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LIE GROUPS IN VARIETIES OF TOPOLOGICAL GROUPS

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1. Introduction. In [1] the authors investigated the following question:

If Ω is a class of topological groups, what topological groups are in the variety $V(\Omega)$ generated by Ω; that is, what topological groups can be "manufactured" from Ω using repeatedly the operations of taking subgroups, quotient groups and arbitrary cartesian products?

This investigation was furthered in [2] where attention was focussed on Lie groups. Our paper is a sequel to this. We seek answers to the questions:

If Ω is a class of Lie groups, can every Lie group in $V(\Omega)$ be "manufactured" from Ω without going outside the class of Lie groups? For what Lie groups G is it true that if $G \in V(\Omega)$, for some class Ω of Lie groups, then G must be isomorphic to a subgroup of a member of Ω?

We obtain some information about the first of these questions and a complete answer for the second question for the case where G is assumed compact.

2. Preliminaries. If Ω is a class of (not necessarily Hausdorff) topological groups, then $S(\Omega)$ denotes the class of all topological groups isomorphic to subgroups of members of Ω. Similarly we define the operators $\bar{S}$, Q, $\bar{Q}$, C and D where they respectively denote closed subgroup, quotient group, separated quotient group, arbitrary cartesian product and finite product.

A non-empty class Ω of topological groups is said to be a *variety* if $Q(\Omega) \subseteq \Omega$, $S(\Omega) \subseteq \Omega$ and $C(\Omega) \subseteq \Omega$. The smallest variety containing a class Ω of topological groups is said to be the *variety generated by* Ω and is denoted by $V(\Omega)$.

The basic theorem [1] on generating varieties is

THEOREM A. *If Ω is a class of topological groups and G is a Hausdorff group in $V(\Omega)$, then $G \in SC\bar{Q}\bar{S}D(\Omega)$.*

COROLLARY A [6]. *If Ω is a class of topological groups and G is a discrete group in $V(\Omega)$, then $G \in \bar{Q}\bar{S}D(\Omega)$.*

For Lie groups we have the following theorem which is essentially proved in [2]:

THEOREM B. *If Ω is a class of Lie groups and G is a Lie group in $V(\Omega)$, then G is locally isomorphic to a member of $\bar{Q}\bar{S}D(\Omega)$.*

Theorem B suggests the

QUESTION 1. If Ω is any class of Lie groups and G is a Lie group in $V(\Omega)$, does G (necessarily) belong to $\bar{Q}\bar{S}D(\Omega)$? (**P 897**)

The following result is proved in [1]:

If Ω is a class of locally compact Hausdorff groups, then the topological group R of reals is in $V(\Omega)$ if and only if $R \in S(\Omega)$.

This suggests the

QUESTION 2. What Lie groups G have the property that if Ω is any class of Lie groups such that $G \in V(\Omega)$, then $G \in S(\Omega)$? (**P 898**)

3. Results. We have

THEOREM 1. *Let Ω be a class of locally compact Hausdorff groups Then the discrete group Z of integers is in $V(\Omega)$ if and only if $Z \in \bar{S}(\Omega)$.*

Proof. Let $Z \in V(\Omega)$. Then, by Corollary A, $Z \in \bar{Q}\bar{S}D(\Omega)$, which clearly implies that $Z \in \bar{S}D(\Omega)$. Thus Z is a subgroup of $A_1 \times \ldots \times A_n$, where each $A_i \in \Omega$. For each $i \in I$, let B_i be the closure of the projection of Z on A_i. Then B_i is a monothetic locally compact abelian group which, by Theorem 9.1 of [3], implies B_i is either compact or isomorphic to Z. If each B_i were compact, then Z would be isomorphic to a subgroup of the compact group $B_1 \times \ldots \times B_n$ — which is impossible. Therefore, for some $i \in I$, B_i is isomorphic to Z. Hence $Z \in \bar{S}(\Omega)$. Noting that the statement $Z \in \bar{S}(\Omega)$ implies $Z \in V(\Omega)$ is obvious, the proof is complete.

