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*POINTWISE INDUCED SEMIGROUPS OF σ-ENDOMORPHISMS**

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1. Introduction. Let X be a set with a separable (countably generated and separating points) σ-algebra $\mathscr{B}$ of its subsets and with a σ-ideal $\mathscr{I}$ of $\mathscr{B}$. A $\mathscr{B}$-measurable transformation f of X with the domain $D(f) \in \mathscr{B}$ is called *non-singular* if $f^{-1}(I) \in \mathscr{I}$ whenever $I \in \mathscr{I}$. We call it *two-sided non-singular* if also $f(I) \in \mathscr{I}$ for any $I \in \mathscr{I}$. By $\mathscr{T}$ we denote the semigroup of all non-singular transformations with the multiplication defined by

$$(fg)(x) = g\big(f(x)\big) \quad \text{for } x \in D(fg) = f^{-1}\big(D(g)\big).$$

We say that a semigroup S *acts on* X if there is a homomorphism $s \to f_s$ of S into $\mathscr{T}$.

By $\boldsymbol{B}$ we denote the quotient Boolean σ-ring $\mathscr{B}/\mathscr{I}$. A mapping F from $\boldsymbol{B}$ into $\boldsymbol{B}$ is called a *σ-endomorphism* if it preserves differences and countable unions of elements. The semigroup of all σ-endomorphisms of $\boldsymbol{B}$ with the multiplication defined by

$$(FG)(a) = F\big(G(a)\big)$$

will be denoted by $\boldsymbol{T}$. We say that a semigroup S *acts on* $\boldsymbol{B}$ if there is a homomorphism $s \to F_s$ of S into $\boldsymbol{T}$; S *acts automorphically on* $\boldsymbol{B}$ if the F_s are automorphisms of the Boolean σ-algebra $\boldsymbol{B}$.

Every element f of $\mathscr{T}$ induces an element F of $\boldsymbol{T}$ by means of the formula

$$F([A]) = [f^{-1}(A)],$$

where $A \in \mathscr{B}$, and $[A]$ is the equivalence class of A. The mapping $f \to F$ is clearly a homomorphism of $\mathscr{T}$ into $\boldsymbol{T}$. In the opposite direction, Sikorski has proved that every element of $\boldsymbol{T}$ is pointwise induced by an element of $\mathscr{T}$, provided X is a Borel space (see Theorem 5.1 of [6] or Theorem 6.3 of [7]; a simple proof using a characteristic function of a sequence of sets is presented in [8], 32.5). We recall that X is called a *Borel space* if it is Borel isomorphic to a Borel subset of the Hilbert cube. If two trans-

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formations f and g induce the same σ-endomorphism of $\boldsymbol{B}$, then they are equal almost everywhere: the symmetric difference $D(g)\,\Delta\,D(f)$ and the set $\{x \in D(f)\cap D(g): f(x) \neq g(x)\}$ are in $\mathscr{I}$ (see the proof of 4.5 in [6]).

The question arises whether for a Borel space X and for a given semigroup S acting by $s\to F_s$ on $\boldsymbol{B}$ there is some action $s\to f_s$ of S on X such that f_s induces F_s for any $s \in S$. The difficulty in finding such a simultaneous choice of the f_s lies in the fact that transformations chosen individually for the σ-endomorphisms F_s need not form a semigroup. If such a choice $s\to f_s$ exists, we say that S is *pointwise induced* by the action $s\to f_s$ on X or, simply, by the transformations f_s.

In his paper [4], Mackey has given a positive solution of this problem under the additional assumptions that S is a locally compact second countable group acting automorphically on $\boldsymbol{B}$, $\mathscr{I}$ is the σ-ideal of zero sets for some finite Borel measure, and the action of S on $\boldsymbol{B}$ is measurable in the sense that the real function $s\to m\big(F_s([A])\big)$ is Borel measurable for any A in $\mathscr{B}$ and for any finite measure m on $\boldsymbol{B}$.

Ours is a different approach as we are concentrating on discrete rather than topological semigroups. The results thus obtained differ essentially from those of Mackey. In this paper we present positive solutions if S is a countable semigroup (Theorem 1), if S is a free product of countable semigroups (Theorem 2), and, under some additional conditions, if S is a direct sum of $\aleph_1$ countable semigroups (Theorem 3). This last result enables us to obtain, under the continuum hypothesis, positive solutions for additive groups R^n, $n = 1, 2, \ldots, \aleph_0$, acting automorphically on $\boldsymbol{B}$ (Corollary 1). Finally, we indicate some relations to the existence of a lower density (Section 7) and shortly discuss connections with deterministic sub-Markov operators (Section 8).

