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ON CLOSED TIMELIKE DISTRIBUTIONS

BY

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0. Introduction. The present paper concerns Riemannian manifolds with metric of index one. Such a manifold $(\overline{M}, g)$ always admits a 1-dimensional differentiable distribution d which is timelike in the sense that $g(Y, Y) < 0$ for each non-zero vector $Y \in d$. In this paper the distribution d is, moreover, assumed to be *closed*, i.e. spanned locally by the gradient of a function. We also admit that each maximal integral curve of d is a complete submanifold of $\overline{M}$.

A submanifold M of $\overline{M}$ is called *spacelike* if its co-dimension is one and $g(Y, Y) > 0$ for any non-zero vector Y tangent to M. In Section 2 we consider certain tensor fields v, w, u and a family of operators $d(t, p)$ which are determined in a natural manner by d. In Section 3 it is shown, in particular, that, under some suitable conditions for $\overline{M}$, d and the quantities listed above, for any compact spacelike submanifold M of $\overline{M}$, either $\overline{M}$ is diffeomorphic to $\boldsymbol{R} \times M$ or $\overline{M}$ is the space of a bundle with fibre M over S^1 (Theorem 1). Theorem 2 states that any two compact spacelike submanifolds of $\overline{M}$ are diffeomorphic provided that $\overline{M}$ is non-compact and the norms of the operators $d(t, p)$ are uniformly bounded. Proposition 4 asserts that if $\overline{M}$ is flat and $w = 0$, then each spacelike submanifold of $\overline{M}$ admits a positive-definite metric with flat Riemannian connection.

1. Preliminaries. By a *manifold* we always mean a C^k paracompact, connected Hausdorff manifold, $k = \infty, \omega$. In the sequel, $(\overline{M}, g)$ will denote an $(n+1)$-dimensional Riemannian manifold $\overline{M}$ with a C^k metric g of index one. A vector $Y \in T\overline{M}$ is called *timelike* (respectively, *spacelike*) if $g(Y, Y) < 0$ (respectively, $g(Y, Y) > 0$). By a *time orientation* at $p \in \overline{M}$ we mean a connected component of the set of timelike vectors at p. The manifold $(\overline{M}, g)$ is called *isochronous* or *time-orientable* if it admits a continuous field of time orientations. If such a field is chosen, $(\overline{M}, g)$ is said to be *time oriented*.

A 1-dimensional distribution d on $\overline{M}$ is called *closed* if it is spanned locally by a gradient, i.e. if for each $p \in \overline{M}$ there exist a neighbourhood W

of p and a C^k function f on W such that $0 \neq df \in d$ everywhere in W. Clearly, the orthogonal complement $d^\perp$ of any 1-dimensional timelike distribution d on $\bar{M}$ is an n-dimensional spacelike distribution on $\bar{M}$.

We adopt the following convention for indices: $\alpha, \beta, \ldots = 0, 1, \ldots, n$, and $i, j, \ldots = 1, \ldots, n$. In the sequel we shall identify vector fields on $\bar{M}$ with 1-forms by means of the metric g, so that exterior differentiation can be applied to vector fields.

LEMMA 1. *Let d be a 1-dimensional timelike C^k distribution on $\bar{M}$. Then d is closed if and only if $d^\perp$ is involutive.*

Proof. Suppose that d is closed. For $p \in \bar{M}$ let f be a function on a neighbourhood of p such that df spans d. The submanifold defined by $f = f(p)$ is orthogonal to df, so it is an integral manifold of $d^\perp$ through p.

Now let $d^\perp$ be involutive and let $p \in \bar{M}$. By Theorem 1 of [1], p. 90, we can find a coordinate system $x^0, \ldots, x^n$ at p such that $x^\alpha(p) = 0$ and the equation $x^0 = \xi$ defines an integral manifold of $d^\perp$ whenever $|\xi|$ is sufficiently small. Since dx^0 is orthogonal to the submanifolds $x^0 = \xi$, we have $0 \neq dx^0 \in d$, which completes the proof.

Therefore, in the case $n = 1$, each 1-dimensional timelike C^k distribution is closed, since its 1-dimensional orthogonal complement is involutive.

LEMMA 2. *Suppose that $(\bar{M}, g)$ is time oriented and d is a closed timelike distribution on $\bar{M}$. Then for each $p \in \bar{M}$ there exists a coordinate system $(W, \varphi) = (W, x^0, \ldots, x^n)$ at p such that*

(1) $\varphi(p) = 0$,

(2) $\varphi(W) = \{(y^0, \ldots, y^n) \mid |y^\alpha| < a\}$ *for some* $a > 0$,

(3) *the equation $x^0 = \xi$ defines an integral manifold of $d^\perp$ whenever $|\xi| < a$,*

(4) *the system of equations $x^i = \xi^i$ defines an integral curve of d whenever $|\xi^i| < a$,*

(5) *the field $\partial/\partial x^0$ is coherent with the time orientation.*

Proof. Using Theorem 1 of [1], p. 90, choose coordinate systems $y^0, \ldots, y^n$ and $z^0, \ldots, z^n$ at p such that $y^\alpha(p) = z^\alpha(p) = 0$ and the equations $y^0 = \xi^0$ (respectively, $z^i = \xi^i$) define integral manifolds of $d^\perp$ (respectively, integral curves of d) whenever $|\xi^\alpha|$ are sufficiently small. We have $dy^0, \partial/\partial z^0 \in d$, since dy^0 is orthogonal to each submanifold $y^0 = \xi^0$, and $\partial/\partial z^0$ is tangent to each curve $z^i = \xi^i$. Thus

$$0 \neq g\left(dy^0, \frac{\partial}{\partial z^0}\right) = dy^0\left(\frac{\partial}{\partial z^0}\right) = \frac{\partial y^0}{\partial z^0},$$

hence the transformation $(z^0, \ldots, z^n) \mapsto (y^0, z^1, \ldots, z^n)$ has the non-zero Jacobi determinant. This shows that $x^0, \ldots, x^n$, where $x^0 = y^0$, $x^i = z^i$, is a coordinate system at p. Conditions (1)-(4) are obvious; to obtain (5) it is sufficient to replace x^0 by $-x^0$, if necessary. This completes the proof.

LEMMA 3. *Under the assumptions of Lemma 2, the atlas consisting of coordinate systems with properties* (1)-(5) *satisfies the relations*

$$g_{00} < 0, \quad g_{0i} = g_{i0} = 0, \tag{6}$$

$$A_0^{i'} = A_i^{0'} = 0, \tag{7}$$

$$A_0^{0'} > 0, \tag{8}$$

where $x^0, \ldots, x^n$ and $x^{0'}, \ldots, x^{n'}$ are coordinate systems of the above type and $A_a^{a'} = \partial x^{a'}/\partial x^a$.

Conversely, given an atlas on $\bar{M}$ satisfying (6)-(8), *the base fields $\partial/\partial x^0$, determined by its charts, define a closed timelike distribution and a time orientation on $\bar{M}$.*

Proof. We have $\partial/\partial x^0 \in d$ and $\partial/\partial x^i \in d^\perp$, which yields (6). Clearly, $dx^{i'} \in d^\perp$, so

$$0 = g\left(dx^{i'}, \frac{\partial}{\partial x^0}\right) = \frac{\partial x^{i'}}{\partial x^0} = A_0^{i'}.$$

Similarly, $A_i^{0'} = 0$. The formula

$$\frac{\partial}{\partial x^{0'}} = A_{0'}^a \frac{\partial}{\partial x^a} = A_{0'}^0 \frac{\partial}{\partial x^0} \tag{9}$$

implies (8) in virtue of (5).

Now let an atlas satisfy (6)-(8). Then our assertion follows easily from (9). This completes the proof.

2. The tensor fields v, w and u. In the sequel, the triple $(\bar{M}, g, d)$ is assumed to satisfy the following two conditions:

(10) $(\bar{M}, g)$ is a time oriented $(n+1)$-dimensional C^k Riemannian manifold ($k = \infty, \omega$) with metric g of index one.

(11) d is a 1-dimensional closed timelike C^k distribution on $\bar{M}$, whose maximal integral curves are complete, i.e. each of them is the image set of a C^k mapping x: $\boldsymbol{R} \to \bar{M}$ such that $g(\dot{x}_t, \dot{x}_t) = -1$.

