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SEMILATTICES DO NOT HAVE EQUATIONALLY COMPACT HULLS

BY

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A *semilattice* is an algebra with one binary operation $\wedge$ which is associative, commutative and idempotent. Each semilattice has a natural partial ordering on it, given by $a \leqslant b$ iff $a \wedge b = a$; the symbol $\vee$ denotes the least upper bound (join) under this partial ordering.

Recall that an algebra A is *equationally compact* if every subset $\Sigma \subseteq A[X]^2$ ($A[X]$ is the free extension of A by the set X in any equational class containing A) is contained in the kernel of a homomorphism $A[X] \to A$ over A whenever every finite subset of Σ has this property. An extension B of an algebra A is *pure* if every finite subset of $A[X]^2$ is contained in the kernel of a homomorphism $A[X] \to A$ over A whenever it is contained in the kernel of a homomorphism $A[X] \to B$ over A. These notions were introduced for general algebras by Mycielski [4] and Węglorz [11], respectively; the present algebraic formulations can be found in Banaschewski and Nelson [1].

The equationally compact semilattices were characterized by Grätzer and Lakser [3] as those semilattices S which are conditionally complete (i. e. every non-empty subset has a greatest lower bound), in which every chain has a least upper bound, and in which the distributivity law $a \wedge \bigvee C = \bigvee \{a \wedge c \mid c \in C\}$ holds for all $a \in S$ and for all chains $C \subseteq S$. Bulman-Fleming [2] has proved that every equationally compact semilattice is a retract of a (topologically) compact one, and Taylor [10] has improved this replacing "retract of a compact one" by "retract of a product of finite semilattices".

It is easy to see that every semilattice has an equationally compact extension; this is provided by the usual embedding of a semilattice S into $P(S)$, the power set of S with the operation of set intersection, which maps s to $\{a \in S \mid a \leqslant s\}$, and the fact that $P(S)$ is isomorphic to the S-th power of the two-element semilattice and hence is equationally compact.

A semilattice S has an equationally compact hull (in the class of all semilattices) iff it has a pure, equationally compact semilattice extension (Banaschewski and Nelson [1], Proposition 2). It is the aim of this note to prove that semilattices do not have equationally compact hulls by exhibiting, for each infinite cardinal $\mathfrak{n}$, an $\mathfrak{n}$-element semilattice $S_{\mathfrak{n}}$ which has no pure, equationally compact, semilattice extension. (Note that such a semilattice has no pure equationally compact extension at all; see [1], p. 156, Remark 2.)

By Theorem 3.12 of Taylor [8] or Proposition 4 of Banaschewski and Nelson [1], it will then follow that there are pure-irreducible semilattices of arbitrarily high cardinality; in fact, we will see that all $S_{\mathfrak{n}}$ are actually pure irreducible.

The reader is referred to [1] for all notions concerning purity and equational compactness which are not defined here.

Definition of $S_{\mathfrak{n}}$. For any cardinal number $\mathfrak{n}$ (finite or infinite), let $S_{\mathfrak{n}}$ be the semilattice with underlying set $(\mathfrak{n}\times\{0,1\})\cup\{u,1\}$ (where $u\notin\mathfrak{n}\times\{0,1\}$) and with the operation defined by

$$x\wedge 1 = x \quad \text{for all } x,$$

$$(\lambda,0)\wedge u = (\lambda,1)\wedge u = (\lambda,0) \quad \text{for all } \lambda<\mathfrak{n},$$

$$(\lambda,0)\wedge(\mu,0) = (\lambda,0)\wedge(\mu,1) = \big(\min(\lambda,\mu),0\big) \quad \text{for all } \lambda,\mu<\mathfrak{n},$$

$$(\lambda,1)\wedge(\mu,1) = \begin{cases}\big(\min(\lambda,\mu),0\big) & \text{if } \lambda \text{ is even and } \mu \text{ is odd or } \textit{vice versa},\\ \big(\min(\lambda,\mu),1\big) & \text{otherwise}.\end{cases}$$

The underlying partially ordered set of $S_{\mathfrak{n}}$ is depicted in Fig. 1.