2. Countable semigroups. Let $\mathscr{S}$ be a subsemigroup of $\mathscr{T}$. A subset A of X is called *invariant* with respect to $\mathscr{S}$ if $f^{-1}(A)\subset A$ for any $f\in\mathscr{S}$. Obviously, $B\cup\bigcup_{f\in\mathscr{S}} f^{-1}(B)$ is the smallest $\mathscr{S}$-invariant set containing B.

LEMMA 1. *Let S be a countable semigroup acting by $s\to F_s$ on $\boldsymbol{B}$. If each F_s is pointwise induced, then also S is pointwise induced.*

Proof. For any s in S let f_s be a non-singular transformation inducing F_s and let $\mathscr{S}$ denote the semigroup generated by all the f_s. Since f_sf_t and f_{st} induce the same σ-endomorphism F_{st}, the set

$$I_{s,t} = \{x \in D(f_sf_t)\cap D(f_{st}):\ (f_sf_t)(x) \neq f_{st}(x)\}\cup\big(D(f_sf_t)\,\Delta\,D(f_{st})\big)$$

is in $\mathscr{I}$ for any s, t in S. We denote by I the smallest $\mathscr{S}$-invariant set containing the union of the $I_{s,t}$ for $s, t\in S$. Since $\mathscr{S}$ is countable, we clearly have $I \in \mathscr{I}$. Let us put

$$g_s = f_s|D(f_s) - I \quad \text{for any } s \in S.$$

Obviously, g_s induces F_s, and it is a routine to check that

$$g_{st}(x) = f_{st}(x) = (f_s f_t)(x) = (g_s f_t)(x) = (g_s g_t)(x)$$
$$\text{for } x \in D(g_{st}) \cup D(g_s g_t);\ s, t \in S.$$

Therefore, $g_{st} = g_s g_t$ and $s \to g_s$ is the required action of S on X.

Using the already mentioned result of Sikorski we obtain

THEOREM 1. *If X is a Borel space, then every countable semigroup S acting on $\boldsymbol{B}$ is pointwise induced.*

It should be noted that in case where the F_s are σ-endomorphisms of $\boldsymbol{B}$ regarded as Boolean σ-algebra, i.e. if $F_s([X]) = [X]$ for any $s \in S$, the transformations f_s can be chosen so that $D(f_s) = X$ and, by putting $g_s(x) = x$ for $x \in I$, we obtain transformations g_s acting on the whole X. The action $s \to g_s$ pointwise induces S.

3. Free products. A semigroup S is called the *free product* of its subsemigroups S_i, $i \in I$, if every element of S has a unique representation in the form $s_1 \ldots s_k$, $k \geqslant 1$, with $s_j \in S_{i(j)}$ and $i(j) \neq i(j+1)$ for $j = 1, \ldots$ $\ldots, k-1$ ([1], § 9.4).

LEMMA 2. *Let a semigroup S acting by $s \to F_s$ on $\boldsymbol{B}$ be a free product of its subsemigroups S_i, $i \in I$. If the S_i are pointwise induced, then also S is pointwise induced.*

Proof. Suppose that any S_i acts on X by $s \to f_s$, and f_s induces F_s for $s \in S_i$. For any s in S we take its unique representation $s_1 \ldots s_k$, $s_j \in S_{i(j)}$, $i(j) \neq i(j+1)$, and define f_s by putting $f_s = f_{s_1} \ldots f_{s_k}$. Clearly, f_s induces $F_{s_1} \ldots F_{s_k} = F_s$, so that $s \to f_s$ is the required action of S on X.

In particular, by Theorem 1 we obtain

THEOREM 2. *Let X be a Borel space and let a semigroup S act on $\boldsymbol{B}$. If S is a free product of its countable subsemigroups, then S is pointwise induced.*

Since for any family of countable semigroups we can construct their free product, there exist uncountable semigroups satisfying Theorem 2. It is, however, less trivial that there are such semigroups *acting effectively on $\boldsymbol{B}$*, i.e. such that $s \to F_s$ is one-to-one.