For $p \in \bar{M}$, $N(p)$ will denote the maximal integral manifold of the involutive distribution $d^\perp$ through p.

By an *admissible neighbourhood* $(W, \varphi) = (W, x^0, \ldots, x^n)$ of $p \in \bar{M}$ we shall mean a coordinate system satisfying (1)-(5). In terms of such a coordinate system we shall use the notation N_ξ for the integral manifold of $d^\perp$ defined by $x^0 = \xi$. Clearly, N_ξ is an open submanifold of $N(\varphi^{-1}(\xi, 0, \ldots, 0))$ (see [1], p. 88-95).

Let X be the unique unit vector field on $\overline{M}$ which is coherent with the time orientation and spans d. In view of (11), X is complete, so its flow (φ_t) is defined for each $t \in \boldsymbol{R}$.

Denoting by $\pi\colon T_p\overline{M} \to d_p^\perp$ the orthogonal projection, we define a linear isomorphism $d(t, p)\colon d_p^\perp \to d_q^\perp$, where $q = \varphi_t p$, by

$$d(t, p)Y = \pi(\varphi_t)_* Y.$$

We have $(\varphi_t)_* Y - d(t, p)Y \in d$, so

$$d \ni (\varphi_{s+t})_* Y - (\varphi_s)_* d(t, p)Y.$$

Hence

$$0 = \pi(\varphi_{s+t})_* Y - \pi(\varphi_s)_* d(t, p)Y = d(s+t, p)Y - d(s, \varphi_t p)d(t, p)Y.$$

Thus we obtain

$$d(s+t, p) = d(s, \varphi_t p)\circ d(t, p) \tag{12}$$

and, consequently,

$$\big(d(t, p)\big)^{-1} = d(-t, \varphi_t p). \tag{13}$$

LEMMA 4. *Suppose that* (W, φ) *is an admissible neighbourhood,* $p, q \in W$, *and*

$$\varphi(p) = (a, x^1, \ldots, x^n), \qquad \varphi(q) = (b, x^1, \ldots, x^n).$$

Then $q = \varphi_t p$, *where*

$$t = \int_a^b \sqrt{|g_{00}(x, x^1, \ldots, x^n)|}\, dx.$$

Proof. Let, e.g., $a \leqslant b$, so $t \geqslant 0$. Clearly, $\varphi_t p$ is determined uniquely by the following condition: there exists a C^k curve z of length t, with the origin at p and the end at $\varphi_t p$, whose tangent vectors are vectors of X multiplied by positive scalars. Using the curve $z\colon [a, b] \to \overline{M}$, given by

$$z(x) = \varphi^{-1}(x, x^1, \ldots, x^n),$$

we conclude that $\varphi_t p = z(b) = q$, as desired.

An n-dimensional submanifold M of $\overline{M}$ is called *spacelike* if so is each non-zero vector tangent to M.

LEMMA 5. *Suppose that* M *is a spacelike submanifold of* $\overline{M}$, $p \in M$, $t_0 \in \boldsymbol{R}$, $a \leqslant c \leqslant b$, *and* $x\colon [a, b] \to M$ *is a continuous curve such that* $x(c) = p$. *Then*

(i) *There exist a neighbourhood U of p in M and a C^k function f on U such that*

(14) $f(p) = t_0$ *and all the points* $\varphi_{f(q)}q$ *lie in* $N(\varphi_{t_0}p)$.

(ii) *There exists at most one continuous function f on M which satisfies* (14) *and is of class C^k. If M is an integral manifold of $d^\perp$, then the mapping F: $M \to N(\varphi_{t_0}p)$, given by $F(q) = \varphi_{f(q)}q$, satisfies the condition $F_{*,q} = d(f(q), q)$ for $q \in M$.*

(iii) *There exist a neighbourhood J of c in $[a, b]$ and a continuous function f on J such that*

(15) $f(c) = t_0$ *and the curve* $z(t) = \varphi_{f(t)}x(t)$ *lies in* $N(\varphi_{t_0}p)$.

(iv) *There exists at most one continuous function f on $[a, b]$ which satisfies* (15). *If x is differentiable, then so is f. In this case, if moreover M is an integral manifold of $d^\perp$, then*

$$\dot{z}(t) = d\big(f(t), x(t)\big)\dot{x}(t) \qquad \textit{for } t \in [a, b].$$

Proof. Choose an admissible neighbourhood $(W, x^0, \ldots, x^n)$ of $\varphi_{t_0}p$ and neighbourhoods V' of t_0 in $\boldsymbol{R}$ and U' of p in M such that $\varphi_t q \in W$ whenever $t \in V'$ and $q \in U'$. The function Q: $V' \times U' \to \boldsymbol{R}$ defined by $Q(t, q) = x^0(\varphi_t q)$ satisfies the conditions

$$Q(t_0, p) = 0 \quad \text{and} \quad \frac{\partial Q}{\partial t}(t_0, p) = \frac{d}{dt}x^0(\varphi_t p)\bigg|_{t=t_0} = g(dx^0, X) \neq 0.$$

By the implicit function theorem we obtain the desired existence and uniqueness statements.

Now suppose that M is an integral manifold of $d^\perp$. Let

$$Y = \dot{y}_0 \in T_q M.$$

We have

$$F_* Y = \frac{d}{ds}\varphi_{f(y(s))}y(s)\bigg|_{s=0} = (\varphi_{f(q)})_* Y + df(Y)X,$$

whence

$$F_* Y = \pi F_* Y = \pi(\varphi_{f(q)})_* Y = d\big(f(q), q\big)Y.$$

This completes the proof.

From Lemmas 4 and 5 we obtain easily

LEMMA 6. *Let $(W, \varphi) = (W, x^0, \ldots, x^n)$ be an admissible neighbourhood of p. If $|a|$ is sufficiently small and*

$$t_0 = \int_0^a \sqrt{|g_{00}(x, 0, \ldots, 0)|}\, dx\,,$$

then the function f on N_0, where

$$f(q) = \int_0^a \sqrt{|g_{00}(x, x^1(q), \dots, x^n(q))|}\, dx ,$$

satisfies (14) *and the mapping* $F\colon N_0 \to N_a$, *given by*

$$F(\varphi^{-1}(0, x^1, \dots, x^n)) = \varphi^{-1}(a, x^1, \dots, x^n), \quad \textit{i.e.} \quad F(q) = \varphi_{f(q)} q,$$

satisfies the relation $F_{*,p} = d(t_0, p)$.

Using admissible coordinates we define, on $\overline{M}$, tensor fields v, w and u of type $(0, 1)$, $(0, 2)$ and $(0, 1)$, respectively:

$$v_0 = 0, \quad v_i = \frac{\partial_i g_{00}}{2g_{00}} = \partial_i \log \sqrt{|g_{00}|}\,, \tag{16}$$

$$w_{00} = w_{0i} = w_{i0} = 0, \quad w_{ij} = \frac{\partial_0 g_{ij}}{2\sqrt{|g_{00}|}}, \tag{17}$$

$$u_a = \frac{\partial_0 v_a}{\sqrt{|g_{00}|}}. \tag{18}$$

The tensor transformation rule for (16), (17) and (18) follows easily from (7).

We have clearly $\partial_i v_j - \partial_j v_i = 0$ and $\partial_a v_0 = 0$, so we conclude immediately

(19) $u = 0$ identically if and only if $dv = 0$ identically.

Now we are going to characterize v and w in terms of the operators $d(t, p)$:

LEMMA 7. *The following conditions are equivalent:*

(i) $v = 0$ *identically on* $\overline{M}$;

(ii) $(\varphi_t)_* Y = d(t, p) Y$ *for* $Y \in d_p^\perp$, *i.e. each* φ_t *leaves the distribution* $d^\perp$ *invariant.*

Proof. Let $v = 0$, so that $\partial_i g_{00} = 0$. In the notation of Lemma 6, f is constant and $F = \varphi_{t_0}$. Hence for each $p \in \overline{M}$ there exists $\delta > 0$ such that $(\varphi_t)_* d_p = d_q$, where $q = \varphi_t p$, whenever $|t| < \delta$. It is now easy to conclude (ii) using (12) and (13).

