COLLOQUIUM MATHEMATICUM

VOL. XLIV 1981 FASC. 2

ON THE CAUCHY PROBLEM FOR THE EQUATION OF ONE-DIMENSIONAL NON-STATIONARY FILTRATION WITH NON-CONTINUOUS INITIAL DATA

BY

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We consider the following Cauchy problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u, x)}{\partial x^2} \quad \text{on } S = \{(x, t): x \in R, t > 0\}, \tag{1}$$

$$u(x, 0) = u_0(x) \quad \text{on } R, \tag{2}$$

where $\varphi(u, x)$ is a given function defined on $H = \{(u, x) : u \geqslant 0, x \in R\}$.

Here and in the sequel $\partial\varphi/\partial x$, $\partial\varphi/\partial t$ are the derivatives of $\varphi\big(u(x, t), x\big)$ with respect to x, t, respectively, and φ'_u, φ'_x the derivatives of $\varphi(u, x)$ with respect to u, x.

It is well known that the problem (1), (2) not always has a classical solution. A function $u = u(x, t)$ is said to be a *weak solution of the problem* (1), (2) in S if u is bounded, continuous and non-negative in $\bar{S}$, $\varphi(u, x)$ has a bounded weak (distribution) derivative with respect to x on S, and satisfies the integral identity

$$\iint_{\bar{S}} \left[\frac{\partial f}{\partial t} u - \frac{\partial f}{\partial x} \frac{\partial \varphi(u, x)}{\partial x}\right] dx dt + \int_{-\infty}^{\infty} f(x, 0) u_0(x) dx = 0 \tag{3}$$

for all f which are continuously differentiable in $\bar{S}$ and vanish for large values of $|x|$ and t.

Oleĭnik et al. ([1]) have shown that if u_0 is a bounded continuous non-negative function on R, $\varphi\big(u_0(x), x\big)$ is lipschitzian,

(4) $\varphi \in C^5(\mathrm{Int} H)$ and all the derivatives of the fifth order of $\varphi(u, x)$ are lipschitzian in any compact subset of $\mathrm{Int} H$,

([1]) О. А. Олейник, А. С. Калашников и Чжоу Юй-Линь, *Задача Коши и краевые задачи для уравнений типа нестационарной фильтрации*, Известия Академии наук СССР, серия математическая, 22 (1958), p. 667-704.

(5) $\lim_{u\to\infty}\varphi(u, x) = +\infty$ uniformly with respect to $x \in R$,

(6) the functions φ and φ'_u are continuous on H and bounded for bounded u,

(7) $\varphi(u, x) > 0, \varphi'_u(u, x) > 0$ for $u > 0$, $\varphi(0, x) \equiv \varphi'_u(0, x) \equiv 0$,

then there exists a unique function u defined on $\bar{S}$ and satisfying (3). Moreover, u satisfies equation (1) in a classical sense in a neighborhood of any point of S at which u is positive.

We observe that the boundedness of the function $\partial\varphi/\partial x$ implies that the function $\varphi(u_0(x), x)$ is lipschitzian. Thus, when we look for a solution in the case of a non-continuous function u_0, we must modify the above-given definition of the solution.

In this paper we give a definition of the weak solution of the problem (1), (2) which preserves its sense for non-negative functions u_0 of class $L_\infty(R)$, and we prove the corresponding existence and uniqueness theorems for the problem (1), (2).

Definition. Let $u_0 \in L_\infty(R)$ and

$$\inf_{x\in R} \operatorname{ess} u_0(x) \geqslant 0.$$

A function $u = u(x, t)$ defined on S will be called a *weak solution of the problem* (1), (2) if

(i) u is a bounded continuous non-negative function on S;

(ii) there exists a constant $p > 1$ such that $\partial\varphi/\partial x \in L_p(K)$ for each compact subset $K \subset \bar{S}$;

(iii) for every $\varepsilon > 0$ there exists $\tau > 0$ such that

$$\int_0^\tau \left|\frac{\partial\varphi(u(x, t), x)}{\partial x}\right| dt < \varepsilon$$

for almost every $x \in R$;

(iv) for every $\tau > 0$

$$\frac{\partial\varphi(u(x, t), x)}{\partial x} \in L_\infty(R \times (\tau, +\infty));$$

(v) for each function $f \in C_0^1(\bar{S})$ equality (3) holds.