Example. Let us assume that X is the real plane, $\mathscr{B}$ the Borel σ-algebra of X, and $\mathscr{I}$ the trivial σ-ideal $\{\emptyset\}$. Let X_i, $i \in I$, be the partition of X into the vertical lines. It follows from the isomorphism theorem for Borel sets ([3], § 37, II, Theorem 2) that for any $i \in I$ there exists a Borel isomorphism f_i from X onto X_i. Let us observe that for any x, y not in X_i and for any positive integers m, n we have

$$f_i^m(x) = f_i^n(y) \Rightarrow m = n,\ x = y.$$

In fact, if, e.g., $m < n$, then

$$x = f_i^{-m}\big(f_i^m(x)\big) = f_i^{n-m}(y) \in X_i,$$

a contradiction; for $m = n$ the assertion is trivial. Now we show that the semigroup $\mathscr{S}$ generated by the f_i, $i \in I$, is the free product of its cyclic subsemigroups $\{f_i^n\colon n \geqslant 1\}$, $i \in I$. To this end, suppose that

$$f_{i_1}^{m_1} \dots f_{i_k}^{m_k} = f_{j_1}^{n_1} \dots f_{j_l}^{n_l},$$

where $i_s \neq i_{s+1}$ and $j_t \neq j_{t+1}$ for $1 \leqslant s \leqslant k-1$, $1 \leqslant t \leqslant l-1$. For any element x in X, its image by the left-hand side of this equality is in X_{i_k} and that by the right-hand side in X_{j_l}. Therefore $i_k = j_l$, which implies $m_k = n_l$. Proceeding by induction we obtain $k = l$, $i_s = j_s$, and $m_s = n_s$ for $1 \leqslant s \leqslant k$. Since $\mathscr{I}$ is trivial, the transformations f of X can be viewed as σ-endomorphisms F of $\boldsymbol{B}$, thus the action $f \to F$ of $\mathscr{S}$ on $\boldsymbol{B}$ is effective. The same argument works if $\mathscr{I}$ is the σ-ideal of countable subsets of X.

4. Semigroups of two-sided non-singular transformations. Let $\mathscr{S}$ be a semigroup of two-sided non-singular transformations on X. A subset A of X is called *two-sided invariant* with respect to $\mathscr{S}$ if both $f^{-1}(A)$ and $f(A)$ are subsets of A for any f in $\mathscr{S}$. Let us observe that A is two-sided invariant if and only if $f^{-1}(A) = A$ for any $f \in \mathscr{S}$. For any $B \subset X$ the smallest two-sided invariant set containing B is equal to

$$B \cup \bigcup f_1^{i_1} \dots f_n^{i_n}(B),$$

where $f_1, \dots, f_n$ run over all finite sequences of elements of $\mathscr{S}$, and $i_1, \dots, i_n \in \{-1, 1\}$. By an argument analogous to that in Lemma 1, we can see that any countable semigroup S acting by $s \to \boldsymbol{F}_s$ on $\boldsymbol{B}$ is pointwise induced by a semigroup of two-sided non-singular transformations, provided each $\boldsymbol{F}_s$ is induced by such a transformation.

LEMMA 3. *Suppose that S_0 and S_1 are countable semigroups. Let both S_0 and S_1 act on X by $s \to f_s$, all f_s being two-sided non-singular transformations, and let $f_s f_t = f_t f_s$ almost everywhere for all $s \in S_0$ and $t \in S_1$, Then there exists a (possibly) new action $t \to g_t$ of S_1 on X such that $f_t = g_t$ almost everywhere and $f_s g_t = g_t f_s$ for all $s \in S_0$ and $t \in S_1$.*

Proof. Let for any $s \in S_0$ and $t \in S_1$

$$I_{s,t} = \{x \in D(f_s f_t) \cap D(f_t f_s)\colon (f_s f_t)(x) \neq (f_t f_s)(x)\} \cup \big(D(f_s f_t) \,\varDelta\, D(f_t f_s)\big).$$

Clearly, $I_{s,t} \in \mathscr{I}$ and the union of all $I_{s,t}$ is contained in $J \in \mathscr{I}$, a two-sided invariant set with respect to the semigroup generated by $\{f_s\colon s \in S_0 \cup S_1\}$. Letting

$$g_t = f_t | D(f_t) - J \qquad \text{for } t \in S_1,$$

we obtain the required action of S_1.

