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WEAKLY CONFLUENT MAPPINGS AND A CLASSIFICATION OF CONTINUA

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This note* is a continuation of work done by Epps in [2]. Our main purpose is to answer a question of J. B. Fugate by giving a characterization of the continua which are weakly confluent images of dendrites.

Our notation follows Whyburn [7] and Epps [2]. A *continuum* is a compact, connected, metric space. A function $f: X \to Y$, where X and Y are continua, is said to be *weakly confluent* if for each continuum C in Y there exists a continuum D in X such that $f(D) = C$.

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1. Weakly confluent images of dendrites.

THEOREM 1. *Let X be a continuum. Let x_1 and x_2 be points of X and let ε be a positive number. If X is the weakly confluent image of a dendrite, then X does not contain a sequence $A_1, A_2, \ldots$ of distinct arcs with endpoints x_1 and x_2 such that the arcs A_i agree on the ε-neighbourhoods of both x_1 and x_2.*

The proof is the same as that given in [2], Example 2.

The following corollary provides a characterization of the continua which are weakly confluent images of dendrites.

COROLLARY 1. *A locally connected continuum X is a weakly confluent image of some dendrite if and only if X satisfies the following two conditions:*

(i) *Each true cyclic element of X is a finite graph.*

(ii) *If $E_1, E_2, \ldots$ is a sequence of distinct true cyclic elements of X all of which lie in a cyclic chain C, then the sequence $E_1, E_2, \ldots$ has at most two cluster points. Each cluster point is an endpoint of the cyclic chain C. No cluster point of $E_1, E_2, \ldots$ lies in a true cyclic element of X.*

Proof. Necessity. Let X be a continuum which is the weakly confluent image of some dendrite. Let C be a true cyclic element of X.

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By [5], every connected subset of X is arcwise connected. By [8], p. 150, C contains an arc B with endpoints y and z such that $B\setminus\{y, z\}$ is an open subset of C. Let $x_1, x_2 \in B\setminus\{y, z\}$. Let (x_1x_2) denote the open arc in B from x_1 to x_2. Since C is cyclic, there is an arc $A_1 \subset C\setminus(x_1x_2)$ with endpoints x_1 and x_2. Suppose that $A_1, \ldots, A_n$ are distinct arcs in $C\setminus(x_1x_2)$ with endpoints x_1 and x_2 such that $A_1\cup \ldots \cup A_n\cup(x_1x_2)$ is a finite graph. If $C \not\subset A_1\cup \ldots \cup A_n\cup(x_1x_2)$, let

$$w \in C\setminus\big(A_1\cup \ldots \cup A_n\cup(x_1x_1)\big).$$

Let A'_{n+1} be an arc in C which passes through w and which has endpoints x_1 and x_2. Let D be the component of $A'_{n+1}\setminus(A_1\cup \ldots \cup A_n)$ which contains w. Let A_{n+1} be an arc in $A_1\cup \ldots \cup A_n\cup D$ with endpoints x_1 and x_2 such that $w \in A_{n+1}$. Then $A_1\cup \ldots \cup A_{n+1}\cup(x_1x_2)$ is a finite graph, and the arcs $A_1, A_2, \ldots, A_{n+1}$ are distinct. By Theorem 1,

$$C \subset A_1\cup \ldots \cup A_n\cup(x_1x_2)$$

for some natural number n, so C is a finite graph.

It follows immediately from Theorem 1 that X satisfies condition (ii).

Sufficiency. If X is a locally connected continuum which satisfies (i) and (ii), it is easy to see that there is a sequence $A_1, A_2, \ldots$ of finite graphs in X such that

(a) each A_i is a union of cyclic elements of X,

(b) $A_i \subset A_{i+1}$ for each i, and

(c) $A_1\cup A_2\cup \ldots$ is dense in X.

For each $i = 1, 2, \ldots$ define $\pi_i\colon A_{i+1} \to A_i$ by setting $\pi_i(x) = x$ for each $x \in A_i$ and letting $\pi_i(x)$ to be the unique element of A_i which is in the closure of the component of $A_{i+1}\setminus A_i$ containing x for each $x \in A_{i+1}\setminus A_i$. Then π_i is a monotone retraction of A_{i+1} onto A_i.

Now, $X = \varprojlim(A_i, \pi_i)$ is the inverse limit of finite connected graphs with monotone, simplicial retractions for bonding maps. By [2], Theorem 4, there are a dendrite D and a weakly confluent map of D onto X.

2. Strongly regular continua. In [2] Epps calls a continuum X *strongly regular* if there exists a sequence $S_1, S_2, \ldots$ of finite subsets of X such that, for each natural number n, $X\setminus S_n$ has finitely many components and each component of $X\setminus S_n$ has diameter less than $1/n$. The next theorem provides a characterization of strongly regular curves.

A continuum is said to be *regular* if it has a basis of open sets with finite boundaries.