THEOREM 1. *Assume* $u_0 \in L_\infty$,

$$\inf_{x\in R} \operatorname{ess} u_0(x) \geqslant 0,$$

and let the function $\varphi(u, x)$ *satisfy conditions* (6) *and* (7). *Then the problem* (1), (2) *has at most one weak solution.*

Proof. At first we prove that equality (3) holds for any function $f \in C_0^0(\bar{S})$ for which $\partial f/\partial x, \partial f/\partial t \in L_\infty(\bar{S})$. Let $\{f_n(x,t)\}$ be a sequence of functions of class C_0^1, which is uniformly convergent to $f(x,t)$, and let the supports of all f_n $(n = 1, 2, \ldots)$ and of f be contained in a compact set $Q \subset \bar{S}$. We also assume that the sequences $\{\partial f_n/\partial x\}$ and $\{\partial f_n/\partial t\}$ are weakly convergent in $L_q(\bar{S})$ $(q = 1/(1-p^{-1}))$ to $\partial f/\partial x$ and $\partial f/\partial t$, respectively. Note that (3) holds true if we replace f by any f_n. Since $\partial\varphi(u,x)/\partial x \in L_p(Q)$ (for some $p > 1$), we obtain in the limit the equality (3) also for the function $f(x,t)$.

Let us assume that the problem (1), (2) has two different solutions $u_1(x,t)$ and $u_2(x,t)$. From equality (3) written for $u_2(x,t)$ and for $u_1(x,t)$, respectively, we infer

$$\iint_S \left[\frac{\partial f}{\partial t}(u_1-u_2) - \frac{\partial f}{\partial x}\frac{\partial}{\partial x}\big(\varphi(u_1,x)-\varphi(u_2,x)\big)\right] dx dt = 0. \tag{8}$$

Let $\{a_n(x)\}$ be a sequence of functions which have the following properties: $a_n(x) = 1$ for $|x| \leqslant n-1$, $a_n(x) = 0$ for $|x| \geqslant n$, $0 \leqslant a_n(x) \leqslant 1$ for $n-1 \leqslant |x| \leqslant n$, the functions $a_n'(x)$ for $n = 1, 2, \ldots$ are uniformly bounded.

With the help of $a_n(x)$ we define now a new sequence of functions $\{f_n(x,t)\}$ $(n = 1, 2, \ldots)$ by putting

$$f_n(x,t) = \begin{cases} a_n(x) \int\limits_T^t \big[\varphi\big(u_1(x,\tau),x\big) - \varphi\big(u_2(x,\tau),x\big)\big] d\tau & \text{for } t < T, \\ 0 & \text{for } t \geqslant T, \end{cases}$$

where $T > 0$ is an arbitrary constant. It is easy to verify that $f_n \in C_0^0(\bar{S})$ and $\partial f_n/\partial x, \partial f_n/\partial t \in L_\infty(\bar{S})$. Therefore, equalities (3) and (8) hold for each function f_n.

Equality (8), with f replaced by f_n defined as above, is of the form

$$I_{1,n} + I_{2,n} + I_{3,n} = 0, \tag{9}$$

where

$$I_{1,n} = \iint_{S_T} a_n(x)[\varphi(u_1,x)-\varphi(u_2,x)](u_1-u_2)dxdt,$$

$$I_{2,n} = -\iint_{S_T} a_n\Big\{\int\limits_T^t \big[\varphi\big(u_1(x,\tau),x\big)-\varphi\big(u_2(x,\tau),x\big)\big]d\tau\Big\}[\varphi(u_1,x) - \\ -\varphi(u_2,x)]dxdt,$$

$$I_{3,n} = -\iint_{Q_n} a_n'\Big\{\int\limits_T^t \big[\varphi\big(u_1(x,\tau),x\big)-\varphi\big(u_2(x,\tau),x\big)\big]d\tau\Big\}\left[\frac{\partial\varphi(u_1,x)}{\partial x} - \\ - \frac{\partial\varphi(u_2,x)}{\partial x}\right]dxdt$$

and

$$Q_n = \{(x, t) : n-1 \leqslant |x| \leqslant n,\ 0 < t \leqslant T\},$$
$$S_T = \{(x, t) : x \in R,\ 0 < t \leqslant T\}.$$

Since

$$I_{2,n} = -\frac{1}{2}\iint\limits_{S_T} a_n(x)\frac{\partial}{\partial t}\left\{\int\limits_T^t \left[\frac{\partial\varphi(u_1(x,\tau),x)}{\partial x} - \frac{\partial\varphi(u_2(x,\tau),x)}{\partial x}\right]d\tau\right\}^2 dxdt$$
$$= \frac{1}{2}\int\limits_{-\infty}^{\infty} a_n(x)\left\{\int\limits_T^0 \left[\frac{\partial\varphi(u_1(x,\tau),x)}{\partial x} - \frac{\partial\varphi(u_2(x,\tau),x)}{\partial x}\right]d\tau\right\}^2 dx,$$

we may write equality (9) in the form

$$\text{(10)}\qquad I_{1,n} + \frac{1}{2}\int\limits_{-\infty}^{\infty} a_n(x)\left\{\int\limits_T^0 \left[\frac{\partial\varphi(u_1(x,\tau),x)}{\partial x} - \frac{\partial\varphi(u_2(x,\tau),x)}{\partial x}\right]d\tau\right\}^2 dx = -I_{3,n}.$$