In contrast with Theorem 1 and the similar result mentioned earlier for R we present the following

Example. Let Ω be the class of all simply connected solvable Lie groups. Then the circle group T is in $V(\Omega)$ but $T \notin S(\Omega)$.

Indeed $T \notin SD(\Omega)$. (To see this use Theorem 2.3, p. 138, of [4] and Theorem 2.5 of [2].)

In fact, using Problem 3, p. 140, of [4], we see that, if Ω is a class of solvable Lie groups, the following conditions are equivalent:

(i) $T \in S(\Omega)$.

(ii) Some compact Hausdorff (non-trivial) group is in $S(\Omega)$.

(iii) $T \in SD(\Omega)$.

(iv) Some compact Hausdorff (non-trivial) group is in $SD(\Omega)$.

(v) Some member of Ω is not simply connected.

THEOREM 2. *If Ω is a class of simply connected solvable Lie groups and G is a Lie group in $V(\Omega)$, then $G \in \bar{Q}\bar{S}D(\Omega)$.*

Proof. By Theorem B, G is locally isomorphic to a group $H \in \bar{Q}\bar{S}D(\Omega)$; that is, there are groups $A_1, \ldots, A_n$ in Ω such that $H = B/N$, where B is a closed subgroup of $A_1 \times \ldots \times A_n$. Let B_1 and N_1 be the components of the identities in B and N, respectively. Then N_1 is a connected closed normal subgroup of B_1 and $H_1 = B_1/N_1$ is locally isomorphic to H. Further, $H_1 \in \bar{Q}\bar{S}D(\Omega)$.

We now use a theorem of Malcev (see Hochschild [4], p. 135-137):

If X is a simply connected solvable Lie group and Y is a connected closed subgroup, then Y is simply connected. If Y is also a normal subgroup, then X/Y is also simply connected.

Thus in our case we see that H_1 is simply connected. Since G is locally isomorphic to H and hence also to H_1, we see that G is a quotient group of H_1. Hence $G \in \bar{Q}\bar{S}D(\Omega)$ as required.

Remark. We saw earlier that if Ω is a (non-empty) class of simply connected solvable Lie groups, then $T \in V(\Omega)$ but $T \notin SD(\Omega)$. We now see that $T \in \bar{Q}\bar{S}D(\Omega)$. Our next theorem gives a stronger result.

THEOREM 3. *Let Ω be a class of locally compact Hausdorff groups. Then the following conditions are equivalent:*

(i) *at least one member of Ω is not totally disconnected,*

(ii) $T \in V(\Omega)$,

(iii) $T \in QS(\Omega)$.

Proof. If (i) is true, then there exists a connected locally compact Hausdorff group G in $S(\Omega)$. By Section 4.13 of [5], this implies that either G is compact or G contains R. If the latter is true, then $R \in S\{G\} \subseteq S(\Omega)$ and so $T \in QS(\Omega)$. If G is compact, then G has a compact connected (non--trivial) Lie group H as a quotient. By Section 4.13 of [5], $T \in S(H)$. Thus $T \in SQ\{G\} \subseteq SQS(\Omega) = QS(\Omega)$. Hence (i) implies (iii).

Suppose (i) is false. Then every member of Ω is totally disconnected. By Theorem 7.7 of [3], every neighbourhood of the identity, in each member of Ω, contains an open subgroup. Indeed, noting that the operators Q, S and C preserve this property, we see that every member of $V(\Omega)$ has this property. Since T does not, $T \notin V(\Omega)$. That is, (ii) implies (i).

Noting that (iii) trivially implies (ii), the proof is complete.

Our next theorem strengthens Theorem 3 for abelian groups.

THEOREM 4. *Let Ω be any class of locally compact Hausdorff abelian groups. Then $T \in V(\Omega)$ if and only if $T \in Q(\Omega)$.*

Proof. By Theorem 3, $T \in V(\Omega)$ if and only if $T \in QS(\Omega)$. Now $T \in QS(\Omega)$ if and only if $Z \in SQ(\Omega^*)$, where Ω^* denotes the class of dual groups of members of Ω. By Theorem 1, $Z \in SQ(\Omega^*)$ if and only if $Z \in S(\Omega^*)$. Finally, we note that $Z \in S(\Omega^*)$ if and only if $T \in Q(\Omega)$, as required.

