COLLOQUIUM MATHEMATICUM

VOL. LVI 1988 FASC. 2

MYCIELSKI–SIERPIŃSKI CONJECTURE AND KOREC–ZNÁM RESULT*

BY

MARC A. BERGER, ALEXANDER FELZENBAUM AND AVIEZRI S. FRAENKEL (REHOVOT)

1. Introduction. We concern ourselves primarily with partitions of a finite group into cosets. Precisely, let

$$\{a_1 S_1, \ldots, a_t S_t\} \tag{1.1}$$

be a partition of G into cosets of the subgroups $S_1, \ldots, S_t$. Our interest here is in obtaining lower bounds for t. In order to describe the known results we introduce a number-theoretic function. For any natural number k with prime factorization

$$k = \prod_{j=1}^{l} p_j^{\alpha_j} \tag{1.2}$$

define

$$f(k) = 1 + \sum_{j=1}^{l} \alpha_j (p_j - 1). \tag{1.3}$$

Mycielski and Sierpiński [4] conjectured that if G is Abelian, then necessarily

$$t \geqslant \max_{1 \leqslant i \leqslant t} f([G:S_i]). \tag{1.4}$$

In the special case where G is cyclic, Znám conjectured that

$$t \geqslant f([G: \bigcap_{i=1}^{t} S_i]). \tag{1.5}$$

(Observe that (1.5) is stronger than (1.4).) This case – G being cyclic – corresponds to disjoint covering systems of the integers into residue classes.

* This research was supported by grant No. 85-00368 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

Here we can always choose G in such a way that

$$\bigcap_{i=1}^{t} S_i = \{e\}.$$

Korec [3] proved that in general if (1.1) is a partition of G and if the subgroups $S_1, \ldots, S_t$ are all normal in G, then (1.5) holds. He also observed that in fact under this normality assumption

$$t \geqslant 1 + \sum_{i=1}^{m} ([H_{i-1}:H_i] - 1), \tag{1.6}$$

where

$$G = H_0 \supset H_1 \supset \ldots \supset H_m = \bigcap_{i=1}^{t} S_i$$

is a composition series from $\bigcap_{i=1}^{t} S_i$ to G. (It follows from the Jordan–Hölder Theorem that this lower bound is independent of the composition series. If G is solvable, then this bound (1.6) coincides with (1.5) – otherwise it is sharper.) Finally, Korec also shows that if G is infinite and (1.1) is a partition of G into cosets of normal subgroups (with $t < \infty$), then necessarily

$$[G: \bigcap_{i=1}^{t} S_i] < \infty$$

and (1.5), (1.6) hold.

To continue our description of known results we introduce the notion of a minimal coset cover. If (1.1) is a covering of G by means of its cosets, we say it is *minimal* if the removal of any one coset $a_i S_i$ would result in G not being covered by the remaining cosets $a_j S_j$. Znám [5] showed that if (1.1) is a minimal coset cover of a cyclic group G, then (1.4) holds. Finally, in [2] Berger et al. extended Znám's lower bound from (1.4) to (1.5) (under the same conditions – that (1.1) be a minimal coset cover of a cyclic group G).

Example 1.A. This exemplifies a minimal coset cover of an Abelian group for which (1.5) fails. Take $G = \sigma_3 \times \sigma_3$, and set

$$S_1 = \{(0,0), (0,1), (0,2)\}, \quad S_2 = \{(0,0), (1,0), (2,0)\},$$

$$S_3 = \{(0,0), (1,1), (2,2)\}, \quad S_4 = \{(0,0), (1,2), (2,1)\}.$$

Then $\{S_1, S_2, S_3, S_4\}$ is a minimal coset cover of G, but

$$f([G: \bigcap_{i=1}^{4} S_i]) = f(9) = 5.$$

Example 1.B. This is due to Korec [3]. It exemplifies a coset partition of a group for which (1.5) fails. Take $G = \mathscr{S}_3$ (the permutation group on 3 elements), and set

$$S_1 = \{e, (12)\}, \quad a_1 = e,$$
$$S_2 = \{e, (13)\}, \quad a_2 = (23),$$
$$S_3 = \{e, (23)\}, \quad a_3 = (13).$$