Each summand on the left-hand side of (10) is non-negative and increasing with n. The right-hand side of (10) is bounded (uniformly with respect to n), thus the summands on the left-hand side have a finite limit. It follows that the function

$$\text{(10')}\qquad w(x, t) = [\varphi(u_1, x) - \varphi(u_2, x)](u_1 - u_2)$$

is integrable on S_T.

Let

$$Q_{n,\tau} = \{(x, t) : n-1 \leqslant |x| \leqslant n,\ 0 < t \leqslant \tau\}, \qquad Q'_{n,\tau} = Q_n \setminus Q_{n,\tau},$$
$$h(x, t) = a'_n(x)\left\{\int\limits_T^t [\varphi(u_1(x,\tau),x) - \varphi(u_2(x,\tau),x)]\,d\tau\right\} \times$$
$$\times\left[\frac{\partial\varphi(u_1, x)}{\partial x} - \frac{\partial\varphi(u_2, x)}{\partial x}\right].$$

It follows from property (iii) in the Definition that for each $\varepsilon > 0$ there exists τ $(0 < \tau < T)$ such that

$$\left|\iint\limits_{Q_{n,\tau}} h(x, t)\,dxdt\right| \leqslant \varepsilon.$$

Hence we have

$$I_{3,n} \leqslant \varepsilon + \left|\iint\limits_{Q'_{n,\tau}} h(x, t)\,dxdt\right|.$$

Let A, B, C be the numbers such that

$$|a_n'(x)| \leqslant A \quad \text{for } x \in R,\ n = 1, 2, \ldots;$$

$$\left|\frac{\partial\varphi(u_i(x,t),x)}{\partial x}\right| \leqslant B \quad \text{for } x \in R, t > \tau, i = 1, 2;$$

$$|\varphi_u'(u_i(x,t),x)| \leqslant C \quad \text{for } (x,t) \in S, i = 1, 2.$$

Using the Cauchy inequality we obtain

$$\left|\iint\limits_{Q_{n,\tau}'} h(x,t)\,dxdt\right| \leqslant 2ABT \iint\limits_{Q_{n,\tau}'} |\varphi(u_1,x)-\varphi(u_2,x)|\,dxdt$$

$$\leqslant 4ABT^{3/2}\Big\{\iint\limits_{Q_{n,\tau}'} [\varphi(u_1,x)-\varphi(u_2,x)](u_1-u_2)\varphi_u'(\bar{u},x)\,dxdt\Big\}^{1/2}$$

$$\leqslant 4ABC^{1/2}T^{3/2}\Big\{\iint\limits_{Q_n} w(x,t)\,dxdt\Big\}^{1/2},$$

where $\bar{u}(x)$ lies between $u_1(x)$ and $u_2(x)$.

The function $w(x,t)$ is integrable on S_T, therefore the integral on the right-hand side of the inequality above converges to zero as $n \to \infty$. Since ε is arbitrary, we infer that

$$\lim_{n\to\infty} I_{3,n} = 0.$$

Therefore, each number on the left-hand side of (10) tends to zero as $n \to \infty$. It follows that

$$\iint\limits_{S_T} w(x,t)\,dxdt = 0, \tag{11}$$

where w is given by (10′).

Since w is non-negative and continuous on S_T and T is arbitrary, equality (11) implies $w = 0$, and hence $u_1(x,t) = u_2(x,t)$ for $(x,t) \in S$.

Remark. Note that if the function $u(x,t)$ is a weak solution in the sense defined by Oleĭnik et al. (op. cit.), then the function u is also a weak solution in our sense.

Thus, if u_0 is continuous, non-negative and bounded and if the function $\varphi(u_0(x),x)$ is lipschitzian, then both solutions are the same and there is no necessity to distinguish them.

Hereafter we assume that the function $\varphi(u,x)$ satisfies conditions (4)-(7).

We shall now prove the existence of a solution of the problem (1), (2).

From our assumptions imposed upon φ and from the implicit function theorem it follows that the equation $\varphi(u, x) = v$ defines

$$u = \Phi(v, x) \tag{12}$$

in a unique way.