Let us note that if all f_t, $t \in S_1$, are defined everywhere on X, then, extending the g_t as in the remark at the end of Section 2, we may assume that also all g_t are defined everywhere. In particular, if some f_t is the identity transformation of X, then g_t can be taken to be the identity.

5. Direct sums. The direct sum of semigroups with identities is a natural generalization of the direct sum of groups. The following definition is adopted from [1], § 9.4:

Let $\{S_i: i \in I\}$ be a non-void family of semigroups with the identity elements $e_i \in S_i$. The set $\sum^* S_i$ of all $(s_i) \in \prod S_i$ such that $s_i = e_i$ for all but finitely many indices is called the *direct sum* of the semigroups S_i.

For any $j \in I$ the isomorphism $s_j \to (s_i)$ with $s_i = e_i$ for $i \neq j$ is called the *natural imbedding* of S_j into $\sum^* S_i$. We say that a semigroup S with the identity element e is the *direct sum* of its subsemigroups S_i containing e if the natural imbeddings $S_j \to \sum^* S_i$ extend to the isomorphism of S onto $\sum^* S_i$. Then we also write $S = \sum^* S_i$. In this case every element $s \neq e$ of S has the unique, modulo the ordering of the factors, representation $s = s_1 \ldots s_n$, where $n \geqslant 1$, $s_j \in S_{i(j)}$, and $i(j) \neq i(k)$ for $i \neq k$; $i, k = 1, \ldots, n$. If $\{J, K\}$ is a non-trivial partition of I, then, clearly, $\sum\limits_{i \in I}^* S_i$ is the direct sum of $\sum\limits_{j \in J}^* S_j$ and $\sum\limits_{k \in K}^* S_k$.

LEMMA 4. *Let S be the direct sum of its two countable subsemigroups S_0 and S_1 containing the identity element e of S. Suppose, in addition, that S acts by $s \to F_s$ on $\boldsymbol{B}$ and that*

(1) *each F_s is induced by a two-sided non-singular transformation,*

(2) *S_0 is pointwise induced by two-sided non-singular transformations f_s, $s \in S_0$,*

(3) *f_e is the identity transformation of X.*

Then there exist two-sided non-singular transformations f_r, $r \in S - S_0$, such that S is pointwise induced by the action $s \to f_s$ on X.

Proof. By Lemma 1, S_1 is pointwise induced by some action $t \to g_t$ on X. By the remark preceding Lemma 3 we may assume that all g_t are two-sided non-singular and, by the remark at the end of Section 4, that g_e is the identity transformation. It follows from Lemma 3 that there exists an action $t \to f_t$ of S_1, such that f_t induces F_t for $t \in S_1$, and $f_s f_t = f_t f_s$ for $s \in S_0$, $t \in S_1$. Each element $r \in S - (S_0 \cup S_1)$ has a unique representation as the product of an element s from S_0 and an element t from S_1. Putting $f_r = f_s f_t$ we obtain the required action of S on X.

THEOREM 3. *Let S be the direct sum of $\aleph_1$ countable semigroups S_α, $\alpha < \omega_1$, containing the identity element e. Suppose that S acts by $s \to F_s$ on $\boldsymbol{B}$ and that*

(1) *each F_s is induced by a two-sided non-singular transformation,*

(2) *F_e is the identity map of $\boldsymbol{B}$.*

Then S is pointwise induced by two-sided non-singular transformations.

Proof. For any $0 < \lambda \leqslant \omega_1$ we put

$$T_\lambda = \sum_{\nu<\lambda}^{*} S_\nu$$

so that $T_\nu \subset T_\lambda$ for $\nu < \lambda$. The construction of the required action of S proceeds by transfinite induction.

1° First step. From the remark preceding Lemma 3 it follows that T_1 is pointwise induced by some action $s \to f_s$, $s \in T_1$, where the f_s are two-sided non-singular transformations and f_e is the identity on X.

2° Non-limit step. Let $0 < \lambda < \omega_1$ and suppose that T_λ is pointwise induced by $s \to f_s$, $s \in T_\lambda$. Since T is a countable semigroup, we may apply Lemma 4: putting $S_0 = T_\lambda$, $S_1 = S_\lambda$, and $S = T_{\lambda+1}$, we obtain an extension $s \to f_s$, $s \in T_{\lambda+1}$, of the action of T_λ, with two-sided non-singular transformations f_s.