Then $\{a_1 S_1, a_2 S_2, a_3 S_3\}$ is a coset partition of G, but

$$f([G: \bigcap_{i=1}^{3} S_i]) = f(6) = 4.$$

This paper contains three parts. In Section 2 we prove the Mycielski–Sierpiński bound (1.4) for any coset partition (1.1) of a finite solvable group G. In Section 3 we prove the bound (1.5) for a minimal coset cover of G assuming that either G is cyclic or else that G is of square-free order and the subgroups S_i are normal. The cyclic group case is really our result (see [2], 3.II), but we prove it here using different techniques altogether. Finally, in Section 4 we use the techniques of Section 3 to provide an alternative proof of Corollary 2.IV(a) of [2], involving minimal product set covers.

2. The Mycielski–Sierpiński bound. Our main result in this section is the Mycielski–Sierpiński bound (1.4) for coset partitions (1.1) of a finite *solvable* group G.

THEOREM 2.I. *Let G be a finite solvable group and let* (1.1) *be a partition of it into some cosets. Then* (1.4) *holds.*

The proof relies on the following

LEMMA 2.II. *Let G be a finite group and let K, S be subgroups of G. Assume that either* (a) *or* (b) *holds*:

(a) *K is a maximal subgroup of G and at least one of K, S is a normal subgroup of G*;

(b) *K is a maximal normal subgroup of G and S is a normal subgroup of G.*

Then either $S \subset K$ or else $S \cap bK$ is a coset of $S \cap K$ for every $b \in G$.

Proof. Let $b_1 K, \ldots, b_m K$ be the distinct cosets of K, ordered so that the first r of them are the ones that intersect S. Then

$$S \cdot K = \bigcup_{i=1}^{r} b_i K. \tag{2.1}$$

Observe that

$$K \subset S \cdot K \subset G. \tag{2.2}$$

(a) Since K or S is normal, $S \cdot K$ is a subgroup of G, and since K is maximal, we conclude from (2.2) that either $S \cdot K = K$ or $S \cdot K = G$. In the first case $S \subset K$; and in the second case, by (2.1), $r = m$. This is our desired conclusion.

(b) Since S and K are both normal, $S \cdot K$ is a normal subgroup of G, and since K is maximal normal, we conclude as above that either $S \subset K$ or $r = m$.

Remark. In words, Lemma 2.II demonstrates that any coset aS of G either lies entirely within a single coset of K or else intersects every coset of K by exactly $|S|/m$ elements, where m is the index of K. Of course, if $m \nmid |S|$, then necessarily the first possibility holds. From this lemma it follows at once that if (1.1) is a coset partition of G and if $S_1 \subset K$, then each coset $b_i K$ entirely contains (at least) one of the cosets from (1.1). This result is due to Korec ([3], Lemma VI). (Although Korec assumes that the S_i are all normal subgroups, this is not necessary when K is both maximal and normal.)

Proof of Theorem 2.I. We use induction on $|G|$. If $|G| = 1$, then necessarily $t = 1$ and $S_1 = G$. Thus (1.4) holds. For the induction step we let K be a normal subgroup of G with prime index p, which exists by virtue of the solvability of G. It suffices to show that $t \geqslant f([G:S_1])$, where S_1 is the first subgroup in (1.1). By multiplying through (1.1) with a_1^{-1} we can assume that $a_1 = e$. Order the cosets in (1.1) so that the first t' of them intersect K. The induced coset partition of K

$$\{a_i S_i \cap K\colon\ 1 \leqslant i \leqslant t'\}$$

satisfies the induction hypothesis since K is solvable, and thus

$$t' \geqslant f([K:S_1 \cap K]).$$

According to Lemma 2.II we need to consider two cases.