First of all, note that, for all $\mathfrak{n}$, there are no elements $x, y\in S_{\mathfrak{n}}$ such that $x\wedge u = y\wedge u = x\wedge y$ and $x\wedge(0,1) = (0,1)$ and $y\wedge(1,1) = (1,1)$. Indeed, if $x\wedge(0,1) = (0,1)$, then either $x = 1$ or $x = (\lambda,1)$ for some even $\lambda<\mathfrak{n}$ and, similarly, $y = 1$ or $y = (\mu,1)$ for some odd $\mu<\mathfrak{n}$; since $x\wedge u = y\wedge u$, it follows that $x = y = 1$, and thus $x\wedge y = 1 \neq x\wedge u$. Thus

$$\Sigma = \big\{(x\wedge u, y\wedge u), (x\wedge u, x\wedge y), \big(x\wedge(0,1),(0,1)\big), \big(y\wedge(1,1),(1,1)\big)\big\}$$

is a finite subset of $S_{\mathfrak{n}}[\{x,y\}]^2$ which is not contained in the kernel of any homomorphism $S_{\mathfrak{n}}[\{x,y\}]\to S_{\mathfrak{n}}$ over $S_{\mathfrak{n}}$.

Now suppose that, for some infinite $\mathfrak{n}$, $T\supseteq S_{\mathfrak{n}}$ is an equationally compact extension. Then it follows from the Grätzer-Lakser result mentioned above that, in T, the sets

$$C_0 = \{(\lambda,1)\mid\lambda \text{ even}\},\qquad C_1 = \{(\lambda,1)\mid\lambda\text{ odd}\}\qquad\text{and}\qquad C_2 = \mathfrak{n}\times\{0\}$$

have least upper bounds. Let $\bar{x} = \bigvee C_0$, $\bar{y} = \bigvee C_1$ and $z = \bigvee C_2$. Then, again by the Grätzer-Lakser result,

$$u\wedge x = \bigvee\{u\wedge(\lambda,1)\mid\lambda\text{ even}\} = \bigvee\{(\lambda,0)\mid\lambda\text{ even}\} = z.$$

(This uses the fact that $\mathfrak{n}$ is infinite.) Similarly, $u \wedge \bar{y} = z = \bar{x} \wedge \bar{y}$. Trivially, $\bar{x} \wedge (0, 1) = (0, 1)$ and $\bar{y} \wedge (1, 1) = (1, 1)$. Thus (with Σ as above), the homomorphism $S_{\mathfrak{n}}[\{x, y\}] \to T$ over $S_{\mathfrak{n}}$, mapping x to $\bar{x}$ and y to $\bar{y}$, contains Σ in its kernel and, consequently, T is not a pure extension of $S_{\mathfrak{n}}$. It follows that $S_{\mathfrak{n}}$ *has no pure, equationally compact extension.*