3° Limit step. Let $\lambda \leqslant \omega_1$ be a limit ordinal. If for any $0 < \nu < \lambda$ the semigroup T_ν is pointwise induced by the already defined two-sided non-singular action $s \to f_s$, $s \in T_\nu$, then the action $s \to f_s$, $s \in T_\lambda$, induces T_λ.

Since $S = T_{\omega_1}$, the action $s \to f_s$, $s \in T_{\omega_1}$, satisfies our theorem.

6. Groups of automorphisms. A one-to-one transformation f from X onto X is called a *point automorphism* if both f and f^{-1} are $\mathscr{B}$-measurable and non-singular.

LEMMA 5. *Let a group G act automorphically on $\boldsymbol{B}$. If G (regarded as a semigroup) is pointwise induced, then it is pointwise induced by point automorphisms of X.*

Proof. Let $g \to f_g$, $g \in G$, be any action that induces G. We may assume that f_e is the identity on X by putting, if necessary, $f_g(x) = x$ whenever $f_e(x) \neq x$ or $x \notin D(f_e)$. Such a modification does not spoil the action of G and we get

$$f_g f_{g^{-1}} = f_e = f_{g^{-1}} f_g,$$

so that $f_g^{-1} = f_{g^{-1}}$.

The following is a consequence of Theorem 3 and Lemma 5:

THEOREM 4. *Let G be the direct sum of $\aleph_1$ countable groups. Suppose, in addition, that G acts automorphically on $\boldsymbol{B}$ by $g \to F_g$. If all automorphisms F_g are pointwise induced, then G is pointwise induced by an action $g \to f_g$, where all f_g are point automorphisms of X.*

In particular, using the Hamel bases for linear spaces over the field of rational numbers, we obtain

COROLLARY 1. *Let us assume the continuum hypothesis ($\aleph_1 = 2^{\aleph_0}$). If X is a Borel space and if $G = R^n$ acts automorphically on $\boldsymbol{B}$ for some $n = 1, 2, \ldots, \aleph_0$, then G is pointwise induced by point automorphisms.*

7. Lower density. If $\mathscr{D}: \boldsymbol{B} \to \mathscr{B}$ is a function satisfying

(1) $[\mathscr{D}(a)] = a$,

(2) $\mathscr{D}(a \cap b) = \mathscr{D}(a) \cap \mathscr{D}(b)$

for all $a, b \in \boldsymbol{B}$, then it is called a *lower density* for $\boldsymbol{B}$. The general problem of the existence of a lower density was studied by von Neumann and Stone in [5].

For any $b \in \boldsymbol{B}$ we denote by E_b the σ-endomorphism of $\boldsymbol{B}$ defined by $E_b(a) = a \cap b$, $a \in \boldsymbol{B}$. The multiplicative semigroup (semilattice) $\boldsymbol{B}$ acts on $\boldsymbol{B}$ by $b \to E_b$.

THEOREM 5. *The following conditions are equivalent:*

(i) *the multiplicative semigroup $\boldsymbol{B}$ acting by $b \to E_b$ on $\boldsymbol{B}$ is pointwise induced;*

(ii) *there is a lower density $\mathscr{D}: \boldsymbol{B} \to \mathscr{B}$.*

Proof. (i) $\Rightarrow$ (ii). Let $\boldsymbol{B}$ be induced by an action $b \to e_b$ on X. We define $\mathscr{D}(b)$ to be the range $R(e_b)$ of e_b. Since $e_b e_b = e_b$ and $\mathscr{B}$ is separable, we have

$$\mathscr{D}(b) = \{x \in D(e_b) : e_b(x) = x\} \in \mathscr{B}.$$

Now it suffices to show (2). From the obvious inclusion $R(e_a e_b) \subset R(e_b)$ and from the commutativity of $\boldsymbol{B}$ it follows that

$$R(e_a e_b) \subset R(e_a) \cap R(e_b).$$

In the opposite direction, for any x on the right-hand side of the inclusion we have $e_a(x) = e_b(x) = x$ so that $x = (e_a e_b)(x)$ and $x \in R(e_a e_b)$.

(ii) $\Rightarrow$ (i). For any $b \in \boldsymbol{B}$ we denote by e_b the identity transformation of $\mathscr{D}(b) \in \mathscr{B}$. From the definition of the lower density it is clear that $b \to e_b$ is the action of $\boldsymbol{B}$ on X.