Case A. $S_1 \subset K$. It follows from the Remark after Lemma 2.II that each coset of K entirely contains some coset $a_i S_i$. Thus $t - t' \geqslant p-1$, and

$$t \geqslant p-1+f([K:S_1]) = f([G:S_1]).$$

Case B. $|S_1 \cap K| = |S_1|/p$. Then $[K:S_1 \cap K] = [G:S_1]$, and thus

$$t \geqslant t' \geqslant f([G:S_1]).$$

3. Bounds for minimal coset covers. Let (1.1) be a minimal coset cover of G and denote by E_i the set of elements in G covered exclusively by $a_i S_i$. The minimality condition, then, amounts to assuming $E_i \neq \emptyset$, $1 \leqslant i \leqslant t$. To conserve notation we may as well assume that

$$a_i \in E_i, \quad 1 \leqslant i \leqslant t. \tag{3.1}$$

THEOREM 3.I. *If G is either a cyclic group or else of square-free order, and if the subgroups S_i in the minimal coset cover* (1.1) *are all normal, then* (1.5) *holds.*

A key step in proving Theorem 3.I is to show that if K is a maximal subgroup of G which is normal in G, and if $S_1 \subset K$, then every coset of K entirely contains (at least) one of the cosets from (1.1). This is always the case where (1.1) is a coset partition of G, as mentioned in the Remark following Lemma 2.II. It is not true in general, though. In Example 1.A consider $K = S_1$. Neither if the cosets $(1, 0)+K$, $(2, 0)+K$ contain any of the subgroups S_2, S_3, S_4.

LEMMA 3.II. *Let G be either a cyclic group or else of square-free order. Let K be a maximal subgroup of G which is normal. Let* (1.1) *be a minimal coset cover of G and assume that the subgroups S_i are all normal. If $S_1 \subset K$, then every coset of K entirely contains* (*at least*) *one of the cosets from* (1.1).

Proof. Assume, by multiplying (1.1) through with a_1^{-1}, that $a_1 = e \in E_1$. Let $p \mid [G:K]$ and let L be a p-Sylow-subgroup of G. Since $L \not\subset K$, it follows from Lemma 2.II that L intersects every coset of K. Let bK, $b \notin K$, be any one of these cosets. Choose $c \in L \cap bK$ and let $a_i S_i$ be a coset from (1.1) containing c. Since $S_1 \subset K$, we have $i \neq 1$. We first show that

$$|L| \nmid |S_i|. \tag{3.2}$$

Otherwise, if $|L| \mid |S_i|$, then S_i, being normal in G, would contain every p-Sylow-subgroup, including L. This would imply that

$$L = cL \subset cS_i = a_i S_i, \tag{3.3}$$

and we would arrive at the conclusion $e \in a_i S_i$, conflicting $e \in E_1$.

If G is cyclic, then $|K| = |G|/p$. Thus

$$|L| \nmid |S_i| \Rightarrow |S_i| \mid |K| \Rightarrow S_i \subset K \Rightarrow a_i S_i \subset bK.$$

If G is of square-free order, then $|L| = p$, and by the Remark following Lemma 2.II we see that

$$p \nmid |S_i| \Rightarrow S_i \subset K \Rightarrow a_i S_i \subset bK.$$

Proof of Theorem 3.I. We use induction on $|G|$. Again the case $|G| = 1$ is immediate, so we proceed to the induction step. Assume as above that $a_1 = e \in E_1$. If $S_1 = G$, then $t = 1$ and we are done. Otherwise, if $S_1 \neq G$, let K be a normal subgroup of G with prime index $[G:K] = p$, containing S_1. It exists since G is solvable and S_1 is normal. Order the cosets in (1.1) so that

$$\{a_i S_i \cap K \colon 1 \leqslant i \leqslant t_1\}$$

is a minimal coset cover of K. Since K is either cyclic or of square-free order, and each $S_i \cap K$ is normal in K, we can use the induction hypothesis to arrive at the bound

$$t_1 \geqslant f([K:K_1]), \tag{3.4}$$

where

$$K_1 = \bigcap_{i=1}^{t_1} S_i.$$

We now define numbers $t_1 < t_2 < \ldots < t_N$ and subgroups

$$K_n = \bigcap_{i=1}^{t_n} S_i, \quad 1 \leqslant n \leqslant N, \tag{3.5}$$

inductively as follows. Suppose we are at stage n, having defined $t_1, \ldots, t_n$. If

$$K_n = \bigcap_{i=1}^{t_n} S_i,$$

we set $N = n$ and stop. Otherwise we let S_k be such that $S_k \not\supset K_n$ and consider the coset $a_k K_n$. Since $a_k \in E_k$ and $K_n \subset S_i$, $1 \leqslant i \leqslant t_n$, it follows that

$$a_k K_n \cap a_i S_i = \emptyset, \quad 1 \leqslant i \leqslant t_n. \tag{3.6}$$