Now suppose that (ii) is satisfied. As in Lemma 6,

$$\varphi_{t_0}\colon N_0 \to N_a \quad \text{and} \quad F\colon N_0 \to N_a,$$

so, by (ii) of Lemma 5, $f(q) = t_0$ for $q \in N_0$. The function

$$Q(a, x^1, \dots, x^n) = \int_0^a \sqrt{|g_{00}(x, x^1, \dots, x^n)|}\, dx$$

depends only on a, whence

$$\partial_i \sqrt{|g_{00}|} = \partial_i \partial_a Q = 0,$$

which completes the proof.

LEMMA 8. *The following conditions are equivalent:*

(i) $w = 0$ *identically on* $\bar{M}$;

(ii) $d(t, p)$ *is an isometry for each* $t \in \boldsymbol{R}$ *and* $p \in \bar{M}$.

Proof. Let $w = 0$, so that $\partial_0 g_{ij} = 0$. In the notation of Lemma 6, $F\colon N_0 \to N_a$ is an isometry, since $g_{ij} = g_{ij}(x^1, \ldots, x^n)$ define the induced metric on both N_0 and N_a. Hence for each $p \in \bar{M}$ there exists $\delta > 0$ such that $d(t, p)$ is an isometry whenever $|t| < \delta$. From (12) and (13) we can easily conclude (ii).

Now let (ii) be satisfied. In terms of Lemma 6, $F\colon N_0 \to N_a$ is an isometry, so

$$g_{ij}(0, x^1, \ldots, x^n) = g_{ij}(a, x^1, \ldots, x^n).$$

Hence $\partial_0 g_{ij} = 0$ which completes the proof.

By $\bar{g}$ we denote the positive-definite metric on $\bar{M}$ which coincides with $-g$ on d, with g on $d^\perp$ and such that d and $d^\perp$ are $\bar{g}$-orthogonal. In admissible coordinates we have clearly $\bar{g}_{00} = -g_{00}$, $\bar{g}_{0i} = \bar{g}_{i0} = 0$, $\bar{g}_{ij} = g_{ij}$. Thus both g and $\bar{g}$ induce the same positive-definite metric on each integral manifold of $d^\perp$. We use the symbols ∇, $\bar{\nabla}$ and $\Gamma^\alpha_{\beta\gamma}$, $\bar{\Gamma}^\alpha_{\beta\gamma}$ for the Riemannian connections and Christoffel symbols of $(\bar{M}, g)$ and $(\bar{M}, \bar{g})$, respectively. For any $Y \in T\bar{M}$ we have $\nabla_Y X$, $\bar{\nabla}_Y X \in d^\perp$, since $\bar{g}(X, X) = -g(X, X) = 1$.

In admissible coordinates the components of X are

$$\left(\frac{1}{\sqrt{|g_{00}|}}, 0, \ldots, 0\right).$$

It is easy to verify that

(20) $\quad v = 0$ identically if and only if $dX = 0$ identically.

Using the obvious relations

$$\Gamma^i_{00} = -\bar{\Gamma}^i_{00} = -g_{00} g^{is} v_s \tag{21}$$

and

$$\frac{1}{\sqrt{|g_{00}|}} g_{ik} \Gamma^k_{j0} = \frac{1}{\sqrt{|g_{00}|}} \bar{g}_{ik} \bar{\Gamma}^k_{j0} = w_{ij},$$

we obtain

$$\nabla_X X = v = -\bar{\nabla}_X X, \quad \text{where } v^\alpha = g^{\alpha\beta} v_\beta, \tag{22}$$

$$g(Y, \nabla_Z X) = w(Y, Z) = \bar{g}(Y, \bar{\nabla}_Z X) \quad \text{for } Y, Z \in d^\perp. \tag{23}$$

For any integral manifold N of $d^{\perp}$, X is a unit normal vector field on N in both metrics g and $\bar{g}$. Hence, for each $p \in \bar{M}$, w_p restricted to $T_pN(p)$ coincides in view of (23) with the second fundamental form of $N(p)$ at p in both Riemannian manifolds $(\bar{M}, g)$ and $(\bar{M}, \bar{g})$.

Thus from (22) and (23) we conclude

LEMMA 9. (i) $v = 0$ *identically if and only if each integral curve of* X *is a geodesic in* $(\bar{M}, g)$ *(or in* $(\bar{M}, \bar{g})$*).*

(ii) $w = 0$ *identically if and only if each integral manifold of* $d^{\perp}$ *is totally geodesic in* $(\bar{M}, g)$ *(or in* $(\bar{M}, \bar{g})$*).*

Example. Let (M^n, g^n) and (M^1, g^1) be complete Riemannian manifolds of dimensions n and 1 with metrics of indices 0 and 1, respectively. Set $(\bar{M}, g) = (M^n, g^n) \times (M^1, g^1)$. The formula

$$d_{(p,q)} = T_{(p,q)}(\{p\} \times M^1) \qquad \text{for } (p, q) \in \bar{M}$$

defines a closed timelike distribution d on $\bar{M}$. It is easy to see that in this case the fields v, w and u vanish identically. Both Riemannian manifolds $(\bar{M}, g)$ and $(\bar{M}, \bar{g}) = (M^n, g^n) \times (M^1, -g^1)$ are complete.

3. Certain connections between the tensor fields v, w, u and the topology of $\bar{M}$. The construction presented in this section gives a useful tool to prove some statements about the topology of $\bar{M}$.

Let M be a spacelike submanifold of $\bar{M}$. It is easy to see that the mapping

$$\Phi = \Phi_M \colon \boldsymbol{R} \times M \to \bar{M},$$

given by $\Phi(t, p) = \varphi_t p$, is locally diffeomorphic and its image set $A_M = \Phi(\boldsymbol{R} \times M)$ is open. By $G = \Phi^* g$ we shall denote the Riemannian metric on $\boldsymbol{R} \times M$, induced from g by Φ. In an obvious manner we define the timelike distribution $D = D_M = \Phi^* d$ on $\boldsymbol{R} \times M$ induced from d, and its G-orthogonal complement $D^{\perp} = D_M^{\perp} = \Phi^* d^{\perp}$. Clearly, D is closed and the flow (L_t) of the unit vector field $\Phi^* X$ on $\boldsymbol{R} \times M$, which spans D, is given by $L_t(s, p) = (t+s, p)$. Therefore, the triple $(\boldsymbol{R} \times M, G, D)$ satisfies (10) and (11), hence all the statements of the preceding sections are valid for it. By $P(t, p)$ we shall denote the maximal integral manifold of $D^{\perp}$ through (t, p). Using Lemma 2 of [4], p. 86, it is easy to see that

(24) for a piecewise C^k curve z in $\boldsymbol{R} \times M$, z lies in an integral manifold of $D^{\perp}$ if and only if $\Phi \circ z$ lies in an integral manifold of $d^{\perp}$.

Hence

(25) for any $(t, p) \in \boldsymbol{R} \times M$, $\Phi\big(P(t, p)\big) \subset N\big(\Phi(t, p)\big)$ and the mapping $\Phi \colon P(t, p) \to N\big(\Phi(t, p)\big)$ is a local isometry.

The natural projections of $\boldsymbol{R}\times M$ onto M and onto $\boldsymbol{R}$ will be denoted by p_M and by $p_{\boldsymbol{R}}$, respectively.

We can define a new positive-definite metric $h = h_M$ on M by

(26) $h(Y,Z) = g(\pi Y, \pi Z)$, where $\pi\colon T_p\bar{M}\to d_p^{\perp}$ is the orthogonal projection for $p\in M$.

By $\|d(t,p)\|$ we shall denote the norm of the operator $d(t,p)$ of the Hilbert space $d_p^{\perp}$ into $d_q^{\perp}$, $q=\varphi_t p$. In view of (13) and Lemma 8, the condition $w=0$ is equivalent to $\|d(t,p)\|\leqslant 1$ for any $t\in\boldsymbol{R}$ and $p\in\bar{M}$.

A spacelike submanifold M of $\bar{M}$ is said to be *covered* by $d^{\perp}$ if, for any maximal integral manifold P of $D_M^{\perp}$, the mapping $p_M\colon P\to M$ is a covering.

