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REGULAR ANALYTIC ARCS AND CURVES

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0. Introduction. In this paper, several results about regular analytic arcs and curves on Riemann surfaces are established; most of the results have already been stated in the literature but the proofs are incomplete. First, it is shown that if two regular analytic arcs, which are defined on compact intervals, intersect an infinite number of times, then they are both subarcs of some regular analytic arc. Next, a classification of real analytic 1-manifolds which satisfy the second axiom of countability is derived. This is used to show that the local and global definitions of a regular analytic arc are equivalent and that every regular analytic arc is contained in such a maximal arc. Riemann surface techniques are used in establishing these results.

1. Preliminaries. In this section we present some of the basic definitions concerning regular analytic arcs and curves.

Definition 1. Let X be a Riemann surface, and I an interval on the real axis. A *regular analytic arc* on X is a function $\gamma: I\to X$ such that γ is analytic at each point of I and $\gamma'(t) \neq 0$ for all $t\in I$. Set $|\gamma| = \{\gamma(t): t\in I\}$ and $T = \{z\in C: |z| = 1\}$. A *regular analytic curve* on X is a function $\gamma: T\to X$ such that γ is analytic at each point of T and $\gamma'(z)\neq 0$ for all $z\in T$. Then $|\gamma| = \{\gamma(z): z\in T\}$.

The derivative $\gamma'(t)$ is computed in terms of any local coordinate on X and the interval I may be open, closed, or half-open. To say that γ is analytic on I means that there is a region $\Omega \supset I$ such that γ extends to an analytic mapping of Ω into X. For this reason it is possible to assume that I is an open interval, if this is desirable.

Definition 2. Suppose that X is a Riemann surface, and $\gamma: I\to X$ and $\delta: J\to X$ are regular analytic arcs on X. The arc γ is called a *subarc* of δ if there is an analytic function f which has a non-vanishing derivative, maps I into J, and such that $\gamma = \delta\circ f$. Two analytic arcs are *equivalent* if each is a subarc of the other.

Clearly, the function $f: I \to J$ is either strictly increasing or strictly decreasing. Two regular analytic arcs $\gamma: I \to X$ and $\delta: J \to X$ are equivalent if and only if there is an analytic function f which has a non-vanishing derivative, maps I onto J, and is such that $\gamma = \delta \circ f$.

Definition 3. Let X be a Riemann surface, and I an interval on the real axis. A *locally regular analytic arc* on X is a function $\gamma: I \to X$ such that for each point $t \in I$ there are an open interval I_t with $t \in I_t$ and a homeomorphism f_t of an open interval J_t onto I_t, $\gamma \circ f_t$ being analytic and having a non-vanishing derivative. A *locally regular analytic curve* on X is defined analogously; just replace the open interval I_t by an open circular arc.

For Jordan arcs and curves, this may be restated as follows. For each point $p \in |\gamma|$ there are a neighborhood U of p and a univalent analytic function g mapping U into the complex plane C such that $g(U \cap |\gamma|) \subset R$, the real axis. Clearly, every regular analytic arc or curve is locally regular. Every locally regular analytic arc $\gamma: I \to X$ has an extension to an open interval if I itself is not open.

2. Intersection of regular analytic arcs. Let X be a Riemann surface. We shall show that two regular analytic arcs γ_1 and γ_2 on X, which are both defined on a compact interval, cannot intersect infinitely many times unless they are both subarcs of some regular analytic arc γ. This is a type of identity theorem for regular analytic arcs. This result is mentioned in [1], p. 192-193, with an indication of its proof; however, it is not cited in a later edition of the same book. The suggested method of proof only shows that γ_1 and γ_2 must fit together to form a single arc or closed curve. This same result is established in [8], p. 243-244. In neither instance it is shown how to obtain a regular analytic parametrization for the union of the two overlapping arcs. One way to obtain such a parametrization will be given in this section. It should be observed that it can be impossible to continue analytically either γ_1 or γ_2 to obtain a parametrization of their union as simple examples show.