Putting (12) into (1) we get

$$\frac{\partial^2 v}{\partial x^2} = \Phi'_v(v, x)\frac{\partial v}{\partial t}, \tag{13}$$

where the function $\Phi'_v(v, x)$ is positive and bounded for $a \leqslant x \leqslant b$ and $v \geqslant \mu$ (for any positive μ and any a and b).

We now consider equation (13) in the domain $G = \{(x, t): a \leqslant x \leqslant b, t \geqslant 0\}$ with the following boundary conditions:

$$\begin{gathered} v(x, 0) = v_0(x) \quad \text{for } x \in [a, b], \\ v(a, t) = v_0(a), \quad v(b, t) = v_0(b) \quad \text{for } t \geqslant 0. \end{gathered} \tag{14}$$

We assume that

$$\begin{gathered} 0 < m \leqslant v_0 \leqslant M \text{ on } [a, b], \quad v_0 \in C^3([a, b]), \\ v_0''' \text{ is lipschitzian}, \quad v_0''(a) = v_0''(b) = 0. \end{gathered} \tag{15}$$

Due to the assumptions above we may use Lemma 1 in the paper by Oleĭnik et al. (op. cit.) from which it follows that there exists a solution $v(x, t)$ of the problem (13), (14) such that v together with its derivatives $\partial v/\partial t$, $\partial v/\partial x$, $\partial^2 v/\partial x^2$ is continuous on G. Furthermore, all the derivatives of v, which appear in the equations obtained by differentiating the equation (13) four times with respect to x and once with respect to t, are continuous in Int$G$. Moreover, the following inequality holds:

$$m \leqslant v(x, t) \leqslant M \quad \text{for } (x, t) \in G. \tag{16}$$

LEMMA 1. *We assume that the function v_0 satisfies* (15), $\varphi(u, x)$ *satisfies* (4)-(7) *and for every $N > 0$ there exists a constant $a > 0$ such that*

$$\frac{\varphi''_{uu}(u, x)}{\varphi'^2_u(u, x)} \geqslant a \quad \text{for } 0 < u \leqslant N, x \in R. \tag{17}$$

Then the solution v of the problem (13), (14) *satisfies the inequality*

$$\frac{\partial v(x, t)}{\partial t} \geqslant -\frac{1}{at} \quad \text{for } x \in [a, b], t > 0, \tag{18}$$

where the constant a depends upon

$$N = \sup_{[a,b]} u_0, \quad u_0 = \Phi(v_0, x).$$

Proof. Differentiating both sides of equation (13) with respect to t we obtain

(19) $$\frac{\partial^2 v'}{\partial x^2} = \Phi''_{vv} v'^2 + \Phi'^2_v \frac{\partial v'}{\partial t} \qquad \left(v' = \frac{\partial v}{\partial t}\right).$$

It follows from equality (12) that $\Phi'_v = 1/\varphi'_u$ and $\Phi''_{vv} = -\varphi''_{uu}/\varphi'^3_u$, thus we may write equation (19) in the form

$$\frac{\partial v'}{\partial t} = \varphi'_u \frac{\partial^2 v'}{\partial x^2} + \frac{\varphi''_{uu}}{\varphi'^2_u} v'^2.$$

Put

$$v' = z - \frac{K}{1+aKt}, \qquad \text{where } K = -\inf_{[a,b]} v'(x, 0) \geqslant 0.$$

The function z satisfies the equation

$$\frac{\partial z}{\partial t} = \varphi'_u \frac{\partial^2 z}{\partial x^2} + \frac{\varphi''_{uu}}{\varphi'^2_u}\left(z - \frac{K}{1+aKt}\right)^2 - \frac{K^2 a}{(1+aKt)^2}.$$

Since $v'(x, 0) \geqslant -K$ for $x \in [a, b]$ and $v'(a, t) = v'(b, t) = 0$ for $t \geqslant 0$, we have $z|_{\partial G} \geqslant 0$, which together with (13) implies that $z(x, t) \geqslant 0$ on G. Therefore

$$\frac{\partial v(x, t)}{\partial t} = z(x, t) - \frac{K}{1+aKt} \geqslant -\frac{1}{at} \quad \text{for } x \in [a, b], t > 0.$$

LEMMA 2. *If the assumptions of Lemma 1 hold true and if for any $N > 0$ there exist constants λ and β ($\lambda > 0$, $0 < \beta < 1$) such that*

(20) $$\varphi'_u(u, x) \geqslant \lambda \varphi^\beta(u, x) \quad \textit{for } 0 < u \leqslant N, x \in R,$$

then

(21) $$\left|\frac{\partial v(x, t)}{\partial x}\right| \leqslant \max\left\{\frac{2M}{x-a}, \frac{2M}{b-x}, \left(\frac{8M^{1-\beta}}{3a\lambda(1-\beta)t}\right)^{1/2}\right\}$$

for $(x, t) \in \operatorname{Int} G$,

where

$$M = \sup_{x\in[a,b]} v_0(x).$$

Proof. Let $(x_0, t) \in \operatorname{Int} G$.