PROPOSITION 1. *Assume that one of the following conditions is satisfied:*

(i) $u=0$ *identically* (*e.g.*, $v=0$);

(ii) $\|d(t,p)\|\leqslant C$ *for some* $C\geqslant 0$ *and for each* $t\in\boldsymbol{R}$, $p\in\bar{M}$ (*e.g.*, $w=0$) *and* $\bar{M}$ *is complete in* $\bar{g}$.

Then each spacelike submanifold M *of* $\bar{M}$ *is covered by* $d^{\perp}$.

We prove first the following

LEMMA 10. *The assumptions of Proposition* 1 *being satisfied, suppose that* M *is a spacelike submanifold of* $\bar{M}$, $p\in M$, *and* $y\colon[a,b]\to P(t_0,p)$ *is a curve of class* C^k *such that* $y(a)=(t_0,p)$. *Set* $y(s)=\big(L(s),x(s)\big)$. *Then for each* $t\in\boldsymbol{R}$ *there exists a unique* C^k *function* $f_t\colon[a,b]\to\boldsymbol{R}$ *such that* $f_t(a)=t$ *and*

(27) *the curve* $s\mapsto\big(f_t(s),x(s)\big)$ *lies entirely in an integral manifold of* $D_M^{\perp}$.

Proof. If we show the existence and continuity of f_t, then from (iv) of Lemma 5, applied to the triple $(\boldsymbol{R}\times M, G, D)$, it will follow that f_t is unique, of class C^k and $f_{t_1}(s)<f_{t_2}(s)$ whenever $t_1<t_2$.

(i) Suppose that $u=0$. Set

$$f_t(s)=f(t,s)=L(s)+(t-t_0)e^{R(s)},\qquad\text{where } R(s_0)=\int_a^{s_0}v\big(\dot{x}(s)\big)\,ds.$$

For each $t\in\boldsymbol{R}$, $f_t(a)=t$. Let E be the set of all $t\in\boldsymbol{R}$ such that f_t satisfies (27). Clearly, $t_0\in E$ and

$$E=\bigcap_{s\in[a,b]}\left\{t\;\middle|\;g\left(X,\frac{d}{ds}\Phi\big(f(t,s),x(s)\big)\right)=0\right\}$$

is closed in $\boldsymbol{R}$. We are going to show that E is open. Let $r\in E$. Using the compactness of $[a,b]$, it is sufficient to prove

(28) For any $s_0\in[a,b]$ there exist $\delta>0$ and a connected neighbourhood J of s_0 in $[a,b]$ such that f_t restricted to J satisfies (27) whenever $|t-r|<\delta$.

Let $z(s) = \Phi(f_r(s), x(s))$. In view of (19) we may choose an admissible neighbourhood $(W, \varphi) = (W, x^0, \ldots, x^n)$ of $z(s_0)$ and a C^k function F on W such that $F(z(s_0)) = 0$ and $dF = v$. Thus

$$\partial_0 F = 0 \quad \text{and} \quad \partial_i F = \partial_i \log \sqrt{|g_{00}|},$$

so

$$\log \sqrt{|g_{00}(x^0, \ldots, x^n)|} = F(x^1, \ldots, x^n) + Q(x^0)$$

for some function Q. Hence

$$\begin{aligned} Q(x) &= \log \sqrt{|g_{00}(x, 0, \ldots, 0)|}, \\ \sqrt{|g_{00}(x^0, \ldots, x^n)|} &= \exp[F(x^1, \ldots, x^n)]\exp[Q(x^0)]. \end{aligned} \tag{29}$$

Now let J be a connected neighbourhood of s_0 in $[a, b]$ such that $z(J) \subset W$. We have $F(z(s_1)) = R(s_1) - R(s_0)$ for $s_1 \in J$. In fact, let, e.g., $s_0 < s_1$ and $A = [s_0, s_1] \times [0, 1]$. Define $K\colon A \to \overline{M}$ by

$$K(s, t) = \Phi(tf_r(s), x(s)).$$

Using the Stokes formula and the fact that v is orthogonal to d, we obtain

$$0 = \int_A K^* dv = \int_{\partial A} K^* v = \int_{s_0}^{s_1} v(\dot{x}(s))\,ds - \int_{s_0}^{s_1} v(\dot{z}(s))\,ds.$$

Thus

$$R(s_1) - R(s_0) = \int_{s_0}^{s_1} v(\dot{x}(s))\,ds = \int_{s_0}^{s_1} v(\dot{z}(s))\,ds = \int_{s_0}^{s_1} dF(\dot{z}(s))\,ds = F(z(s_1)),$$

as desired.

If $|c|$ is sufficiently small, then the curve

$$J \ni s \mapsto z_c(s) = \varphi^{-1}\big(c, x^1(z(s)), \ldots, x^n(z(s))\big)$$

lies in an integral manifold of $d^\perp$. In view of Lemma 4,

$$z_c(s) = \varphi_{t(c,s)} z(s) = \Phi(f_r(s) + t(c, s), x(s)),$$

where

$$t(c, s) = \int_0^c \sqrt{\left|g_{00}\big(x, x^1(z(s)), \ldots, x^n(z(s))\big)\right|}\,dx,$$

so, by (29),

$$t(c, s) = \exp[F(z(s))] \int_0^c \exp[Q(x)]\,dx = \exp[-R(s_0)]t(c, s_0)\exp[R(s)].$$

Therefore

$$f_r(s) + t(c, s) = f(r + \exp[-R(s_0)]t(c, s_0), s).$$

If $|t-r|$ is sufficiently small, then

$$t-r = \exp[-R(s_0)]t(c, s_0) \quad \text{for some } c.$$

Thus (28) follows in view of (24).

(ii) Suppose that $\bar{M}$ is complete in $\bar{g}$ and $\|d(t, p)\| \leqslant C$ for all $t \in \boldsymbol{R}$ and $p \in \bar{M}$. Denote by E the set of all $t \in \boldsymbol{R}$ such that the desired continuous function f_t exists. Clearly, $t_0 \in E$, since we may set $f_{t_0} = L$. For $r \in E$, let s_0 be the supremum of those $s_1 \in [a, b]$ for which

(30) there exist $\delta > 0$ and a differentiable function

$$(r-\delta, r+\delta) \times [a, s_1] \ni (t, s) \mapsto f(t, s) = f_t(s) \in \boldsymbol{R}$$

such that $f_t(a) = t$ and f_t satisfies (27) whenever $|t-r| < \delta$.

Set $z(s) = \big(f_r(s), x(s)\big)$ and choose an admissible neighbourhood $(W, \varphi) = (W, x^0, \ldots, x^n)$ of $z(s_0)$ in $\boldsymbol{R} \times M$. We have

$$z([a, s_1]) \subset W \quad \text{and} \quad (r-\delta, r+\delta) \times \{x(a)\} \subset W$$
$$\text{for some } s_1 > a \text{ and } \delta > 0.$$

If $|t-r| < \delta$, then the curve $z_t\colon [a, s_1] \to \boldsymbol{R} \times M$, given by

$$z_t(s) = \varphi^{-1}\Big(x^0\big(t, x(a)\big), x^1\big(z(s)\big), \ldots, x^n\big(z(s)\big)\Big),$$

lies on an integral manifold of $D^\perp$ and

$$p_M\big(z_t(s)\big) = p_M\Big(\varphi^{-1}\Big(x^0\big(z(s)\big), \ldots, x^n\big(z(s)\big)\Big)\Big) = p_M\big(z(s)\big) = x(s).$$

By setting $f_t(s) = p_{\boldsymbol{R}}\big(z_t(s)\big)$ we show that $s_0 \geqslant s_1 > a$, since

$$f_t(a) = p_{\boldsymbol{R}}\Big(\varphi^{-1}\Big(x^0\big(t, x(a)\big), \ldots, x^n\big(t, x(a)\big)\Big)\Big) = t.$$