Theorem 17 of [5] guarantees the existence of a lower density for $\boldsymbol{B}$ provided the continuum hypothesis holds. Therefore, under the continuum hypothesis, the semigroup $\boldsymbol{B}$ acting on $\boldsymbol{B}$ is pointwise induced. Moreover, if X is an uncountable Borel space and $\mathscr{I}$ the σ-ideal of countable subsets of X, then from the isomorphism theorem for Borel sets, from Theorem 18 of [5], and from the remark on p. 378 of [5] it follows that the continuum hypothesis is equivalent to the existence of a lower density. Thus we get

COROLLARY 2. *Let X be an uncountable Borel space and $\mathscr{I}$ the σ-ideal of its countable sets. The semigroup $\boldsymbol{B}$ acting by $b \to E_b$ on $\boldsymbol{B}$ is pointwise induced if and only if the continuum hypothesis holds.*

8. Sub-Markov operators. Let m be a positive σ-finite measure on $\mathscr{B}$ and let $\mathscr{I}$ be the σ-ideal of zero sets with respect to m. A linear operator P acting on $L_1(m)$, the space of (equivalence classes of) integrable real functions on X, is called *sub-Markov* if

(1) $Pu \geqslant 0$ for $u \geqslant 0$,

(2) $\|P\| \leqslant 1$.

The adjoint operator P^* acts on the space $L_\infty(m)$ of equivalence classes of essentially bounded real measurable functions on X. The operator P is called *deterministic* if

(3) P^*1_A is a characteristic function for any $A \in \mathscr{B}$.

A sub-Markov operator P is called *Markov* if

(4) $P^*1 = 1$.

For more details on sub-Markov operators see, e.g., [2].

It is a routine to show that every deterministic sub-Markov operator defines a σ-endomorphism of the ring $\boldsymbol{B}$. It is also clear that this representation is one-to-one. Conversely, every σ-endomorphism F of $\boldsymbol{B}$ induces a deterministic sub-Markov operator P. This can be shown by putting $P^*1_A = 1_B$ if $F([A]) = [B]$ and by extending P^* to the whole of $L_\infty(m)$. The operator P^* so defined is indeed an adjoint of some operator P. To show this we take a u, $0 \leqslant u \in L_1(m)$, and we choose arbitrarily a representative A' in $F([A])$ for any $A \in B$. It is easily seen that the function p defined on $\mathscr{B}$ by

$$p(A) = \int_{A'} u\,dm$$

is a finite measure on $\mathscr{B}$, absolutely continuous with respect to m. Therefore, by the Radon-Nikodym theorem there exists a function v, $0 \leqslant v \in L_1(m)$, such that

$$p(A) = \int_A v\,dm.$$

The operator P taking u into v extends to a sub-Markov operator on $L_1(m)$; its adjoint is P^*.

The above discussion shows that there is a natural one-to-one correspondence between semigroups of σ-endomorphisms of $\boldsymbol{B}$ and semigroups of deterministic sub-Markov operators on $L_1(m)$. Similarly, there is a natural correspondence between groups of automorphisms of $\boldsymbol{B}$ and groups of invertible deterministic Markov operators (and also groups of automorphisms of the Banach algebra $L_\infty(m)$). In the last statement the word "deterministic" can be dropped, as it is shown by the following lemma whose proof was kindly communicated to us by Professor S. Gładysz.

LEMMA 6. *If P is invertible on $L_1(m)$ and both P and P^{-1} are sub-Markov operators, then P is a deterministic Markov operator.*

Proof. P is a Markov operator since it is isometric. Let $A \in \mathscr{B}$ and $P^*1_A = h$. It suffices to show that h is a characteristic function. For any positive integer n we have

$$1_{B_n} \leqslant nh, \qquad \text{where } B_n = \{x: h(x) \geqslant 1/n\}.$$

Thus,

$$0 \leqslant (P^*)^{-1}1_{B_n}/n \leqslant (P^*)^{-1}h = 1_A,$$

so that $(P^*)^{-1} 1_{B_n} \leqslant 1_A$ since $(P^*)^{-1} 1_{B_n} \leqslant 1$. Letting $n \to \infty$ we obtain

$$(P^*)^{-1} 1_B \leqslant 1_A, \quad \text{where } B = \{x: h(x) > 0\}.$$

Multiplying both sides of the last inequality by P^* we obtain

$$1_B \leqslant P^* 1_A = h \leqslant 1_B,$$

the last inequality in this chain being a consequence of $h \leqslant 1$.

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