LEMMA 1. *Let X be a Riemann surface and d any metric on X which is compatible with the topology. Suppose that Ω is a region in the complex plane C, K is a non-empty compact subset of Ω, $\gamma: \Omega \to X$ is analytic, and $\gamma'(z) \neq 0$ for all $z \in \Omega$.*

(a) *There is an $\varepsilon > 0$ such that γ is univalent in the disk*

$$B(z, \varepsilon) = \{w \in C: |z-w| < \varepsilon\} \quad \textit{for all } z \in K.$$

(b) *Given $\varepsilon > 0$ there exists $\delta > 0$ and $\eta > 0$ such that*

$$\gamma(B(z, \eta)) \subset B_d(\gamma(z), \delta) \subset \gamma(B(z, \varepsilon)) \quad \textit{for any } z \in K,$$

where $B_d(p, \delta) = \{q \in X: d(p, q) < \delta\}$ for any $p \in X$.

Proof. (a) Since $\gamma'(z) \neq 0$, for each z there is an $\varepsilon(z) > 0$ such that γ is univalent in $B(z, \varepsilon(z))$. Now $\{B(z, \varepsilon(z)): z \in K\}$ is an open covering of the compact set K. If $\varepsilon > 0$ is a Lebesgue number for this cover, then γ is univalent in $B(z, \varepsilon)$ for all $z \in K$.

(b) First, we demonstrate the existence of δ. Since γ is non-constant, $\gamma(B(z, \varepsilon))$ is an open set containing $\gamma(z)$. Set

$$\delta(z) = \frac{1}{2} \sup\{r: B_d(\gamma(z), r) \subset \gamma(B(z, \varepsilon))\} \quad \text{and} \quad \delta = \inf\{\delta(z): z \in K\}.$$

It suffices to show that $\delta > 0$. Let $(z_n)_{n=1}^{\infty}$ be a sequence in K with $\delta(z_n) \to \delta$. Since K is compact, we may assume that $z_n \to z \in K$ and $z_n \in B(z, \varepsilon/2)$ for all n. Take any $r > 0$ such that

$$B_d(\gamma(z), r) \subset \gamma(B(z, \varepsilon/2));$$

we may assume that $\gamma(z_n) \in B_d(\gamma(z), r/2)$ for all n. Then

$$B_d(\gamma(z_n), r/2) \subset B_d(\gamma(z), r) \subset \gamma(B(z, \varepsilon/2)) \subset \gamma(B(z_n, \varepsilon)),$$

which yields $\delta(z_n) \geqslant r/4$ for all n. Hence $\delta \geqslant r/4 > 0$.

Next, we show how to determine η. Since γ is uniformly continuous on the compact set K, there is an $\eta > 0$ such that if $z \in K$, then

$$\gamma(B(z, \eta)) \subset B_d(\gamma(z), \delta) \quad \text{for all } z \in K.$$

THEOREM 1. *Let $\gamma_1: I_1 \to X$ and $\gamma_2: I_2 \to X$ be regular analytic arcs on a Riemann surface X, where I_1 and I_2 are compact intervals on the real axis. Either $|\gamma_1| \cap |\gamma_2|$ is finite or else there is a regular analytic arc γ on X such that γ_1 and γ_2 are both subarcs of γ.*

Proof. Without loss of generality we may assume that $I_1 = I_2 = I = [0, 1]$. Set $M = |\gamma_1| \cap |\gamma_2|$ and $M_j = \gamma_j^{-1}(M)$ $(j = 1, 2)$. If both M_1 and M_2 are finite sets, then M itself is a finite set. If either M_1 or M_2 is an infinite set, then we will show that γ_1 and γ_2 are subarcs of the same regular analytic arc. Henceforth, we assume that the set M_1 is infinite. By Lemma 1, there is an $\varepsilon > 0$ such that both γ_1 and γ_2 are analytic and univalent in $B(t, \varepsilon)$ for any $t \in I$. The same lemma then permits us to select δ and η, $0 < \eta < \delta < \varepsilon$, such that

$$\gamma_j(B(t, \eta)) \subset B_d(\gamma_j(t), \delta) \subset \gamma_j(B(t, \varepsilon)) \quad (j = 1, 2) \tag{1}$$

for all $t \in I$. Here d is any metric on X which is compatible with the topology.