(a) Assume that

$$\frac{\partial v(x_0, t)}{\partial x} > 0.$$

If

$$\frac{\partial v(x, t)}{\partial x} > \frac{1}{2}\frac{\partial v(x_0, t)}{\partial x} \quad \text{for each } x \in [x_0, b],$$

then

$$M \geqslant \int_{x_0}^{b} \frac{\partial v(x,t)}{\partial x}\,dx \geqslant \frac{\partial v(x_0,t)}{\partial x}(b-x_0),$$

and hence we infer

$$\frac{\partial v(x_0,t)}{\partial x} \leqslant \frac{2M}{b-x_0}.$$

Now assume that

$$\frac{\partial v(x_1,t)}{\partial x} \leqslant \frac{1}{2}\frac{\partial v(x_0,t)}{\partial x} \quad \text{for certain } x_1 \in [x_0, b]$$

and

$$\frac{\partial v(x,t)}{\partial x} \geqslant 0 \quad \text{for } x \in [x_0, x_1].$$

From equation (13) and from inequalities (18) and (20) we infer that

$$\frac{\partial^2 v(x,t)}{\partial x^2} \geqslant -\frac{1}{a\lambda t} v^{-\beta}(x,t).$$

Multiplying both sides of this inequality by $\partial v(x,t)/\partial x \geqslant 0$ and integrating with respect to x over the interval $[x_0, x_1]$, we obtain

$$\left[\frac{\partial v(x_1,t)}{\partial x}\right]^2 - \left[\frac{\partial v(x_0,t)}{\partial x}\right]^2 \geqslant -\frac{2}{a\lambda(1-\beta)t}[v^{1-\beta}(x_1,t) - v^{1-\beta}(x_0,t)],$$

whence

$$\frac{\partial v(x_0,t)}{\partial x} \leqslant \left[\frac{8M^{1-\beta}}{3a\lambda(1-\beta)t}\right]^{1/2}.$$

(b) In the case $v(x_0,t) < 0$, we prove in a similar way that either

$$\frac{\partial v(x_0,t)}{\partial x} \geqslant \frac{2M}{a-x_0} \quad \left(\text{i.e. } \left|\frac{\partial v(x_0,t)}{\partial x}\right| \leqslant \frac{2M}{x_0-a}\right)$$

or

$$\left|\frac{\partial v(x_0,t)}{\partial x}\right| \leqslant \left[\frac{8M^{1-\beta}}{3a\lambda(1-\beta)t}\right]^{1/2}.$$

Then (a) and (b) together give our lemma.

Assume now that $u_0 \in L_\infty(R)$ and $\inf\operatorname{ess}_{x\in R} u_0(x) \geqslant 0$. Put

$$N = \sup_{x\in R}\operatorname{ess} u_0(x), \qquad M = \sup_{x\in R}\varphi(N,x)$$

and let a, β, λ be the corresponding constants appearing in (17) and (20).

We observe that the boundedness of $\varphi'_u(u,x)$ for bounded u (condition (6)) implies that $M < \infty$.

Let

$$X_n = n + \frac{2M\sqrt{n}}{A} \quad (n = 1, 2, \ldots), \qquad \text{where } A = \left[\frac{8M^{1-\beta}}{3a\lambda(1-\beta)}\right]^{1/2},$$

and let $\{v_{0,n}(x)\}$ be a sequence of functions which have the following properties for any n:

$v_{0,n} \in C^3([-X_n, X_n])$,

$v_{0,n}'''$ satisfies the Lipschitz condition on $[-X_n, X_n]$,

$0 < v_{0,n}(x) \leqslant \varphi(N, x)$ for $x \in [-X_n, X_n]$,

$v_{0,n}''(-X_n) = v_{0,n}''(X_n) = 0$,

$v_{0,n+1}|_{[-X_n,X_n]}$, $v_{0,n+2}|_{[-X_n,X_n]}, \ldots$ tends to $\varphi(u_0(x), x)|_{[-X_n,X_n]}$ in $L_2([-X_n, X_n])$.

