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A NON-STANDARD METRIC IN THE GROUP OF REALS

(Corrigendum to the paper "Concerning the characterization of linear spaces" by S. Hartman, Jan Mycielski, S. Rolewicz and A. Schinzel, Colloquium Mathematicum 13 (1965), p. 199–208)

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It has been pointed out by Mrs. T. C. Stevens [3] that the proof of Theorem 5 given in [1] is wrong. Indeed, the function $D(r)$ defined at the bottom of p. 205 does not fulfil the crucial triangle inequality, e.g. $D(7/16) > D(1/2)+D(-1/16)$.

Therefore, I give here a different proof of the main part of Theorem 5 based on an unpublished result of P. Erdős, mentioned on p. 207 in [1], which I show in a slightly stronger form.

THEOREM 5′. *In the group L of reals an invariant metric ϱ can be effectively constructed, so that L is complete, non-discrete and non-separable. The function $\varrho(\alpha, 0)$ is measurable and satisfies the Baire condition.*

Proof. For $\alpha \in L$ we define $N(\alpha)$ by the formula similar to that used in the proof of Theorem 4 in [1]

$$N(\alpha) = \inf \sum_{i=1}^{\infty} \frac{1}{\log\log 2a_i},$$

where the infimum (possibly equal to infinity) is taken over all representations of α in the form $\sum_{i=1}^{\infty} \varepsilon_i/a_i$, $\varepsilon_i \in \{1, -1\}$, $a_i > 1$, a_i integers. We note that

$$(1) \qquad N(\alpha) \geqslant |\alpha| \quad \text{and} \quad N(\alpha+\beta) \leqslant N(\alpha)+N(\beta).$$

THEOREM T. *If $N(\alpha) < \infty$ then either α is rational or for every $n \geqslant 4$ there exist infinitely many integer pairs (p, q) such that*

$$|\alpha - p/q| < \exp\left(-(\log q)(\log_2 q)^{\log_4 q \cdots \log_n q}\right),$$

where $\log_1 q = \log q$, $\log_j q = \log\log_{j-1} q$ *for* $j = 2, 3, \ldots$

For the proof of the theorem it is convenient to introduce functions

$$L_0(t) = t, \quad L_j(t) = 1+\log L_{j-1}(t), \quad j = 1, 2, \ldots$$

which for $t \geqslant 1$ satisfy the inequality

$$1 \leqslant L_j(t) \leqslant L_{j-1}(t).$$

LEMMA 1. *For $n > 2$, $d \geqslant n^{2(n-1)}$ we have for all $l \geqslant 1$ the inequality*

$$d\prod_{j=1}^{n-2} L_j(l) > 2\prod_{j=1}^{n-2} L_j(dl^{n-1}).$$

Proof. Taking $d = \delta^{n-1}$ it is enough to show that for all $l \geqslant 1$ and all positive $j \leqslant n-2$

$$\delta L_j(l) > L_j(\delta^{n-1} l^{n-1}).$$

For $l = 1$ we have

$$\delta L_j(1) = \delta \geqslant 1+(n-1)\log\delta = L_1(\delta^{n-1}) \geqslant L_j(\delta^{n-1}).$$

On the other hand, the derivative with respect to l of

$$\delta L_j(l) - L_j(\delta^{n-1} l^{n-1})$$

equals

$$\frac{\delta}{L_{j-1}(l)\ldots L_1(l)l} - \frac{n-1}{L_{j-1}(\delta^{n-1}l^{n-1})\ldots L_1(\delta^{n-1}l^{n-1})l},$$

which is positive for $l \geqslant 1$.

LEMMA 2. *Let b_k be a non-decreasing sequence of integers greater than 1. If*

$$\sum_{k=1}^{\infty} \frac{1}{\log\log 2b_k} < \infty \tag{2}$$

then for every $n \geqslant 4$ there exists an $l > 1$ such that

$$\sum_{k=l}^{\infty} \frac{1}{b_k} < \exp\Big(-L_1(b_1 b_2 \ldots b_{l-1}) L_2(b_1 b_2 \ldots b_{l-1})^{\prod_{j=4}^{n} L_j(b_1 b_2 \ldots b_{l-1})}\Big). \tag{3}$$