The first step is to show that γ_1 and γ_2 have a subarc in common. Since M_1 is compact and infinite, there is a sequence $(t_n)_{n=1}^{\infty}$ of distinct points in M_1 which converges to some point $t_0 \in M_1$ and satisfies $t_n \in B(t_0, \varepsilon)$ for all n. For each $n \geqslant 1$ there is at least one point $\tau_n \in M_2$ with $\gamma_2(\tau_n) = \gamma_1(t_n)$. The points τ_n are all distinct since

$$\gamma_2(\tau_m) = \gamma_1(t_m) \neq \gamma_1(t_n) = \gamma_2(\tau_n) \quad \text{if } m \neq n.$$

By extracting a subsequence of $(t_n)_{n=1}^{\infty}$, if necessary, we may assume that $\tau_n \to \tau_0 \in M_2$ and $\tau_n \in B(\tau_0, \eta)$ for all n. Clearly, $\gamma_2(\tau_0) = \gamma_1(t_0)$. By (1) we have

$$\gamma_2\big(B(\tau_0, \eta)\big) \subset B_d\big(\gamma_2(\tau_0), \delta\big) = B_d\big(\gamma_1(t_0), \delta\big) \subset \gamma_1\big(B(t_0, \varepsilon)\big).$$

Consequently, the function $f = [\gamma_1 | B(t_0, \varepsilon)]^{-1} \circ \gamma_2$ is analytic and univalent in $B(\tau_0, \eta)$, maps this disk into $B(t_0, \varepsilon)$, and $\gamma_2 = \gamma_1 \circ f$ on $B(\tau_0, \eta)$. Also $f(\tau_n) = t_n \in I$ for all n. The function $\overline{f(\bar{\tau})}$ is analytic in $B(\tau_0, \eta)$ and agrees with $f(\tau)$ on the infinite set $\{\tau_n : n = 0, 1, \ldots\}$ which accumulates at τ_0, so the identity theorem allows us to conclude that $f(\tau) \equiv \overline{f(\bar{\tau})}$. In particular, f is real valued on the real axis, $f(\tau_0 - \eta, \tau_0 + \eta) \subset (t_0 - \varepsilon, t_0 + \varepsilon)$ and $f'(\tau) \neq 0$ since f is univalent. This implies that either $f' > 0$ or $f' < 0$ on the interval $(\tau_0 - \eta, \tau_0 + \eta)$. We may assume that $f' > 0$ on this interval; if not, simply replace $\gamma_2(\tau)$ by $\gamma_2(1 - \tau)$. Thus f is strictly increasing on $(\tau_0 - \eta, \tau_0 + \eta)$. Define b_1 by $[t_0, b_1) = f\big([\tau_0, \tau_0 + \eta)\big)$; then $t_0 < b_1 \leqslant t_0 + \varepsilon$, and γ_1 and γ_2 share the arc

$$\gamma_2\big([\tau_0, \tau_0 + \eta)\big) = \gamma_1\big([t_0, b_1)\big).$$

Next, we want to continue f analytically along the interval $[\tau_0, 1]$ as far as possible. We can continue f until either $[\tau_0, 1] \subset \mathrm{domain}(f)$ or $[t_0, 1] \subset \mathrm{range}(f)$. The continuation is stopped when we reach one end of an arc which γ_1 and γ_2 have in common. If

