identity mapping. Then $B = X_n$ and we have $\dim X = \dim B + \dim f + 1$. Therefore we cannot improve the inequality in Theorem 1 in general.

Example 2. Let Y be a weakly 1-dimensional space and let $X = I^n \times Y$, where I^n is the n-dimensional cube. Then let $f: X \to Y$ be the projection mapping. Then by [1], $\dim X = n+1$. But $B = Y_1$ and thus $\dim B = 0$. Therefore we again have $\dim X = \dim B + \dim f + 1$.

THEOREM 2. *Suppose that* $f(X) = Y$ *is closed with* Y *compact and* $\dim X = n \geqslant \dim Y = m$ *with* m *finite. Let* $B = \{y \in Y: n - \dim_y Y \leqslant \dim f^{-1}(y)\}$. *Then*

$$\dim X \leqslant \dim B + \dim f.$$

Proof. The proof is similar to that of Theorem 1. We may assume that $n < \infty$ again. In case $m = 0$ the theorem holds by the Hurewicz theorem. Suppose the theorem holds for all lesser values of m and let $m > 0$.

Case (i). $\dim f = n$.

This case is again trivial. Let $D = \{y \in Y: \dim f^{-1}(y) = n-1\}$.

Case (ii). $\dim f < n$ and $\dim D > 0$.

Since D is an F_σ, it is σ-compact. Thus D must contain a non-degenerate continuum $E \subset D$. But then $E \subset B$ and $\dim B > 0$. Thus in this case we have $\dim X \leqslant \dim B + \dim f$.

Case (iii). $\dim f < n$ and $\dim D \leqslant 0$.

Then let C be closed in Y with C satisfying properties (1), (2), (3), and (4) as in case (iii) of the proof of Theorem 1. Also assume that $C \cap B = \emptyset$ if $\dim B = 0$. Let $B^* = \{y \in C\colon \dim f^{-1}(C) - \dim_y C \leqslant \dim f^{-1}(y)\}$. Then by the induction assumption we have

$$\dim f^{-1}(C) \leqslant \dim B^* + \dim f | f^{-1}(C)$$

since $\dim C < \dim Y$. Again it must be that $B^* \subset B$ and thus $\dim B^* \leqslant \dim B - 1$ or $B^* = \emptyset$, since $B^* \subset B \cap C$. Now $B^* \neq \emptyset$ since otherwise $\dim f = n$. Thus $\dim B^* \leqslant \dim B - 1$. Therefore

$$n - 1 \leqslant \dim B - 1 + \dim f | f^{-1}(C)$$

and hence

$$n \leqslant \dim B + \dim f.$$

Now the second theorem is completely proved.

COROLLARY. *If* $f(X) = Y$ *is continuous with* X *compact and* $\dim X = n \geqslant \dim Y$ *with* n *finite, then* $\dim f \leqslant n - k$ *implies* $\dim B \geqslant k$.

This is similar to Lelek's result [6].

REFERENCES

[1] W. Hurewicz, *Sur la dimension des produits Cartésiens*, Annales Mathematicae 36 (1935), p. 194–197.

[2] — and H. Wallman, *Dimension theory*, Princeton 1941.

[3] C. Jung, *Mappings on compact metric spaces*, Colloquium Mathematicum 19 (1968), p. 73–76.

[4] J. Keesling, *Closed mappings which lower dimension*, ibidem 20 (1969), p. 237–241.

[5] K. Kuratowski, *Topology I*, New York 1966.

[6] A. Lelek, *On mappings of Cantorian manifolds*, Colloquium Mathematicum 17 (1967), p. 315–318.

[7] S. Mazurkiewicz, *Sur les ensembles de dimension faible*, Fundamenta Mathematicae 13 (1929), p. 210–217.

[8] K. Menger, *Dimensionstheorie*, Leipzig 1928.

Reçu par la Rédaction le 21. 9. 1968