Therefore we can order the cosets in (1.1) so that $k = t_n + 1$ and

$$\{a_i S_i \cap a_k K_n\colon\ t_n + 1 \leqslant i \leqslant t_{n+1}\}$$

is a minimal coset cover of $a_k K_n$. By multiplying through with a_k^{-1} we arrive at a minimal coset cover of K_n, and thus we conclude from the induction hypothesis that

$$t_{n+1} - t_n \geqslant f([K_n : K_{n+1}]). \tag{3.7}$$

Since the sets K_n are reduced at each stage, this process terminates at some stage N where

$$K_N = \bigcap_{i=1}^{t_N} S_i.$$

Each subgroup K_i lies inside K, so it follows from Lemma 3.II that

$$t - t_N \geqslant p - N. \tag{3.8}$$

Thus, putting $K_0 = K$, we have

$$t \geqslant p - N + \sum_{n=1}^{N} f([K_{n-1} : K_n]) = f([G : \bigcap_{i=1}^{t} S_i]). \tag{3.9}$$

Since the conclusion of Lemma 3.II holds for any group G when (1.1) is a coset partition, we can use the same proof above to prove Korec's theorem.

THEOREM 3.III. *Let G be any finite group, let* (1.1) *be a coset partition of G and let the subgroups S_i all be normal. Then* (1.6) *holds.*

A seemingly different application of the proof above is given in the next section.

4. Product set covers. A *product set* P in Z^n is any finite nonempty set of the form

$$P = P_1 \times \ldots \times P_n, \tag{4.1}$$

where $P_1, \ldots, P_n \subset Z$. The set P_i is referred to as the i-th *projection* of P, denoted by

$$P_i = \pi_i(P), \quad 1 \leqslant i \leqslant n. \tag{4.2}$$

We define the *dimension* of P as

$$\dim(P) = \big|\{i\colon\ |\pi_i(P)| \geqslant 2\}\big|. \tag{4.3}$$

If $R \subset P$ is also a product set, we write

$$\operatorname{index}(R\colon P) = \{i\colon\ \pi_i(R) = \pi_i(P) \text{ and } |\pi_i(R)| \geqslant 2\}. \tag{4.4}$$

We will be working with a fixed product set P in what follows, and when referring specifically to it we shall simply denote this index set by $\operatorname{index}(R)$. The product set $R \subset P$ is called a *cell* of P if $|\pi_i(R)| = 1$ whenever $i \notin \operatorname{index}(R)$. A one-dimensional cell is also called a *line*.

Let $R \subset P$ be a product set and let $C \subset P$ be a cell. If $R \cap C \neq \emptyset$, then

$$\operatorname{index}(R \cap C\colon C) = \operatorname{index}(R) \cap \operatorname{index}(C). \tag{4.5}$$

Furthermore, if $\operatorname{index}(C) \subset \operatorname{index}(R)$, then

$$R \cap C \neq \emptyset \Rightarrow C \subset R. \tag{4.6}$$

A *product set cover* of P is a finite family of product sets $\{R_1, \ldots, R_t\}$, each a subset of P, which covers P. For such a cover we denote by E_i the set of points in P covered exclusively by R_i, and we say the cover is *minimal* if $E_i \neq \emptyset$, $1 \leqslant i \leqslant t$.

THEOREM 4.I. *Let $\{R_1, \ldots, R_t\}$ be a minimal product set cover of the product set P. Then*

$$t \geqslant \dim(P) - \Big|\bigcap_{i=1}^{t} \operatorname{index}(R_i)\Big| + 1. \tag{4.7}$$