$$1 \in [\tau_0, \tau_0 + \eta) \cup [t_0, b_1),$$

then we stop. Otherwise, the function $\big[\gamma_1\big(B(b_1, \varepsilon)\big)\big]^{-1} \circ \gamma_2$ is analytic and univalent in $B(\tau_0 + \eta, \eta)$ and coincides with f on $(\tau_0, \tau_0 + \eta)$. This function provides an analytic continuation of f to the disk $B(\tau_0 + \eta, \eta)$; the extended function will still be denoted by f. Note that f remains real valued on the real axis. f is now analytic on $B(\tau_0, \eta) \cup B(\tau_0 + \eta, \eta)$, $\gamma_2 = \gamma_1 \circ f$ on the domain of f, and $f' > 0$ on $[\tau_0, \tau_0 + 2\eta)$. Set

$$f\big([\tau_0, \tau_0 + 2\eta)\big) = [t_0, b_2).$$

We terminate this process if $1 \in [\tau_0, \tau_0 + 2\eta) \cup [t_0, b_2)$. If not, then we proceed as before. After a finite number of steps we continue f analytically to a neighborhood of $[\tau_0, \tau_0 + k\eta)$ for some integer $k \geqslant 1$ such that $f' > 0$ on $[\tau_0, \tau_0 + k\eta)$, f maps $[\tau_0, \tau_0 + k\eta)$ onto $[t_0, b_k)$, $1 \in [\tau_0, \tau_0 + k\eta) \cup [t_0, b_k)$, and $\gamma_2 = \gamma_1 \circ f$ on the domain of f.

In a similar fashion, f may be extended analytically to a neighborhood of $(\tau_0 - j\eta, \tau_0]$ for some integer $j \geqslant 1$ such that $f' > 0$ on $(\tau_0 - j\eta, \tau_0]$, f maps $(\tau_0 - j\eta, \tau_0]$ onto $(a_j, t_0]$, $0 \in (\tau_0 - j\eta, \tau_0] \cup (a_j, t_0]$, and $\gamma_2 = \gamma_1 \circ f$. Therefore, the function f may analytically be continued to a neighborhood of $(\tau_0 - j\eta, \tau_0 + k\eta)$ so that $f' > 0$ on the real axis and f maps $(\tau_0 - j\eta, \tau_0 + k\eta)$ onto (a_j, b_k). Set

$$[\alpha, \beta] = I \cap f^{-1}\big((a_j, b_k) \cap I\big) \quad \text{and} \quad [a, b] = f([\alpha, \beta]).$$

Both $[\alpha, \beta]$ and $[a, b]$ are subintervals of I; either $\alpha = 0$ or $a = 0$, and either $\beta = 1$ or $b = 1$. If $[\alpha, \beta] = I$, then γ_2 is a subarc of γ_1 and we can take $\gamma = \gamma_1$. Similarly, if $[a, b] = I$, then γ_1 is a subarc of γ_2 and we may use $\gamma = \gamma_2$. The two remaining possibilities are

(i) $0 < \alpha < \beta = 1$, $0 = a < b < 1$,

(ii) $0 = \alpha < \beta < 1$, $0 < a < b = 1$.

If case (ii) occurs, then by interchanging the roles of γ_1 and γ_2 we obtain a situation analogous to (i). Consequently, we need only to consider the case in which $f\colon [\alpha, 1] \to [0, b]$, $0 < \alpha < 1$, $0 < b < 1$.