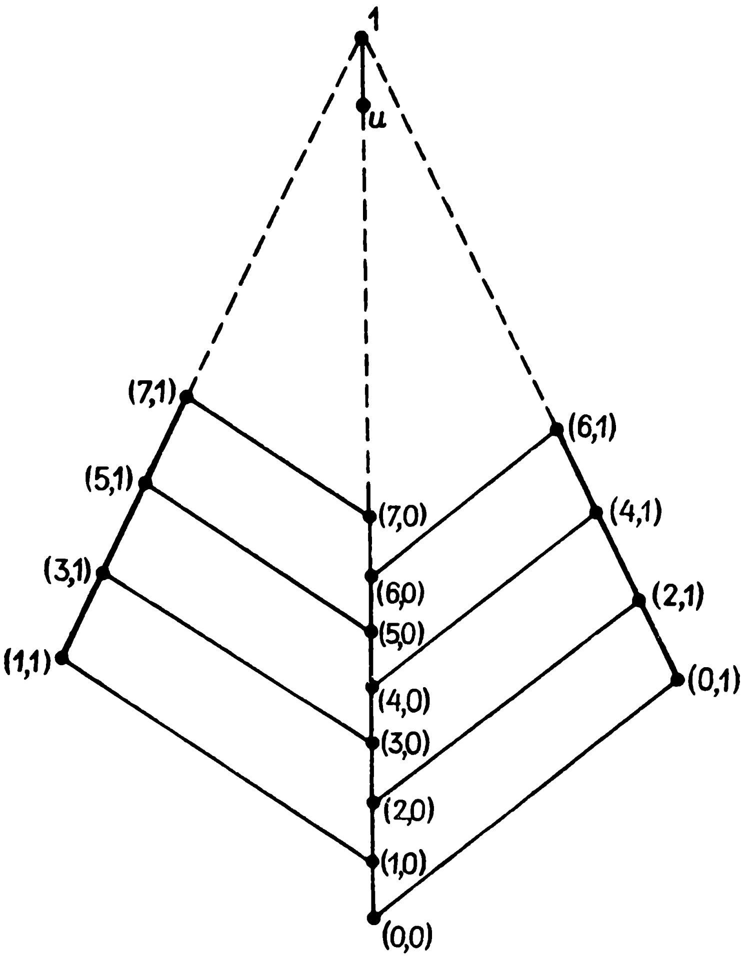

Fig. 1

Now suppose that $\mathfrak{n} > 0$ and that θ is a proper congruence on $S_{\mathfrak{n}}$. Then it is easy to see that either $(u, 1) \in \theta$ or $\big((\lambda, 0), (\lambda+1, 0)\big) \in \theta$ for some $\lambda < \mathfrak{n}$ or $\big((0, 0), (0, 1)\big) \in \theta$ or $\big((1, 0), (1, 1)\big) \in \theta$.

In case $(u, 1) \in \theta$, let $\bar{x} = \bar{y} = 1$.

In case $\big((\lambda, 0), (\lambda+1, 0)\big) \in \theta$ for λ even, let $\bar{x} = (\lambda, 1)$ and $\bar{y} = (\lambda+1, 1)$.

In case $\big((\lambda, 0), (\lambda+1, 0)\big) \in \theta$ for λ odd, let $\bar{x} = (\lambda+1, 1)$ and $\bar{y} = (\lambda, 1)$.

In case $\big((0, 0), (0, 1)\big) \in \theta$, let $\bar{x} = (1, 0)$ and $\bar{y} = (1, 1)$.

In case $\big((1, 0), (1, 1)\big) \in \theta$, let $\bar{x} = (2, 1)$ and $\bar{y} = (2, 0)$.

Then, in all cases, $(\bar{x} \wedge u, \bar{y} \wedge u)$, $(\bar{x} \wedge \bar{y}, \bar{x} \wedge u)$, $\big(\bar{x} \wedge (0, 1), (0, 1)\big)$ and $\big(\bar{y} \wedge (1, 1), (1, 1)\big)$ belong to θ.

It follows from this (see Taylor [8], Lemma 3.4, or Banaschewski and Nelson [1], Lemma 5) that $S_\mathfrak{n}$ *is pure irreducible for all* (*finite or infinite*) $\mathfrak{n} > 0$.

As mentioned above, Taylor has proved that every equationally compact semilattice is a retract of a product of finite semilattices. His short proof uses some rather complicated results, including the one of Numakura [6] that every totally disconnected topological semigroup is an inverse limit of finite discrete ones. We close by giving a modification of Taylor's proof which is straightforward and self-contained, and does not use this result of Numakura. The following proposition is essentially a corollary to Theorem 3.1 of Pacholski and Węglorz [7] (see also Taylor [9], Lemma 7.1); the present proof uses algebraic, rather than model-theoretic, techniques.

PROPOSITION. *If A is a compact Hausdorff topological algebra, and if $\mathfrak{C}$ is a down-directed set of closed congruences on A with*

$$\bigcap \mathfrak{C} = \Delta_A = \{(a, a) \mid a \in A\},$$

then the embedding $A \to \prod A/\theta$ ($\theta \in \mathfrak{C}$) is pure (*and hence is retractable*).