Remark. Theorem 4 cannot be extended to the non-abelian case. For example, if $\Omega = \{SL(2, K)\}$, then $T \notin Q(\Omega)$, since by p. 350 of [3],

$SL(2, K)$ has no non-trivial finite-dimensional unitary representations. However, Theorem 2.5 of [2] shows that $T \in V(\Omega)$.

We now turn to Question 2 of Section 2. For convenience, we say that a topological group G has *property* S if for any class Ω of Lie groups such that $G \in V(\Omega)$, we have $G \in S(\Omega)$.

LEMMA 1. *Let G be any connected locally compact group which is not a Lie group. Then G does not have property S.*

Proof. By Section 4.6 of [5], there exists a family $\{H_i: i \in I\}$ of Lie groups such that $G \in SC\{H_i: i \in I\}$. Thus $G \in V\{H_i: i \in I\}$, however $G \notin S\{H_i: i \in I\}$.

LEMMA 2. *Let G be any connected Lie group which is not simply connected. Then G does not have property S.*

Proof. Let H be the simply connected covering group of G. Then $G \in Q\{H\} \subset V\{H\}$. Noting that G and H have the same dimension, it is clear that $G \notin S\{H\}$.

LEMMA 3. *Let G be any compact connected non-simple Lie group. Then G does not have property S.*

Proof. Let $L(G)$ be the Lie algebra of G. By p. 144 of [4], $L(G) = L(H_1) \oplus L(H_2)$, where H_1 and H_2 are simply connected Lie groups, H_1 is abelian and H_2 is semisimple. Since $H_1 \oplus H_2$ is simply connected and locally isomorphic to G, we see that $G \in Q\{H_1 \oplus H_2\}$, which implies that $G \in V\{H_1, H_2\}$.

Suppose that G has property S. Then $G \in S\{H_1, H_2\}$. So $\dim G \leqslant \dim H_1$ or $\dim G \leqslant \dim H_2$. Noting that G is locally isomorphic to $H_1 \oplus H_2$ we see that $\dim G = \dim H_1 + \dim H_2$. So we have a contradiction unless H_1 or H_2 is the trivial group.

Firstly consider the case H_1 is trivial. Then G is locally isomorphic to H_2 and $G \in S\{H_2\}$. Dimension arguments show that G must be isomorphic to H_2; that is, G is a semisimple simply connected compact Lie group. Thus G is isomorphic to $A_1 \oplus A_2 \oplus \ldots \oplus A_n$, where each A_i is a compact simple simply connected Lie group. So $G \in V\{A_1, \ldots, A_n\}$, which by our supposition implies $G \in S\{A_1, \ldots, A_n\}$. By dimension arguments again, this is a contradiction unless all the A_i except one of them is the trivial group. Thus $G = A_i$ for some i; that is, G is a simple Lie group — which is a contradiction.

We are left with the case H_2 is trivial. Then G is locally isomorphic to H_1. By Lemma 2, G must be simply connected and hence G is isomorphic to H_1; that is, G is a compact abelian Lie group. Thus $G \in V\{R\}$. However, $G \notin S\{R\}$. This final contradiction shows that G does not have property S.

LEMMA 4. *Let Ω be a class of Lie groups and G a simple simply connected Lie group in $Q(\Omega)$. Then $G \in S(\Omega)$.*

Proof. There exists a continuous open homomorphism f of H onto G, where $H \in \Omega$. Let $L(G)$ and $L(H)$ be the Lie algebras of G and H, respectively. Then f induces an algebra homomorphism θ of $L(H)$ onto $L(G)$. By p. 126 of [4], there exists an algebra homomorphism φ of $L(G)$ into $L(H)$ such that $\theta\varphi$ acts identically on $L(G)$. Put $L = \varphi(L(G))$. Then there exists a connected closed subgroup A of G such that L is the Lie algebra of A. Noting that φ is an isomorphism of $L(G)$ onto L, we see that there exists a local isomorphism of G into A. Since G is simply connected, this local isomorphism can be extended to a continuous homomorphism Γ of G onto A. Clearly, $f\Gamma$ acts identically on G and thus G is isomorphic to A; that is, $G \in S(\Omega)$, as required.