Now choose $a < s_2 < s_1 < s_0 \leqslant s_3 \leqslant b$ such that $z([s_2, s_3]) \subset W$, and find a $\delta > 0$ and a function f for s_1 as in (30). Let $\delta_1 \in (0, \delta)$ be such that $\big(f_t(s_2), x(s_2)\big) \in W$ if $|t-r| < \delta_1$. For such t, the curve $z_t\colon [s_2, s_3] \to \boldsymbol{R} \times M$, given by

$$z_t(s) = \varphi^{-1}\Big(x^0\big(f_t(s_2), x(s_2)\big), x^1\big(z(s)\big), \ldots, x^n\big(z(s)\big)\Big),$$

lies on an integral manifold of $D^\perp$. We have

$$p_M\big(z_t(s)\big) = p_M\big(z(s)\big) = x(s) \quad \text{and} \quad p_{\boldsymbol{R}}\big(z_t(s_2)\big) = p_{\boldsymbol{R}}\big(f_t(s_2), x(s_2)\big) = f_t(s_2).$$

Therefore, $p_{\boldsymbol{R}} \circ z_t$ defines a C^k extension of f_t from $[a, s_1]$ to $[a, s_3]$ for $|t-r| < \delta_1$, which yields $s_0 \geqslant s_3$. Thus $s_0 = b$, since, otherwise, we could choose $s_3 > s_0$. Moreover, b satisfies (30), i.e., $(r-\delta, r+\delta) \subset E$ for some $\delta > 0$. Hence E is open in $\boldsymbol{R}$. Let $(\bar{r}, r)$ be the maximal interval in E, containing t_0.

Suppose that $r < \infty$. Using (iii) of Lemma 5, choose the maximal $s_0 \in (a, b]$ such that the desired function f_r can be defined on $[a, s_0)$. Let

$$z(s) = \Phi\big(f_r(s), x(s)\big),$$

so that

$$x(s) = \Phi\big(-f_r(s), z(s)\big) \quad \text{and} \quad \dot{x}(s) = (\varphi_{-f_r(s)})_* \dot{z}(s) - f_r'(s) X.$$

Hence $\pi\dot{x}(s) = d\big(-f_r(s), z(s)\big)\dot{z}(s)$, so, by (13),

$$\dot{z}(s) = d\big(f_r(s), x(s)\big)\pi\dot{x}(s) \quad \text{and} \quad \bar{g}\big(\dot{z}(s), \dot{z}(s)\big) \leqslant C^2 h\big(\dot{x}(s), \dot{x}(s)\big)$$

in view of our assumption. The h-length of x is finite and, consequently, so is the $\bar{g}$-length of z. Since $\bar{M}$ is complete in $\bar{g}$, it follows that the limit

$$q = \lim_{s \to s_0} z(s)$$

exists. Let $(W, \varphi) = (W, x^0, \ldots, x^n)$ be an admissible neighbourhood of q. For $s_1 \in [a, s_0)$, in view of (30) we obtain

$$z(s_1) = \lim_{t \to r} \Phi\big(f_t(s_1), x(s_1)\big).$$

Hence we may choose $s_1 < s_0$ and $r_1 < r$ such that

$$z\big([s_1, s_0)\big) \subset W \quad \text{and} \quad \Phi\big(f_t(s_1), x(s_1)\big) \in W$$

whenever $t \in [r_1, r]$. The curve $\bar{y}\colon [s_1, s_0] \to \bar{M}$, given by

$$\bar{y}(s) = \varphi^{-1}\Big(x^0\Big(\Phi\big(f_{r_1}(s_1), x(s_1)\big)\Big), x^1\big(z(s)\big), \ldots, x^n\big(z(s)\big)\Big),$$

lies in an integral manifold of $d^\perp$. In view of Lemma 4,

$$\bar{y}(s) = \varphi_{t(s)} z(s),$$

where $[s_1, s_0] \ni s \mapsto t(s) \in \boldsymbol{R}$ is a continuous function. Using the fact that the function

$$(\bar{r}, r) \ni t \mapsto f_t(s_1) - f_r(s_1)$$

is monotone increasing, we conclude that $f_{r_1}(s_1) - f_r(s_1) = t(s_1)$. Both curves $\Phi\big(t(s), z(s)\big)$ and

$$\Phi\big(f_{r_1}(s), x(s)\big) = \Phi\big(f_{r_1}(s) - f_r(s), z(s)\big)$$

lie in an integral manifold of $d^\perp$, hence $f_{r_1}(s) - f_r(s) = t(s)$ for $s \in [s_1, s_0)$ in view of (iv) of Lemma 5. Therefore, the limit $\lim\limits_{s \to s_0} f_r(s)$ exists. From (iii) of Lemma 5 we conclude easily that $s_0 = b$. Thus $(r - \delta, r + \delta) \subset E$ for some $\delta > 0$, which contradicts our choice of $(\bar{r}, r)$. Hence $r = \infty$ and, similarly, $\bar{r} = -\infty$, so that $E = \boldsymbol{R}$, which completes the proof of Lemma 10.

Proof of Proposition 1. We have

(31) Each $p \in M$ has a connected neighbourhood U in M such that $p_M^{-1}(U)$ is a disjoint union of integral manifolds of $D^\perp$ each of which is mapped diffeomorphically by p_M onto U.

In fact, let $B^n(r)$ denote the closed ball of centre 0 and radius r in $\boldsymbol{R}^n$. We may choose a neighbourhood V of $(0, p)$ in $P(0, p)$ and a diffeomorphism $H\colon B^n(1) \to \bar{V}$ such that $H(0) = (0, p)$, and $p_M\colon \bar{V} \to \bar{U}$ is a diffeomorphism for some neighbourhood U of p in M. For $z \in S^{n-1} = \partial B^n(1)$ we define a curve $y_z\colon [0, 1] \to P(0, p)$ by $y_z(s) = H(sz)$, so that $y_z(0) = (0, p)$. Set $x_z = p_M \circ y_z$. Applying Lemma 10 to $y = y_z$, $[a, b] = [0, 1]$ and $t_0 = 0$, we obtain a family of continuous functions

$$f_t^z\colon [0, 1] \to \boldsymbol{R}, \qquad t \in \boldsymbol{R}, z \in S^{n-1}.$$

For $t \in \boldsymbol{R}$ define $F_t\colon U \to \boldsymbol{R} \times M$ by

$$F_t\big(x_z(s)\big) = \big(f_t^z(s), x_z(s)\big)$$

and set $V_t = F_t(U)$. If $F_t(q) = F_{t_1}(q_1)$, then $q = q_1$ and we may write $q = q_1 = x_z(s)$, so $f_t^z(s) = f_{t_1}^z(s)$. By (iv) of Lemma 5, $f_t^z = f_{t_1}^z$, hence $t = t_1$. Therefore, the sets V_t are pairwise disjoint. It is easy to see that $p_M\colon V_t \to U$ is a one-one and onto map. Now, let

$$(t, q) \in p_M^{-1}(U) = \boldsymbol{R} \times U \qquad \text{and} \qquad q = x_z(s_0).$$

Define $y\colon [0, s_0] \to V$ by $y(s) = y_z(s_0 - s)$ and set $x = p_M \circ y$, so that $x(s) = x_z(s_0 - s)$, $x(0) = q$, $x(s_0) = p$. Let $\bar{f}_t$ be the function on $[0, s_0]$ determined for y as in Lemma 10 and set $t_0 = \bar{f}_t(s_0)$. The curve $\bar{z}\colon [0, s_0] \to \boldsymbol{R} \times M$, given by $\bar{z}(s) = \big(\bar{f}_t(s), x(s)\big)$, lies in an integral manifold of $D^\perp$, and $\bar{z}(s_0) = (t_0, p)$. It follows from (iv) of Lemma 5 that

$$\bar{z}(s) = \big(f_{t_0}^z(s_0 - s), x_z(s_0 - s)\big).$$

Hence

$$(t, q) = \bar{z}(0) = \big(f_{t_0}^z(s_0), q\big) = F_{t_0}(q)$$

which shows that

$$p_M^{-1}(U) = \bigcup_{t \in \boldsymbol{R}} V_t.$$

Now it is sufficient to prove that each V_t is an integral manifold of $D^\perp$. Let r_0 be the supremum of those $r \in [0, 1]$ for which F_{t_0} is differentiable on $p_M\big(H(B^n(r))\big)$. From (i) and (iv) of Lemma 5 it follows easily that $r_0 > 0$. For any $z_0 \in S^{n-1}$ choose a neighbourhood W of z_0 in S^{n-1}, and a neighbourhood J of r_0 in $(0, 1]$ such that on the neighbourhood

$$U' = \{x_z(s) \mid z \in W, s \in J\}$$

of $x_{z_0}(r_0)$ in $\bar{U}$ there exists a C^k function f which satisfies (14) with p replaced by $x_{z_0}(r_0)$, t_0 by $f_{t_0}^{z_0}(r_0)$, and φ_t by L_t. By (iv) of Lemma 5 this function coincides with F_{t_0}. Therefore, F_{t_0} is differentiable on a neighbourhood of $p_M\big(H(B^n(r_0))\big)$ in $\bar{U}$. Now it follows easily that $r_0 = 1$.

