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MULTIPLIERS ON WEIGHTED HARDY SPACES OVER LOCALLY COMPACT VILENKIN GROUPS, II

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1. Introduction. In [7] L. Saloff-Coste introduced a new class of operators T_σ on a locally compact Vilenkin group G, where T_σ is determined by a function $\sigma : G \times \Gamma \to \mathbf{C}$ (Γ is the dual group of G). The operators T_σ are the natural analogue on G of the pseudodifferential operators on $\mathbf{R}^n$ belonging to the Hörmander class $S^m_{\rho,\delta}$. The main result in [7] is Theorem VI.7 in which Saloff-Coste proves that the operators T_σ are continuous from $L^p(G)$ to $L^p(G)$, $1 < p < \infty$, when $\sigma \in S^m_{\rho,\delta}(G)$, with $0 \le \rho < \delta \le 1$ or $0 < \rho = \delta < 1$ and $-m \ge (1-\rho)|1/p - 1/2|$. This theorem is the analogue on G of various results for pseudodifferential operators on $\mathbf{R}^n$ that are due to Calderón–Vaillancourt [1], C. Fefferman [2] and Päivärinta and Somersalo [6].

In this paper we continue the work of Saloff-Coste. We consider here pseudodifferential operators T_σ on G where the function σ is a function of only one variable, $\sigma : \Gamma \to \mathbf{C}$. In this case the continuity of T_σ is equivalent to the fact that σ is a multiplier. In Section 2 we discuss the continuity of the operators T_σ on the Hardy spaces $H^p(G)$, $0 < p \le 1$. In Section 3 we consider the continuity of T_σ on certain power-weighted Hardy spaces $H^p_\alpha(G)$. Our main result in this section is obtained by means of an application of a multiplier theorem for $H^p_\alpha(G)$ spaces obtained earlier by the present authors.

In order to state our results more precisely we first introduce some notation; by and large, we shall use the same notation as in [4] or [5] and the reader is referred especially to [5] for more details. Thus, G will denote a locally compact Vilenkin group, that is, G is a locally compact Abelian topological group containing a strictly decreasing sequence of compact open subgroups $(G_n)_{n=-\infty}^{\infty}$ so that

(i) $\sup\{\operatorname{order} G_n/G_{n+1} : n \in \mathbf{Z}\} < \infty$,
(ii) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$.

The Haar measure on G is denoted by μ and μ is normalized so that

$\mu(G_0) = 1$. We set $\mu(G_n) = (m_n)^{-1}$. Let $|0| = 0$ and for $x \in G_n \setminus G_{n+1}$, let $|x| = (m_n)^{-1}$. The dual group of G is denoted by Γ and we set

$$\Gamma_n = \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

The Haar measure on Γ, denoted by λ, is chosen so that $\lambda(\Gamma_0) = 1$. Then $\lambda(\Gamma_n) = (\mu(G_n))^{-1} = m_n$ for all $n \in \mathbf{Z}$. Let $|\gamma| = m_n$ if $\gamma \in \Gamma_{n+1} \setminus \Gamma_n$ and let $|\gamma_0| = 0$, where $\gamma_0(x) = 1$ for all $x \in G$; furthermore, we set $\langle\gamma\rangle = \max\{1, |\gamma|\}$.

The Herz spaces $K(\alpha, p, q)$ and the generalized Lipschitz spaces $\Lambda(\alpha, p, q)$ are defined in [4] or [5]. The following result will be used later on (see [4, Theorem 1*]).

THEOREM 0. *Let $\alpha \in \mathbf{R}$, $1 \le p \le 2$ and $0 < q \le \infty$. Then $(\Lambda(\alpha, p, q; \Gamma))^\vee \hookrightarrow K(\alpha, p', q; G)$. Moreover, for every $f \in \Lambda(\alpha, p, q; \Gamma)$ we have*

$$\begin{aligned}\|f^\vee\|_{K(\alpha,p',q;G)} &= \Big(\sum_{l=-\infty}^{\infty} ((m_l)^{-\alpha}\|f^\vee \chi_{G_l\setminus G_{l+1}}\|_{p'})^q\Big)^{1/q} \\ &\le C\Big(\sum_{l=-\infty}^{\infty} (m_l)^{-\alpha q} \sup\{\|\tau_\xi f - f\|_p^q : \xi \in \Gamma_l\}\Big)^{1/q},\end{aligned}$$

where $\tau_\xi f(\gamma) = f(\gamma - \xi)$.