LEMMA 5. *Let G be a compact simple simply connected Lie group. Let Ω be a class of Lie groups such that* (i) $G \notin S(\Omega)$ *and* (ii) *each non-simply connected Lie group locally isomorphic to G is isomorphic to a member of Ω. Then $G \in SC(\Omega)$ if and only if the intersection of all the proper non-trivial subgroups of the centre $Z(G)$ of G is the identity element e.*

Proof. Assume

$$G \leqslant \prod_{j \in J} H_j,$$

where each $H_j \in \Omega$. If p_j is the projection mapping of G onto H_j, then, clearly, we must have

$$\bigcap_{j \in J} N_j = \{e\},$$

where N_j is the kernel of the mapping p_j. Since G is compact and simple, $p_j(G)$ is either the trivial group or a connected Lie group locally isomorphic to G. Indeed, if $p_j(G)$ is not the trivial group, then $N_j \leqslant Z(G)$ for all $j \in J$. Noting that condition (i) implies $N_j \neq \{e\}$ for any $j \in J$, we infer that the intersection of all the proper non-trivial subgroups of $Z(G)$ is $\{e\}$, as required.

Conversely, let $A_1 \cap A_2 \cap \dots \cap A_n = \{e\}$, where $A_1, \dots, A_n$ are proper non-trivial subgroups of $Z(G)$. The quotient groups G/A_i $(i = 1, \dots, n)$ are in Ω and it is obvious that $G \leqslant G/A_1 \times G/A_2 \times \dots \times G/A_n$. The proof is complete.

As an immediate consequence of the proof of Lemma 5 we have

LEMMA 6. *Let G be a compact simple simply connected Lie group. If Ω is a class of Lie groups such that $G \in SC(\Omega)$, then $G \in SD(\Omega)$.*

LEMMA 7. *Let G be a compact simple simply connected Lie group. If Ω is a class of Lie groups such that $G \in V(\Omega)$, then $G \in SD(\Omega)$.*

Proof. By Theorem A, we have $G \in SC\bar{Q}\bar{S}D(\Omega)$. By Lemma 6, then, $G \in SDQSD(\Omega) \subseteq SQDSD(\Omega) \subseteq QSDSD(\Omega) \subseteq QSSDD(\Omega) = QSD(\Omega)$. Now, using Lemma 4, we see that $G \in SSD(\Omega) = SD(\Omega)$.

THEOREM 5. *Let G be a compact connected Hausdorff group. Then G has property S if and only if G is a simple simply connected Lie group with the property that the intersection of all the proper non-trivial subgroups of $Z(G)$ is not $\{e\}$.*

Proof. If G has property S, then Lemmas 1, 2 and 3 imply that G is a simple simply connected Lie group. Let Ω be the class of all non--simply connected Lie groups locally isomorphic to G. Then $G \notin S(\Omega)$. However, if the intersection of all the proper non-trivial subgroups of $Z(G)$ is $\{e\}$, then Lemma 5 implies that G belongs to $SC(\Omega)$ and hence also to $V(\Omega)$. So G does not have property S.

Now let G be a compact simple simply connected Lie group having the intersection of all proper non-trivial subgroups of $Z(G)$ not equal to $\{e\}$. If Γ is any class of Lie groups such that $G \in V(\Gamma)$, then Lemma 7 implies $G \in SD(\Gamma) \subseteq SC(\Gamma)$. Now, Lemma 5 implies that $G \in S(\Gamma)$, as required.

Remark. We note that the compact simple simply connected Lie groups having the property that the intersection of all proper non-trivial subgroups of $Z(G)$ is *not* $\{e\}$ are precisely those with the following Lie algebras (see p. 504-506 of [7] and [8]):

(i) A_n, $n+1$ a prime power.
(ii) B_n, $n \geqslant 2$.
(iii) C_n, $n \geqslant 3$.
(iv) D_n, $n \geqslant 4$ and n an odd prime power.
(v) G_2.
(vi) F_4.
(vii) E_6.
(viii) E_7.
(ix) E_8.

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