Proof. Let $\sum_{k=1}^{\infty} 1/\log\log 2b_k = c$. We shall show first that for all l we have

$$\sum_{k=l}^{\infty} \frac{1}{b_k} < \frac{3c}{b_l^{1/3}}. \tag{4}$$

Indeed, the number of integers $b_k \leqslant b$ does not exceed $c \log\log 2b < cb^{1/3}$ $(b \geqslant 2)$. Hence

$$\sum_{k=l}^{\infty} \frac{1}{b_k} \leqslant \sum_{j=1}^{\infty} \sum_{b_l^j \leqslant b_k < b_l^{j+1}} \frac{1}{b_k} < \sum_{j=1}^{\infty} \frac{cb_l^{(j+1)/3}}{b_l^j} = \frac{c}{b_l^{1/3} - b_l^{-1/3}} < \frac{3c}{b_l^{1/3}}.$$

Suppose now that inequality (3) does not hold for any $l > 1$. Then by (4)

$$(5) \qquad b_l^{1/3} < 3c \exp\left(L_1(b_1 \ldots b_{l-1}) L_2(b_1 \ldots b_{l-1})^{\prod_{j=4}^{n} L_j(b_1 \ldots b_{l-1})}\right).$$

We shall show by induction that for

$$d = \max\{\log\log 2b_1,\ \log 54c^3,\ n^{2(n-1)}\}$$

we have for all $k \geqslant 1$

$$(6) \qquad \log\log 2b_k \leqslant d \prod_{j=0}^{n-2} L_j(k).$$

For $k = 1$ this follows at once, since $\prod_{j=0}^{n-2} L_j(1) = 1$. Assume that (6) is true for all $k < l$, where $l > 1$. We have

$$\log 2b_k \leqslant \exp\left(d \prod_{j=0}^{n-2} L_j(k)\right);$$

hence

$$L_1(b_1 \ldots b_{l-1}) \leqslant 1 + \sum_{k=1}^{l-1} \log 2b_k \leqslant \frac{\exp\left(d \prod_{j=0}^{n-2} L_j(l-1)\right)}{1 - e^{-d}}$$

$$< \exp\left(d \prod_{j=0}^{n-2} L_j(l) - d \prod_{j=1}^{n-2} L_j(l)\right),$$

$$L_2(b_1 \ldots b_{l-1}) \leqslant 1 + d \prod_{j=0}^{n-2} L_j(l) - d \prod_{j=1}^{n-2} L_j(l) \leqslant d \prod_{j=0}^{n-2} L_j(l) \leqslant dl^{n-1},$$

$$L_j(b_1 \ldots b_{l-1}) \leqslant L_{j-2}(dl^{n-1}) \qquad (2 \leqslant j \leqslant n).$$

It follows now from (4) that

$$\log 2b_l \leqslant \log 54c^3 + 3L_1(b_1 \ldots b_{l-1})\, L_2(b_1 \ldots b_{l-1})^{\prod_{j=4}^{n} L_j(b_1 \ldots b_{l-1})}$$

$$\leqslant \log 54c^3 + 3\exp\left(d \prod_{j=0}^{n-2} L_j(l) - d \prod_{j=1}^{n-2} L_j(l) + \prod_{j=1}^{n-2} L_j(dl^{n-1})\right),$$

hence in virtue of Lemma 1 and by the choice of d

$$\log 2b_l \leqslant 4\exp\Big(d\prod_{j=0}^{n-2} L_j(l)-\tfrac{1}{2}d\prod_{j=1}^{n-2} L_j(l)\Big) < \exp\Big(d\prod_{j=0}^{n-2} L_j(l)\Big)$$

and the inductive proof of (6) is complete.

The series $\sum_{k=1}^{\infty}\prod_{j=0}^{n-2} L_j(k)^{-1}$ is divergent since

$$\int_1^x \prod_{j=0}^{n-2} L_j(t)^{-1}\,dt = L_{n-1}(x)-1,$$

thus (6) contradicts (2).

Proof of Theorem T. By the definition of $N(\alpha)$ we have, for suitable integers $a_i > 1$ and $\varepsilon_i \in \{1, -1\}$

$$\alpha = \sum_{i=1}^{\infty}\frac{\varepsilon_i}{a_i}, \quad \sum_{i=1}^{\infty}\frac{1}{\log\log 2a_i} < \infty.$$

Since $\sum_{i=1}^{\infty} 1/a_i < \infty$, we order $a_1, a_2, \ldots$ in a non-decreasing sequence $b_1, b_2, \ldots$ $(b_i \geqslant 2)$ and find

$$\alpha = \sum_{k=1}^{\infty}\frac{\eta_k}{b_k}, \quad \eta_k \in \{1, -1\}, \quad \sum_{k=1}^{\infty}\frac{1}{\log\log 2b_k} < \infty.$$