Each F_t is differentiable and maps U into an integral manifold of $D^\perp$. It is now easy to see that F_t is an embedding, so V_t is an integral manifold of $D^\perp$, which proves (31).

Now we are in a position to complete the proof of Proposition 1. Suppose that $p \in M$ and that P is a maximal integral manifold of $D^\perp$. Let a connected neighbourhood U of p in M and its decomposition

$$U = \bigcup_{q \in \boldsymbol{R} \times U} V_q$$

into integral manifolds V_q of $D^\perp$ with $q \in V_q$ be chosen as in (31). Clearly, for $q \in p_M^{-1}(U) \cap P$ we have

$$V_q \subset p_M^{-1}(U) \cap P,$$

which shows that $p_M^{-1}(U) \cap P$ is a disjoint union of open subsets of P each, of which is mapped by p_M diffeomorphically onto U. This completes the proof.

We need the following

LEMMA 11. *Let A be a subset of $\bar{M}$ such that*

(32) *A is a union of maximal integral curves of d as well as a union of maximal integral manifolds of $d^\perp$.*

Then either A is empty or $A = \bar{M}$.

Proof. Clearly, (32) holds also for the complement $\bar{M} - A$. Using admissible neighbourhoods, it is easy to verify that both A and its complement are open. This completes the proof.

PROPOSITION 2. *Suppose that a compact spacelike submanifold M of $\bar{M}$ is covered by $d^\perp$ and, for any $(t, p) \in \boldsymbol{R} \times M$, the mapping*

$$\Phi_M\colon P(t, p) \to N(\varphi_t p)$$

is a covering. Then $\bar{M} = A_M$ and one of the following cases holds:

(I) *Any maximal integral curve of X intersects M for exactly one parameter value. In this case, Φ is a diffeomorphism of the product $\boldsymbol{R} \times M$ onto $\bar{M}$; in particular, $\bar{M}$ is not compact.*

(II) *Any maximal integral curve of X intersects M for infinitely many parameter values. In this case, $\bar{M}$ is the space of a locally trivial bundle with base S^1 and fibre M; in particular, $\bar{M}$ is compact and its fundamental group contains the group of integers.*

Proof. Let a maximal integral manifold N of $d^{\perp}$ intersect A_M, say $N = N(\varphi_t p)$. Then $N = \Phi(P(t,p)) \subset A_M$. By Lemma 11, $A_M = \overline{M}$, hence each maximal integral curve of X intersects M.

Suppose that

(I) if $p \in M$ and $\varphi_t p \in M$, then $t = 0$.

In this case $\Phi\colon \boldsymbol{R}\times M \to \overline{M}$ is clearly one-one and onto, so it is a diffeomorphism. Each maximal integral curve of X meets M for exactly one parameter value.

The remaining case is now

(II) $\varphi_t p \in M$ for some $p \in M$ and $t \neq 0$.

We can clearly claim that $t > 0$ in (II). Since M is compact, there exists $\varepsilon > 0$ such that the conditions $q \in M$ and $\varphi_s q \in M$ imply that either $s = 0$ or $|s| \geqslant \varepsilon$.

Now choose $p \in M$ and $t > 0$ such that $\varphi_t p \in M$. For any $q \in M$, let $x\colon [0,1] \to M$ be a continuous curve with $x(0) = p$, $x(1) = q$. Set $N = N(\varphi_t p)$. The mappings

$$p_M\colon P(t,p) \to M \quad \text{and} \quad \Phi\colon P(0,\varphi_t p) \to N$$

are coverings. Let

$$y\colon [0,1] \to P(t,p)$$

be the p_M-lift of x with $y(0) = (t,p)$, say

$$y(s) = \big(T_0(s), x(s)\big),$$

and let

$$z\colon [0,1] \to P(0,\varphi_t p)$$

be the Φ-lift of $\Phi\circ y$ with $z(0) = (0,\varphi_t p)$, say

$$z(s) = \big(T_1(s), x_1(s)\big).$$

We have

$$\Phi\big(T_0(s), x(s)\big) = \Phi\big(y(s)\big) = \Phi\big(z(s)\big) = \Phi\big(T_1(s), x_1(s)\big),$$

so

$$\Phi\big(T(s), x(s)\big) = x_1(s) \in M, \quad \text{where } T(s) = T_0(s) - T_1(s). \tag{33}$$

We assert that $T(s) > 0$ for $s \in [0,1]$. Otherwise, choose the smallest $s_0 \in [0,1]$ with $T(s_0) = 0$. Clearly, $s_0 > 0$, since $T(0) = t > 0$, so $0 < T(s) < \varepsilon$ for s sufficiently near s_0, a contradiction with (33).

Therefore $\varphi_{T(1)} q \in M$, $T(1) > 0$. We have thus shown that for each $q \in M$ there exists $s > 0$ such that $\varphi_s q \in M$. Using the compactness of M we can define a function $T\colon M \to \boldsymbol{R}$ by

$$T(p) = \min\{t > 0 \mid \varphi_t p \in M\} \geqslant \varepsilon.$$

Let $p \in M$. Choose a coordinate system $(W, x^0, \ldots, x^n)$ at $\varphi_{T(p)}p$ such that $W \cap M$ is defined by $x^0 = 0$, and let V_1, U_1 be neighbourhoods of $T(p)$ in $\boldsymbol{R}$ and of p in M such that $\Phi(V_1 \times U_1) \subset W$. The function $F\colon V_1 \times U_1 \to \boldsymbol{R}$, given by $F(t, q) = x^0(\varphi_t q)$, satisfies the conditions

$$F\big(T(p), p\big) = 0 \quad \text{and} \quad \frac{\partial F}{\partial t}\big(T(p), p\big) = g(dx^0, X) \neq 0.$$

By the implicit function theorem there exist neighbourhoods $U \subset U_1$ of p and $V \subset V_1$ of $T(p)$, and a C^k function T_1 on U such that, for any $q \in U$, $T_1(q)$ is the unique element of V with $\Phi\big(T_1(q), q\big) \in M$. Since $T_1(p) = T(p) > 0$, we may assert $T_1 > 0$ by taking U smaller, if necessary.

Now suppose that $p_m \to p$, $p_m \in U$. Clearly, $\varepsilon \leqslant T(p_m) \leqslant T_1(p_m)$, so

$$\limsup T(p_m) \leqslant T(p).$$

Suppose that $\liminf T(p_m) < T(p)$. Then some number $t_0 \in [\varepsilon, T(p))$ is the limit of a subsequence of $T(p_m)$, hence $\Phi(t_0, p) \in M$ in view of compactness, which contradicts our choice of $T(p)$. Therefore, $T(p) = \lim T(p_m)$ which proves that T is continuous. Let $U' \subset U$ be a neighbourhood of p such that $T(U') \subset V$. Clearly, $T = T_1$ on U', hence T is of class C^k.

Similarly, we can prove that for each $q \in M$ there exists $s < 0$ such that $\varphi_s q \in M$ and that the function $S\colon M \to \boldsymbol{R}$, where

$$S(q) = \max\{s < 0 \,|\, \varphi_s q \in M\} \leqslant -\varepsilon,$$

is of class C^k. Define $Z\colon M \to M$ by $Z(q) = \varphi_{T(q)} q$. It is easy to see that Z is a diffeomorphism, since $Z^{-1}(q) = \varphi_{S(q)} q$.