Finally, we want to determine a parametrization of a regular analytic arc which contains both γ_1 and γ_2 as subarcs. To accomplish this we will "blend" two regions by making use of the function f. Let Ω_2 be a Jordan region containing I which is symmetric about the real axis. Suppose that σ_2 is a symmetric crosscut of Ω_2 such that $[\alpha, 1]$ is contained in one component ω_2 of $\Omega_2 - \sigma_2$. By selecting Ω_2 and σ_2 appropriately we may assume that f is analytic and univalent in ω_2. Then $\omega_1 = f(\omega_2)$ is a Jordan region containing $[0, b]$ which is symmetric about the real axis. By adding σ_1, a symmetric Jordan arc which loops around 1, to a part of $\partial\omega_1$, we obtain a Jordan region Ω_1 which contains both I and ω_1. The function f maps ω_2, the right-hand end of Ω_2, conformally onto ω_1, the left-hand end of Ω_1. The crucial fact is that there exist conformal maps g_1 and g_2 of Ω_1 and Ω_2, respectively, into the unit disk $\boldsymbol{B}$ such that

$$g_1(\Omega_1) \cup g_2(\Omega_2) = \boldsymbol{B},$$

$$g_1(\Omega_1) \cap g_2(\Omega_2) = g_1(\omega_1) = g_2(\omega_2) \quad \text{and} \quad g_2 | \omega_2 = g_1 \circ f.$$

Moreover, it is possible to select g_1 and g_2 to be symmetric about the real axis. This result can be established in a variety of ways. It can be demonstrated by the method of blending domains (see [6], p. 98-99) or the welding of Riemann surfaces (see [2], p. 118-119). The third possibility is sketched below; the same method will be used in later sections.

Form the disjoint or free union of Ω_1 and Ω_2. Identify two points if they correspond under the mapping f. The resulting quotient space W is a Riemann surface; it has a natural conformal structure which makes the canonical injections $p_j\colon \Omega_j \to W$ $(j = 1, 2)$ into analytic functions. W is a simply connected Riemann surface which is conformally equivalent to the unit disk $\boldsymbol{B}$. Since both Ω_1 and Ω_2 are symmetric about the real axis, W naturally inherits an anticonformal involution. Under the conformal mapping g of W onto $\boldsymbol{B}$ we can assume that this involution is carried into complex conjugation. Then the functions $g_j = g \circ p_j$ $(j = 1, 2)$ have the desired properties.

At last we obtain the arc γ. Clearly,

$$g_1(I) \cup g_2(I) = [t_1, t_2] \subset (-1, 1).$$

Set

$$\gamma(t) = \begin{cases} \gamma_1\big(g_1^{-1}(t)\big) & \text{for } t \in g_1(I), \\ \gamma_2\big(g_2^{-1}(t)\big) & \text{for } t \in g_2(I). \end{cases}$$

$\gamma: [t_1, t_2] \to X$ is well defined. Indeed, if $t \in g_1(I) \cap g_2(I)$, then

$$\gamma_2\big(g_2^{-1}(t)\big) = \gamma_1 \circ f\big(g_2^{-1}(t)\big) = \gamma_1\big(g_1^{-1}(t)\big)$$

since $g_2 = g_1 \circ f$ and $\gamma_2 = \gamma_1 \circ f$. Now, γ is a regular analytic arc on X, and γ_1, γ_2 are subarcs of γ.

3. Classification of real analytic 1-manifolds. In [9], p. 55-57, it is shown that any connected smooth (C^∞) 1-manifold without boundary which satisfies the second axiom of countability is diffeomorphic either to the open interval $(-1, 1)$ or to the unit circle $T = \{z \in \boldsymbol{C}: |z| = 1\}$. Making use of this fact and the technique employed at the end of Section 2 we shall show that every connected real analytic 1-manifold whose topology has a countable basis is homeomorphic, in the real analytic sense, either to $(-1, 1)$ or to T endowed with the usual real analytic structure. It is sufficient to show that any real analytic structure on $(-1, 1)$ or T is equivalent to the usual one.