By Lemma 2, for every $n > 0$ there exists an $l > 1$ such that

$$\left|\alpha-\sum_{k=1}^{l-1}\frac{\eta_k}{b_k}\right| \leqslant \sum_{k=l}^{\infty}\frac{1}{b_k} < \exp\Big(-L_1(b_1\ldots b_{l-1})L_2(b_1\ldots b_{l-1})^{\prod_{j=4}^{n} L_j(b_1\ldots b_{l-1})}\Big).$$

Taking $q = b_1\ldots b_{l-1}$, $p = q\sum_{k=1}^{l-1}\eta_k/b_k$ we find $q \geqslant 2$,

(7) $$\left|\alpha-\frac{p}{q}\right| < \exp\Big(-L_1(q)L_2(q)^{\prod_{j=4}^{n} L_j(q)}\Big).$$

However, for $q \geqslant 2$ we have $L_j(q) \geqslant 1+1/(j+1)$ $(j = 0, 1, \ldots)$, hence $\lim_{n\to\infty}\prod_{j=4}^{n} L_j(q) = \infty$ and the right-hand side of (7) tends to 0 when n tends to ∞. It follows that either α is rational or for each $n \geqslant 4$ there exist infinitely many integer pairs (p, q) satisfying (7). Since $L_j(q) > \log_j q$, we get the theorem.

COROLLARY 1. *If $N(\alpha) < \infty$ then either α is rational or it is a Liouville number.*

COROLLARY 2. *There are Liouville numbers α for which $N(\alpha) = \infty$.*

Deduction of Corollary 1 from Theorem T is immediate. A direct proof along the same lines would be considerably shorter than the proof of the said theorem.

An example of Liouville's number α with $N(\alpha) = \infty$ is given by

$$\alpha = \sum_{k=2}^{\infty} 1/2^{k!}.$$

Indeed, the partial sums of the series give best rational approximations of α, and hence if

$$2^{k!} < q \leqslant 2^{(k+1)!}$$

we find

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{1}{2^{(k+2)!}} > \exp\left(-\log q \cdot (\log\log q)^2\right).$$

Therefore, $N(\alpha) = \infty$ in virtue of Theorem T with $n = 4$.

Proof of Theorem 5′. Let us define the metric $\varrho(x, y)$ by the formula

$$\varrho(x, y) = \frac{1}{1 + N(x-y)^{-1}}, \tag{8}$$

where, as usual, $\infty^{-1} = 0$. ϱ is a metric in virtue of (1) and it is clearly invariant. In order to see that L is complete with respect to ϱ, let us take a sequence (α_k) fundamental in ϱ, hence also in N. Without loss of generality (replacing (α_k), if necessary, by its subsequence) we can assume that

$$N(\alpha_{k+1} - \alpha_k) < 1/2^k.$$

Thus there exist sequences $(a_{ik})_{i=1}^{\infty}$ $(k = 1, 2, \ldots)$ of integers greater than 1 such that

$$\alpha_{k+1} - \alpha_k = \sum_{i=1}^{\infty} \frac{\varepsilon_{ik}}{a_{ik}} \quad (\varepsilon_{ik} = \pm 1) \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{1}{\log\log 2a_{ik}} \leqslant \frac{1}{2^{k-1}}.$$

Putting

$$\alpha = \alpha_1 + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{\varepsilon_{ik}}{a_{ik}}$$

we have $\lim \varrho(\alpha_k, \alpha) = 0$.

In fact, it follows directly from the definition of N that, for $x_i \in L$,

$$N\left(\sum_{i=1}^{\infty} x_i\right) \leqslant \sum_{i=1}^{\infty} N(x_i);$$

hence

$$N(\alpha_k-\alpha) = N\Big(\sum_{n=k}^{\infty} (\alpha_{n+1}-\alpha_n)\Big) \leqslant \sum_{n=k}^{\infty} N(\alpha_{n+1}-\alpha_n) \leqslant \sum_{n=k}^{\infty} 1/2^{n-1}$$

and $\lim\limits_{k} N(\alpha_k-\alpha) = 0$, thus also $\lim\limits_{k} \varrho(\alpha_k, \alpha) = 0$.

L with respect to ϱ is non-discrete, since e.g. $\lim\limits_{n} \varrho(1/n, 0) = 0$.