Proof. Choose points $a_i \in E_i$, $1 \leqslant i \leqslant t$. We use induction on $d = \dim(P)$. If $d = 1$, then this theorem follows from the observation that either $t \geqslant 2$ or else $t = 1$ and $|\operatorname{index}(R_1)| = d = 1$. We proceed now to the induction step. If $R_1 = P$, then $t = 1$, $|\operatorname{index}(R_1)| = \dim(P)$ and we are done. Otherwise, if $R_1 \neq P$, let C be a $(d-1)$-dimensional cell of P containing a_1 for which $\operatorname{index}(R_1) \subset \operatorname{index}(C)$. Order the sets R_i so that

$$\{R_i \cap C\colon\ 1 \leqslant i \leqslant t_1\}$$

is a minimal product set cover of C. Set

$$I_1 = \bigcap_{i=1}^{t_1} \operatorname{index}(R_i).$$

By the induction hypothesis, using (4.5) we have

$$t_1 \geqslant (d-1) - \left|\bigcap_{i=1}^{t_1} \operatorname{index}(R_i \cap C{:}C)\right| + 1 = (d-1) - |I_1| + 1. \tag{4.8}$$

Let L be the line containing a_1 with index complementing index(C):

$$\operatorname{index}(L) = \operatorname{index}(P) \setminus \operatorname{index}(C). \tag{4.9}$$

Any product set $R_i \neq R_1$ intersecting L at a point other than a_1 cannot intersect C – otherwise it would intersect a_1. Since $L \not\subset R_1$, this shows that

$$t - t_1 \geqslant 1. \tag{4.10}$$

If I_1 were equal to $\bigcap_{i=1}^{t} \operatorname{index}(R_i)$, then our result (4.7) would follow now from (4.8) and (4.10).

We now inductively define numbers $t_1 < t_2 < \ldots < t_N$ and cells $C_1, \ldots, C_N$ with

$$\operatorname{index}(C_n) = I_n = \bigcap_{i=1}^{t_n} \operatorname{index}(R_i), \qquad 1 \leqslant n \leqslant N-1. \tag{4.11}$$

Suppose we are at stage n, having defined $t_1, \ldots, t_n$ and $C_1, \ldots, C_{n-1}$. If

$$I_n = \bigcap_{i=1}^{t} \operatorname{index}(R_i),$$

we set $N = n$ and stop. Otherwise, let R_k be such that index$(R_k) \not\supset I_n$ and let C_n be the cell containing a_k with index$(C_n) = I_n$. Since $a_k \in E_k$, it follows from (4.6) that

$$R_i \cap C_n = \emptyset, \qquad 1 \leqslant i \leqslant t_n. \tag{4.12}$$

Therefore, we can order the product sets R_i so that $k = t_n + 1$ and

$$\{R_i \cap C_n\colon\ t_n + 1 \leqslant i \leqslant t_{n+1}\}$$

is a minimal product set cover of C_n. Thus we conclude from the induction hypothesis, using (4.5), that

$$\begin{aligned} t_{n+1} - t_n &\geqslant \dim(C_n) - \left|\bigcap_{i=t_n+1}^{t_{n+1}} \operatorname{index}(R_i \cap C_n{:}C_n)\right| + 1 \\ &= |I_n| - |I_{n+1}| + 1. \end{aligned} \tag{4.13}$$

Eventually, at stage N,

$$I_N = \bigcap_{i=1}^{t} \operatorname{index}(R_i).$$

Thus, using (4.8) and (4.13), we get

$$t \geqslant t_N = \sum_{n=1}^{N-1} (t_{n+1} - t_n) + t_1 \geqslant (d-1) - |I_N| + N. \tag{4.14}$$

This establishes (4.7) when $N \geqslant 2$. The case $N = 1$ was already handled above.

REFERENCES

[1] M. A. Berger, A. Felzenbaum and A. S. Fraenkel, *New results for covering systems of residue sets*, Bull. Amer. Math. Soc. 14 (1986), pp. 121–125.

[2] – *Covers of product sets and the Korec–Znám result*, European J. Combin., in press.

[3] I. Korec, *On a generalization of Mycielski's and Znám's conjectures about coset decomposition of Abelian groups*, Fund. Math. 85 (1974), pp. 41–47.

[4] J. Mycielski and W. Sierpiński, *Sur une propriété des ensembles linéaires*, ibidem 58 (1966), pp. 143–147.

[5] Š. Znám, *On properties of systems of arithmetic sequences*, Acta Arith. 26 (1974–1975), pp. 279–283.

FACULTY OF MATHEMATICAL SCIENCES
THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT 76100, ISRAEL

Reçu par la Rédaction le 17.7.1986;
en version modifiée le 23.3.1987