We begin by recalling some of the basic facts about real analytic 1-manifolds. A *real analytic* 1-*manifold* is a pair $\langle Y, \mathscr{H} \rangle$, where Y is a one-dimensional manifold and $\mathscr{H}$ is a maximal real analytic atlas on Y. Such an atlas $\mathscr{H} = (h_a)_{a \in A}$ consists of real analytically compatible charts h_a, where each h_a is a homeomorphism of an open set $U_a \subset Y$ onto an open set $h_a(U_a) \subset \boldsymbol{R}$, the real line. $\mathscr{H}$ is called the *real analytic structure* on Y. If $\mathscr{H}_0$ is any real analytic atlas for a one-dimensional manifold Y, then it uniquely determines a maximal real analytic atlas $\mathscr{H} \supset \mathscr{H}_0$ for Y, and $\mathscr{H}_0$ is called a *basis* for the real analytic structure $\mathscr{H}$ on Y. For example, the usual real analytic structure on $(-1, 1)$ has $\mathscr{H}_0 = (h)$, where h is the identity function as a basis.

Let $\mathscr{H} = (h_a)_{a \in A}$ be any real analytic structure on $(-1, 1)$, and let $M = \langle (-1, 1), \mathscr{H} \rangle$ be the associated real analytic 1-manifold. We want to show that M is equivalent to $(-1, 1)$ with the usual real analytic structure. Determine open intervals $I_n = (a_n, b_n)$ $(n = 0, \pm 1, \pm 2, \ldots)$ such that each closed interval $[a_n, b_n]$ is contained in the domain of some $h_n \in \mathscr{H}$,

$$\bigcup_{-\infty}^{+\infty} I_n = (-1, 1) \quad \text{and} \quad -1 < a_n < b_{n-1} < a_{n+1} < b_n < 1$$

for all n. The functions $(h_n)_{n=-\infty}^{+\infty}$ form a basis for $\mathscr{H}$. Observe that $I_m \cap I_n = \emptyset$ if $|m-n| \geqslant 2$. Set $h_n(I_n) = (\alpha_n, \beta_n)$; we may assume that $h_n(a_n) = \alpha_n$ and $h_n(b_n) = \beta_n$. Each h_n is increasing on $[a_n, b_n]$ and

$$f_n = h_{n+1} \circ h_n^{-1} \colon [h_n(a_{n+1}), \beta_n] \to [\alpha_{n+1}, h_{n+1}(b_n)]$$

is a real analytic homeomorphism with $f_n' > 0$. Thus f_n extends to a conformal mapping of a symmetric Jordan region containing $[h_n(a_{n+1}), \beta_n]$ onto a symmetric Jordan region containing $[\alpha_{n+1}, h_{n+1}(b_n)]$. Thus for each closed interval $[\alpha_n, \beta_n]$ we obtain two Jordan regions, one ω_n^- containing $[\alpha_n, h_n(b_{n-1})]$ and the other ω_n^+ containing $[h_n(a_{n+1}), \beta_n]$. We may assume that the closures of these two regions are disjoint. By making use of a Jordan arc in the upper half plane which connects these two Jordan curves together with its reflection in the real axis, we obtain a Jordan region $\Omega_n \supset [\alpha_n, \beta_n]$ such that ω_n^- and ω_n^+ are both subregions of Ω_n and each is determined by a symmetric crosscut. Then f_n maps ω_n^+, the right-hand end of Ω_n, conformally onto ω_{n+1}^-, the left-hand end of Ω_{n+1} for all n.

We now construct a Riemann surface as in Section 2. Take the disjoint union of all the regions Ω_n and identify two points if they correspond under some f_n. The resulting quotient space W is a Riemann surface with a natural conformal structure which makes the canonical injections $p_n \colon \Omega_n \to W$ into analytic functions. W is simply connected and conformally equivalent to the unit disk $\boldsymbol{B}$. Also, W has a natural anticonformal involution; we may assume that the conformal mapping g of W onto $\boldsymbol{B}$ converts this involution into complex conjugation. Put $g_n = g \circ p_n$; then g_n is analytic and univalent in Ω_n, $g_n' > 0$ on $[\alpha_n, \beta_n]$,

$$\bigcup_{-\infty}^{+\infty} g_n\big((\alpha_n, \beta_n)\big) = (-1, 1) \quad \text{and} \quad g_n = g_{n+1} \circ f_n \quad \text{on } \omega_n^+.$$

Define $\varphi \colon M \to (-1, 1)$ by $\varphi(t) = g_n\big(h_n(t)\big)$ if $t \in I_n$. The map φ is well defined, real analytic, one-to-one and onto. In other words, φ is a real analytic homeomorphism of M onto $(-1, 1)$ with the usual real analytic structure. This result could also be established by the method of blending an infinite number of domains (see [6], p. 77-80).