Let $v_n(x, t)$ be such a solution of equation (13) defined in the domain $\{(x, t): x \in [-X_n, X_n],\ t \geqslant 0\}$ that

$$v_n(x, 0) = v_{0,n}(x) \quad \text{for } x \in [-X_n, X_n],$$

(22)

$$v_n(-X_n, t) = v_{0,n}(-X_n), \qquad v_n(X_n, t) = v_{0,n}(X_n) \quad \text{for } t \geqslant 0.$$

From (21) it follows that

$$\left|\frac{\partial v_n(x, t)}{\partial x}\right| \leqslant \frac{A}{\sqrt{t}} \quad \text{for } -n \leqslant x \leqslant n,\ 0 < t \leqslant n, \tag{23}$$

where the constant A depends upon $\varphi(u, x)$ and N and is independent of n.

In the proof of existence of the solution of the problem (1), (2) (see below) we show that the sequence $\{v_n\}$ contains a subsequence which tends nearly uniformly on S to a function $v(x, t)$ and that the function $u(x, t) = \Phi(v(x, t), x)$ is a solution of the problem (1), (2).

We preced this by proving the following

LEMMA 3. *Let $u_n = \Phi(v_n, x)$, where v_n is a solution of the problem* (13), (14) *and Φ is defined by* (12). *Then there exist a natural number n_0 and a function $\varrho(s, x)$, continuous and increasing with respect to s and defined for $s \geqslant 0$, $x \in R$ ($\varrho(0, x) \equiv 0$), such that*

$$|u_n(x, t') - u_n(x, t)| \leqslant \varrho\left(\frac{t'-t}{t}, x\right) \tag{24}$$

for $x \in [-n, n]$, $0 < t < t' \leqslant n$ and $n \geqslant n_0$.

Proof. We have $\varphi_{uu}''(u, x) > 0$ for $u > 0$ and $x \in R$ (this follows from inequality (17)). Therefore the function $\varphi_u'(u, x)$ for each $x \in R$ is increasing with respect to u, so we can write

$$\varphi(u+u', x) - \varphi(u', x) \geqslant \varphi(u, x) - \varphi(0, x) \quad \text{for } u' \geqslant u \geqslant 0,\ x \in R,$$

which gives

$$\varphi(u+u', x) \geqslant \varphi(u, x)+\varphi(u', x) \quad \text{for } u \geqslant 0, u' \geqslant 0, x \in R.$$

This inequality allows us to write

$$(25) \qquad |v_n(x, t')-v_n(x, t)| \geqslant \varphi\big(|u_n(x, t')-u_n(x, t)|, x\big)$$

for $x \in [-X_n, X_n]$, $0 < t < t'$.

Let

$$M_1 = \sup_{x\in R}\varphi'_u(M, x)$$

and let n_0 be the least natural number for which $n_0 \geqslant (M/4A)^2$.

If $x_0 \in [-n, n]$, $0 < t < t' \leqslant n$, then it follows from (25) that

$$|v_n(x_0, t')-v_n(x_0, t)| \geqslant \varphi\big(|u_n(x_0, t')-u_n(x_0, t)|, x_0\big).$$

The both functions $v_n(x, t')$ and $v_n(x, t)$ satisfy the Lipschitz condition with respect to x on $[-n, n]$ with a constant $A/\sqrt{t}$. It follows that (cf. inequality (23))

$$(26) \qquad |v_n(x, t')-v_n(x, t)| \geqslant \frac{1}{2}\varphi\big(|u_n(x_0, t')-u_n(x_0, t)|, x_0\big)$$

for $x \in [-n, n] \cap [x_0-\Delta, x_0+\Delta]$, where

$$\Delta = \frac{\varphi\big(|u_n(x_0, t')-u_n(x_0, t)|, x_0\big)\sqrt{t}}{4A}.$$

Since $\varphi'_u(u, x)$ is bounded for bounded u and $x \in R$, we obtain

$$|\varphi(u', x)-\varphi(u, x)| \leqslant M_1|u'-u| \quad \text{for } u \leqslant N, u' \leqslant N \text{ and } x \in R,$$

which together with (26) gives

$$(27) \qquad |u_n(x, t')-u_n(x, t)| \geqslant \frac{1}{2M_1}\varphi\big(|u_n(x_0, t')-u_n(x_0, t)|, x_0\big)$$

for $x \in [-n, n] \cap [x_0-\Delta, x_0+\Delta]$.

Assume that $n \geqslant n_0$. From the definition of n_0 it follows that either $x_0+\Delta \in [-n, n]$ or $x_0-\Delta \in [-n, n]$.