Now let us define a continuous mapping $H\colon [0, 1] \times M \to \bar{M}$ by

$$H(s, p) = \Phi\big(sT(p), p\big) = \varphi_{sT(p)} p.$$

We assert that H maps onto $\bar{M} = A_M$. Given $p \in M$ and $t \geqslant 0$ (respectively, $t \leqslant 0$), define the sequence T_m (respectively, S_m) by

$$T_1 = T(p), \quad T_{m+1} = T\big(\Phi(T_1 + \ldots + T_m, p)\big)$$

(respectively, $S_1 = S(p)$, $S_{m+1} = S\big(\Phi(S_1 + \ldots + S_m, p)\big)$). These definitions make sense since $\Phi(T_1 + \ldots + T_m, p) \in M$. For $m = 1$ it is clear. Suppose it holds for an arbitrary m. Then

$$\begin{aligned}\Phi(T_1 + \ldots + T_{m+1}, p) &= \Phi\big(T_{m+1}, \Phi(T_1 + \ldots + T_m, p)\big)\\ &= Z\big(\Phi(T_1 + \ldots + T_m, p)\big) \in M\end{aligned}$$

(similarly, $\Phi(S_1 + \ldots + S_m, p) \in M$). We have $T_1 + T_2 + \ldots = \infty$, since $T_m \geqslant \varepsilon$ (respectively, $S_1 + S_2 + \ldots = -\infty$, since $S_m \leqslant -\varepsilon$). Therefore, we can choose m and $s \in [0, 1]$ such that

$$t = T_1 + \ldots + T_{m-1} + sT_m$$

(respectively, $t = S_1 + \ldots + S_{m-1} + sS_m$). Set

$$q = \Phi(T_1 + \ldots + T_{m-1}, p) \in M,$$

so $T_m = T(q)$ (respectively, $q = \Phi(S_1 + \ldots + S_{m-1}, p)$, so $S_m = S(q)$). We have

$$\varphi_t p = \varphi_{sT_m} q = \Phi(sT(q), q) = H(s, q)$$

(respectively, $\varphi_t p = \Phi((s-1)S_m, \Phi(S_m, q)) = \Phi((s-1)S(q), Z^{-1}(q))$ $= \Phi((1-s)T(Z^{-1}(q)), Z^{-1}(q)) = H(1-s, Z^{-1}(q))$), which proves that H maps onto $\bar{M}$.

Suppose that $H(t, p) = H(s, q)$, so

$$\Phi(tT(p), p) = \Phi(sT(q), q).$$

Let, e.g., $tT(p) \geqslant sT(q)$, hence

$$q = \Phi(tT(p) - sT(q), p),$$

so either $tT(p) - sT(q) \geqslant T(p)$, which yields $s = 0$, $t = 1$ and $q = Z(p)$, or $sT(p) = tT(q)$, hence $p = q$ and $s = t$. Therefore, H induces a homeomorphism of B onto $\bar{M}$, where B is the space obtained from $[0, 1] \times M$ by identifying $\{0\} \times M$ with $\{1\} \times M$ by means of the diffeomorphism Z. Thus $\bar{M}$ is the space of a locally trivial bundle with base S^1 and fibre M. It is easy to see that the homomorphism of fundamental groups, induced by the bundle projection $\bar{M} \to S^1$, is surjective, hence $\pi_1 \bar{M} \supset \pi_1 S^1$. This completes the proof.

Now we are in a position to prove

THEOREM 1. *Suppose that M is a compact spacelike submanifold of $\bar{M}$. Assume that one of the following conditions is satisfied*:

(i) *$\bar{M}$ is complete in $\bar{g}$ and $\|d(t, p)\| \leqslant C$ for each $t \in \mathbf{R}$ and $p \in \bar{M}$*;

(ii) *M is covered by $d^{\perp}$ and its fundamental group $\pi_1 M$ is finite*;

(iii) *$v = 0$ identically on $\bar{M}$ and the first real cohomology group $H^1(M, \mathbf{R})$ is trivial.*

Then the assertion of Proposition 2 *holds.*

Moreover, any of conditions (ii) *and* (iii) *implies that each maximal integral manifold of $d^{\perp}$ is compact.*

Proof. (i) Let $\bar{M}$ be complete in $\bar{g}$ and let $\|d(t, p)\| \leqslant C$ for each t and p. We are going to prove that, for any $(t, p) \in \mathbf{R} \times M$, $P = P(t, p)$ is complete in the metric induced from $\mathbf{R} \times M$, which we denote also by G. Our assertion will then follow from Proposition 2, together with Proposition 1, (25) and Theorem 4.6 of [4], p. 176.

Given a vector $Y \in TP$, say $Y = \dot{y}_0$, where $y_s = (t(s), x_s)$, we have clearly

$$x_s = \varphi_{-t(s)} \Phi(y_s) \quad \text{and} \quad \dot{x}_0 = (\varphi_{-t(0)})_* \Phi_* Y - t'(0) X,$$

hence

$$\pi(p_M)_* Y = \pi\dot{x}_0 = d\big(-t(0),\, \Phi(y_0)\big)\Phi_* Y. \tag{34}$$

Therefore, the metric $p_M^* h$ on P, induced by the covering $p_M\colon P \to M$ from the metric h on M, satisfies the condition

$$p_M^* h(Y, Y) \leqslant C^2 G(Y, Y). \tag{35}$$

In view of the compactness of M and by Theorem 4.6 of [4], p. 176, P is complete in $p_M^* h$. It follows from (35) that each Cauchy sequence in (P, G) is a Cauchy sequence in $(P, p_M^* h)$. Hence P is complete in G.

(ii) Suppose that M is covered by $d^\perp$ and that $\pi_1 M$ is finite. For any $(t, p) \in \boldsymbol{R} \times M$, $p_M\colon P \to M$ is a finite covering, where $P = P(t, p)$, hence P is compact. In view of (25) and Corollary 4.7 of [4], p. 178, $\Phi\colon P \to N(\varphi_t p)$ is a covering and $N(\varphi_t p)$ is compact. Our assertion follows now from Proposition 2.

(iii) Let $v = 0$ and $H^1(M, \boldsymbol{R}) = 0$. It follows easily from Lemma 7 that the flow L_t of $\Phi^* X$ leaves the distribution $D^\perp$ invariant. In view of Proposition 1, M is covered by $d^\perp$. Therefore, for any continuous curve $x\colon [0, 1] \to M$, the value

$$z(x) = p_{\boldsymbol{R}}\big(y(1)\big) - p_{\boldsymbol{R}}\big(y(0)\big),$$

where y is any p_M-lift of x to a maximal integral manifold of $D^\perp$, depends only on x. Clearly, z may be considered as a 1-dimensional real singular co-chain in M. Any 2-dimensional singular simplex in M admits a p_M-lift to a maximal integral manifold of $D^\perp$, hence $\delta z = 0$, that is, z is a co-cycle. By our assumption, there exists a 0-dimensional real singular co-chain f such that $z = \delta f$. In other words, f is a real function on M and $z(x) = f\big(x(1)\big) - f\big(x(0)\big)$ for any continuous curve $x\colon [0, 1] \to M$. It follows that for any maximal integral manifold P of $D^\perp$ the covering projection $p_M\colon P \to M$ is one-one, so it is a diffeomorphism. In fact, for (s, p), (t, p) in P, choose a continuous curve $y\colon [0, 1] \to P$ such that $y(1) = (t, p)$ and $y(0) = (s, p)$. Setting $x = p_M \circ y$, we have

$$0 = f\big(x(1)\big) - f\big(x(0)\big) = z(x) = p_{\boldsymbol{R}}\big(y(1)\big) - p_{\boldsymbol{R}}\big(y(0)\big) = t - s,$$

so $t = s$, as desired. Hence P is compact. The rest of the proof goes as in (ii).

Remark. Suppose that $\bar{M}$ is compact and $\|d(t, p)\| \leqslant C$ for any t and p. If M is a compact spacelike submanifold of $\bar{M}$, then, by Theorem 1, $\bar{M}$ is the space of a bundle with fibre M over S^1. Namely, to obtain $\bar{M}$ we must identify the subsets $\{0\} \times M$ and $\{1\} \times M$ of $[0, 1] \times M$ by the diffeomorphism $Z\colon M \to M$, where $Z(p) = \varphi_{T(p)} p$ (see the proof of Proposition 2).