The equivalence of any two real analytic structures on the unit circle T can be established in a similar manner. We give a brief indication of the proof and omit all details.

Let $\mathscr{H} = (h_a)_{a \in A}$ be a real analytic structure on T and set $M = \langle T, \mathscr{H} \rangle$. Since T is compact, we can use a finite number of circular arcs J_n to cover T, the closure of each J_n being contained in the domain of some $h_n \in \mathscr{H}$. We may assume that the J_n overlap analogous to the I_n and that $(J_n)_{n=1}^{N}$ is the set of arcs ordered clockwise around the circle. We can obtain a Jordan region Ω_n symmetric about the unit circle which contains the closure of J_n and a conformal map $f_n = h_{n+1} \circ h_n^{-1}$ of the right-hand end of J_n

onto the left-hand end of J_{n+1}. "Right-hand" and "left-hand" are as seen from the center of the circle; we take $J_{N+1} = J_1$ for convenience. We now form a Riemann surface W from the regions Ω_n and the maps f_n. The surface W is conformally equivalent to a doubly connected region in the complex plane which contains T and is symmetric about T. In this manner we obtain a real analytic homeomorphism of M onto T. Let us summarize our results in a theorem.

THEOREM 2. *Let M be a connected real analytic* 1-*manifold which satisfies the second axiom of countability. If M is compact, then M is real analytically equivalent to the unit circle T with the usual real analytic structure. If M is non-compact, then M is real analytically equivalent to the open interval* $(-1, 1)$ *with the usual real analytic structure.*

4. Locally regular analytic arcs and curves. In this section we show that the local and global definitions of regular analytic arcs and curves are equivalent. It is enough to show that the local definition implies the global one. In [3], p. 376, Satz 50, this result is established for regular analytic Jordan arcs $\gamma: I \to X$, where I is a compact interval, by making use of the Riemann Mapping Theorem. This same result is mentioned in [7], p. 228-229, but the proof contains an error. We shall establish the result in the general case.

Let $\gamma: I \to X$ be a locally regular analytic arc on a Riemann surface X, where I is an open interval on the real axis. For each $t \in I$ there are an open interval $I_t \subset I$ with $t \in I_t$ and a homeomorphism f_t of an open interval J_t onto I_t such that $\gamma \circ f_t$ is analytic and has a non-vanishing derivative. For each $t \in I$ write $h_t = f_t^{-1}: I_t \to J_t$; then $(h_t)_{t \in I}$ is a basis for a real analytic structure $\mathscr{H}$ on I. By Theorem 2 there is a real analytic homeomorphism φ of $(-1, 1)$ with the usual real analytic structure onto $\langle I, \mathscr{H} \rangle$. Then $\delta = \gamma \circ \varphi$ is a regular analytic arc; in other words, we can reparametrize γ so that it becomes a regular analytic arc.

In the same manner it can be shown that every locally regular analytic curve can be reparametrized as a regular analytic curve.

5. Maximal regular analytic arcs. Let $\gamma: I \to X$ be a regular analytic arc on a Riemann surface X; without loss of generality we may assume that I is an open interval. We will show that γ is contained in a maximal regular analytic arc which is uniquely determined up to equivalence of regular analytic arcs. The method of proof is similar to that employed by Bochner [4] in demonstrating the existence of maximal Riemann surfaces.