From Green's formula we get

$$\int_{\partial\Omega} u_n dx+\frac{\partial v_n}{\partial x} d\tau = \iint_{\Omega}\left[\frac{\partial u_n}{\partial t}-\frac{\partial^2 v_n}{\partial x^2}\right]dx d\tau = 0,$$

where $\Omega = \{(x, \tau) : x_0 \leqslant x \leqslant x_0 + \Delta, t \leqslant \tau \leqslant t'\}$. Hence we infer that

$$(28) \quad \int_{x_0}^{x_0+\Delta} [u_n(x, t') - u_n(x, t)]dx = \int_t^{t'} \left[\frac{\partial v_n(x_0+\Delta, \tau)}{\partial x} - \frac{\partial v_n(x_0, \tau)}{\partial x}\right] d\tau \quad \text{for } n \geqslant n_0.$$

The difference $u_n(x, t') - u_n(x, t)$ has a constant sign on $[x_0, x_0 + \Delta]$ (see (27)), therefore due to inequality (27) we may write

$$\left|\int_{x_0}^{x_0+\Delta} [u_n(x, t') - u_n(x, t)]dx\right| \geqslant \frac{\varphi^2(|u_n(x_0, t') - u_n(x_0, t)|, x_0)\sqrt{t}}{8AM_1}.$$

From (23) it follows that

$$\left|\int_t^{t'} \left[\frac{\partial v_n(x_0+\Delta, \tau)}{\partial x} - \frac{\partial v_n(x_0, \tau)}{\partial x}\right] d\tau\right| \leqslant \int_t^{t'} \frac{2A}{\sqrt{\tau}} d\tau = 2A\frac{t'-t}{\sqrt{t}},$$

whence (see also (28))

$$\frac{\varphi^2(|u_n(x_0, t') - u_n(x_0, t)|, x_0)\sqrt{t}}{8AM_1} \leqslant 2A\frac{t'-t}{\sqrt{t}},$$

which gives

$$\varphi(|u_n(x_0, t') - u_n(x_0, t)|, x_0) \leqslant 4A\sqrt{M_1\frac{t'-t}{t}}.$$

So for $-n \leqslant x \leqslant n$, $0 < t < t' \leqslant n$, $n \geqslant n_0$ we have

$$|u_n(x, t') - u_n(x, t)| \leqslant \varrho\left(\frac{t'-t}{t}, x\right) = \Phi\left(4A\sqrt{M_1\frac{t'-t}{t}}, x\right).$$

It is easy to verify that the function $\varrho(s, t)$ satisfies the conditions of the lemma.

THEOREM 2. *Let* $u_0 \in L_\infty(R)$,

$$\inf_{x\in R} \operatorname{ess} u_0(x) \geqslant 0,$$

and let u_0 *satisfy conditions* (4)-(7), (17) *and* (20). *Then the problem* (1), (2) *has a weak solution* $u(x, t)$, *in the sense given in the Definition, with the following properties*:

(a) $0 \leqslant u(x, t) \leqslant N$ *for* $(x, t) \in S$, *where*

$$N = \sup_{x\in R} \operatorname{ess} u_0(x);$$

(b) *for $t > 0$ the function $\varphi(u(x,t), x)$ satisfies the Lipschitz condition with respect to x, with a constant $A/\sqrt{t}$, where*

$$A = \left[\frac{8M^{1-\beta}}{3a\lambda(1-\beta)}\right]^{1/2}, \quad M = \sup_{x \in R} \varphi(N, x)$$

and a, β, λ are constants from Lemmata 1 *and* 2;

(c) *there exists a function $\varrho(s, x)$ defined for $s \geqslant 0, x \in R$ which is continuous and increasing with respect to s ($\varrho(0, x) \equiv 0$) and such that*

$$|u(x, t') - u(x, t)| \leqslant \varrho\left(\frac{t'-t}{t}, x\right) \quad \textit{for } x \in R, t' > t > 0;$$

(d) *u satisfies equation* (1) *in the classical sense in any neighborhood of every point of S at which u is positive*;

(e) *on each finite interval $[a, b]$, $u(x, t)$ tends weakly in $L_2([a, b])$ to $u_0(x)$ as $t \to 0$.*

Proof. Let v_n be a solution of (13), (22) and let

$$G_n = \left\{(x, t)\colon\ x \in [-n,\ n],\ t \in \left[\frac{1}{n}, n\right]\right\}.$$

From Lemmata 2 and 3 it follows that the functions $\{v_{n+k}\}$ for $k = 1, 2, \ldots$ are uniformly bounded and uniformly continuous on G_n.

It is easy to verify that the sequence $\{v_n\}$ contains a subsequence $\{v_{n_k}\}$ which is convergent uniformly on each G_n.