Proof. Let $A[X]$ be, as usual, the free extension over A by the set X in some convenient equational class containing A. For each $g \in A^X$, let $\bar{g}$: $A[X] \to A$ be the homomorphism over A extending g. Then, for each $p \in A[X]$, the map $\tilde{p}$: $A^X \to A$, given by $\tilde{p}(g) = \bar{g}(p)$, is continuous with respect to the product topology on A^X. Thus, for each $(p, q) \in A[X]^2$, if θ is a closed subset of A^2, then

$$\{g \in A^X \mid (\bar{g}(p), \bar{g}(q)) \in \theta\} = \{g \in A^X \mid (\tilde{p}(g), \tilde{q}(g)) \in \theta\}$$

is a closed subset of A^X.

Now suppose that $\Sigma \subseteq A[X]^2$ is a finite subset which is contained in the kernel of some homomorphism $A[X] \to \prod A/\theta$ ($\theta \in \mathfrak{C}$) over $A \to \prod A/\theta$. For each $\theta \in \mathfrak{C}$, let

$$S_\theta = \{g \in A^X \mid \bar{g}^2(\Sigma) \subseteq \theta\};$$

then the S_θ form a down-directed collection of non-empty closed subsets of A^X. Since A, and hence also A^X, is compact, there exists $g \in \bigcap S_\theta$. But $\bigcap \mathfrak{C} = \Delta_A$, and thus $\Sigma \subseteq \operatorname{Ker} \bar{g}$. This proves that the embedding in question is pure.

COROLLARY 1. *If A is a compact algebra in which the closed congruences θ of finite index* (*i.e. with A/θ finite*) *separate the points of A, then A is the* (*algebraic*) *retract of a product of finite algebras* (*each of which is a quotient of A*).

Proof. In any topological algebra, the set of closed congruences of finite index forms a down-directed set (since there is an embedding $A/\theta_1 \cap \theta_2 \to A/\theta_1 \times A/\theta_2$).

COROLLARY 2. *Every equationally compact semilattice is a retract of a product of finite ones.*

Proof. Bulman-Fleming [2] has shown that every equationally compact semilattice is the retract of its closure $\bar{S}$ in 2^S. The closure $\bar{S}$, being a compact subspace of 2^S, obviously satisfies the hypotheses of Corollary 1.

One final comment. $S_{\mathfrak{n}}$ is the underlying $\wedge$-semilattice of a complete lattice, and we have seen that, despite this, $S_{\mathfrak{n}}$ (for $\mathfrak{n}$ infinite) does not have an equationally compact hull. However, since, by the Grätzer-Lakser characterization, the underlying $\wedge$-semilattice of the lattice of all ideals of any lattice is an equationally compact semilattice, and since the embedding of every distributive lattice into its ideal lattice is pure (Nelson [5]), and hence also pure as a semilattice homomorphism, it follows that any semilattice which is the underlying $\wedge$-semilattice of a distributive lattice has an equationally compact hull in the class of all semilattices.

REFERENCES

[1] B. Banaschewski and E. Nelson, *Equational compactness in equational classes of algebras*, Algebra Universalis 2 (1972), p. 152-165.

[2] S. Bulman-Fleming, *On equationally compact semilattices*, ibidem 2 (1972), p. 146-151.

[3] G. Grätzer and H. Lakser, *Equationally compact semilattices*, Colloquium Mathematicum 20 (1969), p. 27-30.

[4] Jan Mycielski, *Some compactifications of general algebras*, ibidem 13 (1964), p. 1-9.

[5] E. Nelson, *The embedding of a distributive lattice into its ideal lattice is pure*, Algebra Universalis (in print).

[6] K. Numakura, *Theorems on compact totally disconnected semigroups and lattices*, Proceedings of the American Mathematical Society 8 (1957), p. 623-626.

[7] L. Pacholski and B. Węglorz, *Topologically compact structures and positive formulas*, Colloquium Mathematicum 19 (1968), p. 37-42.

[8] W. Taylor, *Residually small varieties*, Algebra Universalis 2 (1972), p. 33-53.

[9] — *Some constructions of compact algebras*, Annals of Mathematical Logic 3 (1972), p. 395-436.

[10] — *Pure irreducible mono-unary algebras*, Algebra Universalis (to appear).

[11] B. Węglorz, *Equationally compact algebras I*, Fundamenta Mathematicae 59 (1966), p. 289-298.

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