Now let $C = 1$, i.e., $w = 0$ identically. Then Z is an isometry of (M, h) onto itself. In fact, let $Y \in T_pM$, say $Y = \dot{y}(0)$. Then

$$Z_*Y = \frac{d}{ds}\varphi_{T(y(s))}y(s)\Big|_{s=0} = (\varphi_{T(p)})_*Y + dT(Y)X,$$

so $Z_*Y = (\varphi_{T(p)})_*\pi Y + aX$ for some a, since X is invariant by $\varphi_{T(p)}$. Therefore

$$\pi Z_*Y = d(T(p), p)\pi Y$$

and our assertion follows from Lemma 8.

Using Theorem 1 we obtain

COROLLARY 1. *Suppose that $\bar{M}$ is complete in $\bar{g}$ and $\|d(t, p)\| \leqslant C$ for each $t \in \boldsymbol{R}$ and $p \in \bar{M}$. If $\bar{M}$ admits a compact spacelike submanifold, then $\bar{M}$ cannot simultaneously be compact and have a finite fundamental group. In particular, $(\bar{M}, \bar{g})$ cannot be a space of constant positive curvature.*

Proof. Suppose that $(\bar{M}, \bar{g})$ is a space of constant positive curvature. By Corollary 2.4.10 of [5], p. 69, $\bar{M}$ is compact and has a finite fundamental group, which contradicts Theorem 1.

THEOREM 2. *Suppose that $\bar{M}$ is non-compact, complete in $\bar{g}$ and $\|d(t, p)\| \leqslant C$ for any $t \in \boldsymbol{R}$ and $p \in \bar{M}$. Then any two compact spacelike submanifolds of $\bar{M}$ are diffeomorphic.*

Proof. Suppose that M is a compact spacelike submanifold of $\bar{M}$. Then (I) of Proposition 2 holds. For another compact spacelike submanifold M' of $\bar{M}$, the mapping $p_M\colon \Phi^{-1}(M') \to M$ is one-one. In fact, otherwise M' would meet some integral curve of X twice which, in view of Theorem 1, would imply that $\bar{M}$ is compact. Hence $\Phi^{-1}(M')$ is diffeomorphic to M, which completes the proof.

Let $\sigma = g^{ij}v_iv_j = \bar{g}^{ij}v_iv_j$ denote the square of length of v. From (21) we obtain easily the relation $\nabla_0 v_0 = -\bar{\nabla}_0 v_0 = \sigma g_{00}$ in admissible coordinates. Since the field v is spacelike, it is easy to verify that v is parallel in $(\bar{M}, g)$ (or in $(\bar{M}, \bar{g})$) if and only if v vanishes.

We shall consider some consequences of the case where v or w vanishes identically.

PROPOSITION 3. *Suppose that $v = 0$ identically on $\bar{M}$, and N is a maximal integral manifold of $d^{\perp}$. Then*

(i) *$\varphi_t N$ is a maximal integral manifold of $d^{\perp}$ for each $t \in \boldsymbol{R}$.*

(ii) *$\bar{M} = \bigcup_{t \in \boldsymbol{R}} \varphi_t N$.*

(iii) *Any two maximal integral manifolds of $d^{\perp}$ are diffeomorphic.*

(iv) *The set $E = \{t \in \boldsymbol{R} \mid \varphi_t N = N\}$ is independent of the choice of N and is an additive subgroup of $\boldsymbol{R}$. Therefore, either $E = 0$, or $E = a\boldsymbol{Z}$ for some $a > 0$, or E is dense in $\boldsymbol{R}$. If $E = 0$, then $\bar{M}$ is diffeomorphic to the* *product* $\boldsymbol{R}\times N$. *If* $E = a\boldsymbol{Z}$, $a > 0$, *then* $\bar{M}$ *is the space of a locally trivial bundle with base* S^1 *and fibre* N. *If* E *is dense in* $\boldsymbol{R}$, *then none of maximal integral manifolds of* $d^{\perp}$ *has the relative topology.*

Proof. (i) follows immediately from Lemma 7, and (ii) from Lemma 11. By (ii), each maximal integral manifold of $d^{\perp}$ is of the form $\varphi_s N$, hence (iii) holds and E is independent of the choice of N.

Let $E = 0$. Then $\Phi_N\colon \boldsymbol{R}\times N \to \bar{M}$ is a diffeomorphism, since $p\in N$ and $\varphi_t p \in N$ implies $t = 0$.

Suppose that $E = a\boldsymbol{Z}$, $a > 0$. Define a continuous mapping

$$H\colon [0,1]\times N \to \bar{M}$$

by

$$H(t,p) = \varphi_{ta}p.$$

Let B be the space obtained from $[0,1]\times N$ by identifying $(1,p)$ with $(0,\varphi_a p)$ for each $p\in N$. Thus B is the space of a locally trivial bundle with base S^1 and fibre N, and H induces a continuous one-one mapping $\bar{H}$ of B onto $\bar{M}$. For the point q of B, determined by $(1,p)$ and $(0,\varphi_a p)$, the sets of the type

$$V(\varepsilon, U) = (1-\varepsilon, 1]\times U \cup [0,\varepsilon)\times\varphi_a U,$$

where $\varepsilon\in(0,1)$, and U runs over neighbourhoods of p in N, determine a base of neighbourhoods of q in B. Clearly, the $\bar{H}$-image of such a base neighbourhood,

$$\begin{aligned} H\big(V(\varepsilon, U)\big) &= \Phi_N\big((a-\varepsilon a, a]\times U\big)\cup\Phi_N\big([a, a+\varepsilon a)\times U\big)\\ &= \Phi_N\big((a-\varepsilon a, a+\varepsilon a)\times U\big), \end{aligned}$$

is open, since Φ_N is locally diffeomorphic. Now it is easy to verify that $\bar{H}$ is open, hence it is a homeomorphism. If E is dense, we may choose a sequence $t_m\in E$, $t_m\neq 0$, $t_m\to 0$. For an admissible neighbourhood $(W, x^0, \ldots$ $\ldots, x^n)$ of p, the equation $x^0 = 0$ defines a neighbourhood of p in $N(p)$. This neighbourhood is not of the form $N(p)\cap U$ for any open subset U of $\bar{M}$, since $p\in U$ implies $\varphi_{t_m}p\in N(p)\cap U$ for sufficiently large m. This completes the proof.

Now suppose that w vanishes identically on $\bar{M}$. By (34) and Lemma 8, $p_M\colon P\to M$ is a local isometry for any maximal integral manifold P of $D_M^{\perp}$, M being a spacelike submanifold of $\bar{M}$ considered with the metric $h = h_M$. Therefore we have

PROPOSITION 4. *Suppose that* $w = 0$ *identically and* M *is a spacelike submanifold of* $\bar{M}$. *If any of the Riemannian manifolds* $(\bar{M}, g)$ *and* $(\bar{M}, \bar{g})$ *is flat, then so is* (M, h). *If* M *is complete in the induced metric, then in this case the k-th homotopy group of* M *is trivial for each* $k\geqslant 2$. *If, moreover,* M *is compact and* $\bar{M}$ *is orientable, then the Euler characteristic of* M *is even.*

Proof. (M, h) is flat in view of the preceding remark, (25) and Lemma 9. If M is complete, then the relation $h(Y, Y) = g(Y, Y) + (g(Y, X))^2$ for $Y \in TM$ shows that M is complete also in h, hence our assertion on homotopy groups follows, e.g., from Hadamard-Cartan theorem (cf. [2], p. 206). If $\bar{M}$ is orientable, then so is M, since X is a normal vector field on M. From Proposition 6.13 of [3], p. 41, it follows now that $\chi(M)$ is even if M is compact. This completes the proof.

COROLLARY 2. *Suppose that $\bar{M}$ is 3-dimensional, complete in $\bar{g}$, and $w = 0$ identically. If one of $(\bar{M}, g)$ and $(\bar{M}, \bar{g})$ is flat and $\bar{M}$ admits a compact spacelike submanifold, then the fundamental group of $\bar{M}$ contains the group of integers.*

Proof. Let M be a compact spacelike submanifold of $\bar{M}$. In view of Proposition 4, (M, h) is flat, hence M is homeomorphic either to the torus or to the Klein bottle. Considering both cases of Theorem 1, we conclude our assertion.

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