THEOREM 2. *A regular continuum X is not strongly regular if and only if there exist points $a, b \in X$ and a countable set D such that every finite cutting of X between a and b meets D and, for each $d \in D$, $X\setminus\{d\}$ has infinitely many components.*

Proof. Necessity. We say that a subset T of X has *property G* provided that T is finite and $X\setminus T$ has finitely many components. Let T_1 and T_2 be two subsets of X with property G. Then the set $T_1\cup T_2$ is finite and at most finitely many components of $X\setminus(T_1\cup T_2)$ have closures which meet at least two points of $T_1\cup T_2$, since it is well known (see [4], 1.6) that in a regular continuum, for every infinite collection of disjoint connected sets, the diameters of the sets converge to zero. Hence, all but finitely many components of $X\setminus(T_1\cup T_2)$ are components either of $X\setminus T_1$ or of $X\setminus T_2$. Thus, $T_1\cup T_2$ also has property G. If X is not strongly regular, then it follows from the fact that the union of two sets with property G also has property G and from the compactness of X that there exist $a, b\in X$ such that no set with property G separates a and b. Let D be the set of points each of which separates X into infinitely many components. By [7], III.8.4, D is countable. Since no subset of X with property G separates a and b in X, every finite cutting of X between a and b meets D.

Sufficiency is clear.

COROLLARY 2. *Every cyclic regular continuum is strongly regular.*

3. Classification of continua. Epps considered the following six classes of continua in [2]:

(I) regular continua;

(II) strongly regular continua;

(III) continua in which all connected sets are arcwise connected;

(IV) inverse limits of connected finite graphs with monotone simplicial bonding maps;

(V) inverse limits of connected finite graphs with monotone simplicial retractions as bonding maps;

(VI) weakly confluent images of dendrites.

We have the following relations among these six classes of continua and no other relations hold:

$$\begin{array}{ccc} (\mathrm{I}) & \supset & (\mathrm{II}) \\ \cup & & \cup \\ (\mathrm{III}) & & (\mathrm{IV}) \\ \cup & & \cup \\ (\mathrm{V}) & = & (\mathrm{VI}) \end{array}$$

Epps proved in [2] that (I) $\supsetneqq$ (II), (IV) $\supsetneqq$ (V) and (V) $\subset$ (VI). A proof that (IV) $\subset$ (II), due to A. Lelek, is given in [1], Theorem 2.3. That proof shows that every continuum in (IV) has uncountably many local cut-points. Hence, Sierpiński's triangular continuum (see [3], p. 276) is a continuum which is in (II) but not in (IV). Sierpiński's triangular continuum is also an example of a continuum which is in (II), but not in (III) (see, for example, [6], Theorem 1). Tymchatyn proved in [6] that (III) $\subset$ (I).

It follows from [5] that (V) $\subset$ (III). This also follows from Corollary 1 of this paper and the fact (see Whyburn [7], IV.11.2) that the property of containing no connected subset which is not arcwise connected is cyclicly extensible. By the proof of Corollary 1 of this paper, (VI) $\subset$ (V). To complete the classification we give an example of a continuum which is in (III) but not in (II).

Example. Let X' be the Knaster dyadic continuum, i.e., X' is the plane continuum consisting of the segment $0 \leqslant x \leqslant 1$, $y = 0$ and of the semi-circles

$$\left(x - \frac{2k-1}{2^n}\right)^2 + y^2 = \frac{1}{4^n}, \quad y \geqslant 0,$$

where $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2^{n-1}$. Let M be the union of all line segments in the plane where one endpoint has polar coordinates $\{0, 0\}$ and the other endpoint has polar coordinates $\{\pi/n, \pi/n\}$, $n = 1, 2, \ldots$

For each ordered pair (k, n) such that $n = 1, 2, \ldots$ and $k = 1, \ldots, 2^{n-1}$, let $M(k, n)$ be a space homeomorphic to M and of diameter at most $1/n$. The sets $M(k, n)$ are taken to be disjoint from each other and from X'. Identify the point $\{0, 0\}$ of $M(k, n)$ with the point $\big((2k-1)/2^n, 0\big)$ in X'. Let

$$X = X' \cup \bigcup \{M(k, n) \mid n = 1, 2, \ldots; \ k = 1, 2, \ldots, 2^{n-1}\}$$

with the identifications indicated above and with the identification topology.

It is easy to see that X is a locally connected continuum. By Theorem 2, X is not strongly regular. To prove that X is in (III) it suffices, by [7], IV.11.2, to prove that the true cyclic element X' of X is in (III).

Let C be a connected set in X'. We may suppose that if $(x, y) \in C$ for some $y > 0$, then the entire semi-circle in X' which contains (x, y) is in C, since it is clear that C may be retracted monotonicly onto a set with this property. If $(0, 0)$ and $(1, 0)$ are in C for each $x \in [0, 1]$, let $(x, c_x) \in C$ be such that $(x, d) \in C$ implies $d \leqslant c_x$. It is easy to see that $\{(x, c_x) \mid x \in [0, 1]\}$ is an arc in C with endpoints $(0, 0)$ and $(1, 0)$. It is not difficult to show that if (x, a) and (y, b) are any two points of C which lie on semi-circles in X', then (x, a) and (y, b) lie in an arc in C. It follows that C has a dense arc component. By [8], Theorem 34, C is arcwise connected.

Remark. Epps [2], Theorem 5, proved that every strongly regular continuum is the monotone image of a continuum in (IV). Epps' proof can be modified to show that every regular continuum X is the monotone image of a continuum Y in (IV). The continuum Y is strongly regular and has the property that every subcontinuum of Y has a non-void interior in Y.

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