We shall only sketch the proof of this fact.

From the sequence $\{v_n\}$ we choose a subsequence $\{v_{m_k}\}$ which is convergent uniformly on G_1. Denote the first element of this subsequence by v_{n_1}. Now, from $v_{m_2}, v_{m_3}, \ldots$ we choose a new subsequence which is uniformly convergent on G_2. We denote the first element of this new subsequence by v_{n_2}. Proceeding in this way we obtain a subsequence $\{v_{n_k}\}$ which has the desired properties.

Let

$$v(x, t) = \lim_{k \to \infty} v_{n_k}(x, t), \quad u(x, t) = \Phi(v(x, t), x) \quad \text{for } (x, t) \in S.$$

By the definition of v_{n_k}, $v(x, t)$ (as well as $u(x, t)$) is continuous on S.

It follows from the remarks above and from (23) that

$$\sup_{x \in R} \operatorname{ess} \left|\frac{\partial v(x, t)}{\partial x}\right| \leqslant \frac{A}{\sqrt{t}} \quad \text{for } t > 0. \tag{29}$$

This inequality implies conditions (ii) (for any $p \in (1, 2)$), (iii) and (iv) from the Definition and condition (b) from Theorem 2.

Since the functions u_n satisfy (24), condition (c) from Theorem 2 holds.

Since the functions $u_n(x, t) = \Phi(v_n, x)$ satisfy (1) on $[-X_n, X_n]\times\times[0, +\infty)$ and since $0 \leqslant u_n \leqslant N$ on the boundary of this domain, by Theorem 12 in the paper by Oleĭnik et al. (op. cit.) we have $0 \leqslant u_n(x, t) \leqslant N$ for $x \in [-X_n, X_n]$, $t \geqslant 0$. Hence $0 \leqslant u \leqslant N$ on S.

Let $f \in C_0^1(\bar{S})$ and let $K = \operatorname{supp} f$. The sequence $\{\partial\varphi(u_{n_k}, x)/\partial x\}$ (beginning with some k) is contained in $L_p(K)$ and is bounded in $L_p(K)$-norm (for $1 < p < 2$), therefore it is weakly compact. Thus there exists a subsequence $\{\partial\varphi(u_{n_{k_i}}, x)/\partial x\}$ weakly convergent in $L_p(K)$ to $\partial\varphi(u, x)/\partial x$. Equality (3) holds true for $u_{n_{k_i}}$; by passing to the limit as $i \to \infty$ we infer that (3) also holds for u.

In this way we have proved that

$$(30)\qquad \int_{\tau}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial f}{\partial t}u - \frac{\partial f}{\partial x}\frac{\partial\varphi(u, x)}{\partial x}\right]dxdt + \int_{-\infty}^{\infty} f(x, \tau)u(x, \tau)dx = 0$$

for any $f \in C_0^1\big(R\times[\tau, +\infty)\big)$ and $\tau > 0$.

From this and from (29) we infer that $u(x, t)$ is a weak solution of the problem (1), (2) in the sense defined by Oleĭnik et al. in the domain $R\times[\tau, +\infty)$. This solution takes the value $u(x, \tau)$ for $t = \tau$.

Part (d) of the theorem follows from Theorem 2 in op. cit.

We have only to prove (e). Since $C_0^1([a, b])$ is a closed subset of $L_2([a, b])$, it suffices to prove that

$$(31)\qquad \lim_{\tau\to 0}\int_{-\infty}^{\infty} h(x)[u_0(x) - u(x, \tau)]dx = 0 \quad \text{for any } h \in C_0^1(R).$$

Let $h \in C_0^1(R)$ and let f be a function with $C_0^1(\bar{S})$ such that $f(x, 0) = h(x)$ for $0 \leqslant t \leqslant \tau$.

From (29) and from (3) we get

$$\int_{0}^{\tau}\int_{-\infty}^{\infty}\left[\frac{\partial f}{\partial t}u - \frac{\partial f}{\partial x}\frac{\partial\varphi(u, x)}{\partial x}\right]dxdt + \int_{-\infty}^{\infty} f(x, \tau)[u_0(x) - u(x, \tau)]dx = 0,$$

i.e.

$$\int_{-\infty}^{\infty} h(x)[u_0(x) - u(x, \tau)]dx = \int_{-\infty}^{\infty} h'(x)\left(\int_{0}^{\tau}\frac{\partial\varphi(u, x)}{\partial x}dt\right)dx.$$

The right-hand side of this equality tends to zero as $\tau \to 0$, so (31) holds true.

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Reçu par la Rédaction le 24. 11. 1977