Let $\Gamma = (\gamma_a)_{a \in A}$ be the family of all regular analytic arcs $\gamma_a: I_a \to X$ which contain γ as a subarc; I_a denotes an open interval. With each $a \in A$ there is an associated analytic function $f_a: I \to I_a$ with non-vanishing derivative and such that $\gamma = \gamma_a \circ f_a$. We define a partial ordering on the set Γ as follows: We write $\gamma_a < \gamma_\beta$ if γ_a is a subarc of γ_β and the function

$f_{\alpha\beta}\colon I_\alpha \to I_\beta$ which exhibits γ_α as a subarc of γ_β (that is, $\gamma_\alpha = \gamma_\beta \circ f_{\alpha\beta}$) satisfies $f_\beta = f_{\alpha\beta} \circ f_\alpha$. It is elementary to verify that this defines a partial ordering on the set Γ and that $f_{\beta\delta} \circ f_{\alpha\beta} = f_{\alpha\delta}$ if $\gamma_\alpha < \gamma_\beta < \gamma_\delta$.

Zorn's Lemma will be used to show that Γ contains maximal elements; we will prove that every chain in Γ has an upper bound.

Let $(\gamma_\alpha)_{\alpha\in B}$ be a chain in Γ. Let I^∞ be the direct limit of the topological spaces $(I_\alpha)_{\alpha\in B}$ with respect to the functions $f_{\alpha\beta}$. The direct limit I^∞ is just the free union of the open intervals $(I_\alpha)_{\alpha\in B}$ with two points $t_\alpha \in I_\alpha$ and $t_\beta \in I_\beta$ identified if $\gamma_\alpha < \gamma_\beta$ and $t_\beta = f_{\alpha\beta}(t_\alpha)$. It is straightforward to verify that I^∞ is a connected non-compact one-dimensional manifold. There is a natural real analytic structure induced on I^∞ which makes each of the canonical injections $p_\alpha\colon I_\alpha \to I^\infty$ a real analytic function. Thus I^∞ is a connected real analytic manifold. The function $\delta\colon I^\infty \to X$,

$$\delta\big(p_\alpha(t_\alpha)\big) = \gamma_\alpha(t_\alpha),$$

is well defined and real analytic. From [5], p. 116, Corollary 2, it follows that the topology for I^∞ has a countable base. By Theorem 2, we know that I^∞ is real analytically equivalent to $(-1, 1)$ with the usual real analytic structure. Hence we may simply assume that I^∞ is $(-1, 1)$ with the usual real analytic structure. Then $\delta\colon (-1, 1) \to X$ is a regular analytic arc which contains γ as a subarc and $\gamma_\alpha < \delta$ for all $\alpha \in B$. Thus we may conclude that Γ contains maximal elements.

Suppose that $\delta_1\colon I_1 \to X$ and $\delta_2\colon I_2 \to X$ are both maximal elements in Γ. Each of these arcs has γ as a subarc. This implies that there is an analytic function f, defined on $(\alpha_1, \beta_1) \subset J_1$, mapping this interval into J_2 so that

$$\delta_1 | (\alpha_1, \beta_1) = \delta_2 \circ f \quad \text{and} \quad f' \neq 0 \quad \text{on } (\alpha_1, \beta_1).$$

Extend f analytically to the largest subinterval of J_1 as possible so that $f' \neq 0$ on this subinterval. If $\operatorname{domain}(f) = J_1$, then δ_1 is a subarc of δ_2. Since δ_1 is maximal, this means that δ_1 and δ_2 are equivalent arcs. If $\operatorname{range}(f) = J_2$, then δ_2 is a subarc of δ_1 and we obtain the same conclusion. Otherwise, use the function f as in Section 2 to piece δ_1 and δ_2 together to obtain a regular analytic arc δ containing both δ_1 and δ_2 as subarcs. Since δ_j $(j = 1, 2)$ is maximal, we find that δ_j and δ are equivalent regular analytic arcs. Consequently, δ_1 and δ_2 are equivalent.

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