Suppose that L with respect to ϱ is separable and let $\{r_1, r_2, \ldots\}$ be a dense sequence. For every $x \in L$ there is an r_k such that $\varrho(r_k, x) < 1$, hence $N(x-r_k) < \infty$ and, by Lemma 2, $x-r_k$ is either rational or a transcendental Liouville number. But the set of Liouville numbers has measure 0, hence the real line would be a countable union of sets of measure 0, which is impossible. The same argument shows that $\varrho(\alpha, 0)$ is measurable. It remains to show that $\varrho(\alpha, 0)$ satisfies the Baire condition. To this end let us observe that there is a one-to-one correspondence between the set $\mathrm{Irr}(-\frac{1}{2}, \frac{1}{2})$ of all irrational numbers in the interval $(-\frac{1}{2}, \frac{1}{2})$ and the set of all pairs of sequences (ε_i) and (a_i) satisfying the condition

$$\varepsilon_i \in \{1, -1\}, \quad a_i \text{ integer}, \quad a_i \geqslant 2, \quad a_i+\varepsilon_{i+1} \geqslant 2 \quad (i \geqslant 1).$$

The correspondence is given by the expansion of an irrational number ξ into the continued fraction according to the nearest integer:

$$\xi = \frac{\varepsilon_1|}{|a_1} + \frac{\varepsilon_2|}{|a_2} + \ldots$$

The numerators $\varepsilon_i(\xi)$ and the denominators $a_i(\xi)$ are piecewise constant functions of ξ, hence they are of the first Baire class. Let

$$X = \left\{\xi \in \mathrm{Irr}(-\tfrac{1}{2}, \tfrac{1}{2}) \colon \sum_{i=1}^{\infty} \frac{1}{\log\log 2a_i(\xi)} < \infty\right\}$$

and let for $\xi \in X$

$$f(\xi) = \sum_{i=1}^{\infty} \frac{\varepsilon_i(\xi)}{a_i(\xi)}, \qquad g(\xi) = \sum_{i=1}^{\infty} \frac{1}{\log\log 2a_i(\xi)}.$$

Both the set X and the functions f, g are B-measurable.

We can rewrite the definition of $N(\alpha)$ in the form

$$N(\alpha) = \inf\{g(\xi) \colon \xi \in X,\ f(\xi) = \alpha\}.$$

The condition $\varepsilon_{i+1}+a_i \geqslant 2$ does not occur in the original definition of $N(\alpha)$ but can be added there. Indeed, if $\pm 1/2$ occurs in the representation of α as $\sum_{i=1}^{\infty} \varepsilon_i/a_i$, we can replace it by $\pm 1/4 \pm 1/4$ diminishing $\sum_{i=1}^{\infty} 1/\log\log 2a_i$, since $2/\log\log 8 < 1/\log\log 4$.

In order to prove that $\varrho(\alpha, 0)$ satisfies the Baire condition it suffices to show that for each real a the sets

$$S_1(a) = \{\alpha \in L\colon\ N(\alpha) \leqslant a\} \quad \text{and} \quad S_2(a) = \{\alpha \in L\colon\ N(\alpha) \geqslant a\}$$

satisfy the Baire condition.

We have

$$S_1(a) = \bigcap_{m=1}^{\infty} \{f(\xi)\colon\ \xi \in X,\ g(\xi) < a+1/m\} = \bigcap_{m=1}^{\infty} f(g^{-1}(0, a+1/m)).$$

Since X, f, g are B-measurable, $g^{-1}(0, a+1/m)$ is a Borel set for every m, thus $f(g^{-1}(0, a+1/m))$ is analytic and so is $S_1(a)$. Similarly,

$$S_2(a) = L \setminus f(g^{-1}(0, a)).$$

Thus $S_2(a)$ is an analytic complement and so it satisfies the Baire condition as well as $S_1(a)$ does. Of course in this way we obtain another proof of measurability of ϱ.

Remark 1. In the Remark after Theorem 5 in [1] it is stated that a separable complete Borel metric in L must be topologically equivalent to the usual metric. However it is actually the Baire condition and not the B-measurability that is responsible for this statement.

Remark 2. The numbers $\alpha \in L$ with $\varrho(\alpha, 0) < 1$ or, equivalently, $N(\alpha) < \infty$, form an additive group G. Since $G \neq L$, it is totally disconnected in the topology induced by ϱ. On the other hand it is generated by every neighbourhood of zero, since if

$$\alpha = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{a_i}, \quad \sum_{i=1}^{\infty} \frac{1}{\log\log 2a_i} < \infty$$

then also for every m

$$\alpha = \sum_{i=1}^{\infty} \underbrace{\left(\frac{\varepsilon_i}{a_i m} + \ldots + \frac{\varepsilon_i}{a_i m}\right)}_{m \text{ times}}, \quad \sum_{i=1}^{\infty} \frac{m}{\log\log 2a_i m} < \infty$$

and, for m large enough, $\varepsilon_i/a_i m$ is in any given neighbourhood. Therefore G furnishes another example of the phenomenon exhibited by T. C. Stevens [3], which yields a negative answer to Mazur's problem 160 [2].

REFERENCES

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