We now define the constant coefficient pseudodifferential operators on locally compact Vilenkin groups. This definition is the one variable version of Definition II.4 in [7].

DEFINITION 1. Let $m \in \mathbf{R}$ and $\rho \ge 0$. A function $\sigma : \Gamma \to \mathbf{C}$ belongs to S_ρ^m if $\sigma \in L^\infty(\Gamma)$ and

(i) there exists a $C > 0$ so that for all $\gamma \in \Gamma$, $|\sigma(\gamma)| \le C(\langle\gamma\rangle)^m$,

(ii) for each $\kappa > 0$ there exists a $C_\kappa > 0$ so that for $\xi, \gamma \in \Gamma$ with $|\xi| < \langle\gamma\rangle$ we have

$$|\sigma(\gamma - \xi) - \sigma(\gamma)| \le C_\kappa |\xi|^\kappa (\langle\gamma\rangle)^{m-\rho\kappa}.$$

For $\sigma \in S_\rho^m$ we define the operator T_σ (formally) by

$$T_\sigma f(x) = \int_\Gamma \sigma(\gamma)\hat{f}(\gamma)\gamma(x)\,d\lambda(\gamma) = (\sigma\hat{f})^\vee(x).$$

Thus, formally, T_σ is a Fourier multiplier operator and σ is a Fourier multiplier.

2. Multipliers on Hardy spaces. In this section we prove a theorem which is a version on G of Theorem 1.1 in [6]. Although in [5] we characterized the Hardy spaces on G only in terms of (p, ∞) atoms, these spaces can also be characterized in terms of $(p,2)$ atoms; we shall use this fact without further elaboration.

THEOREM 1. *Let* $0 < p \le 1$ *and* $0 \le \rho < 1$. *If* $\sigma \in S_\rho^{-\mu}$ *for some* $\mu \ge (1-\rho)(1/p - 1/2)$, *then* σ *is a Fourier multiplier on* H^p, *that is,* $\sigma \in \mathcal{M}(H^p)$.

Proof. We shall assume that $\mu = (1-\rho)(1/p-1/2)$. Let a be a $(p,2)$ atom such that $\operatorname{supp} a \subset G_n$ for some $n \in \mathbf{Z}$. This implies that $\widehat{a}$ is constant on the cosets of Γ_n in Γ so that $\widehat{a}(\gamma) = 0$ for all $\gamma \in \Gamma_n$. Next let $\sigma = \sigma\chi_{\Gamma_n} + \sigma\chi_{\Gamma\setminus\Gamma_n} = \sigma_1 + \sigma_2$. Then $\|(\sigma\widehat{a})^\vee\|_2 = \|\sigma\widehat{a}\|_2 = \|\sigma_2\widehat{a}\|_2$. We now distinguish two cases: $n \ge 0$ and $n < 0$.

(i) Assume $n \ge 0$. If $\gamma \notin \Gamma_n$, then $\langle\gamma\rangle = |\gamma| \ge m_n$, so that

$$\|(\sigma\widehat{a})^\vee\|_2 \le \|\sigma_2\|_\infty \|a\|_2 \le C(m_n)^{-\mu}(m_n)^{1/p-1/2} = C(m_n)^{\rho(1/p-1/2)}.$$

Next, consider $\|\,|\cdot|^b(\sigma_2\widehat{a})^\vee\|_2$ for some $b > 0$. We have, according to Theorem O,

$$\begin{aligned}\|\,|\cdot|^b(\sigma_2\widehat{a})^\vee\|_2 &= \|(\sigma_2\widehat{a})^\vee\|_{K(b,2,2;G)}\\ &\le \Big(\sum_{l=-\infty}^{\infty}(m_l)^{-2b}\sup\{\|\tau_\xi(\sigma_2\widehat{a}) - \sigma_2\widehat{a}\|_2^2 : \xi \in \Gamma_l\}\Big)^{1/2}\\ &= \Big(\sum_{l=-\infty}^{[n\rho]}\cdots + \sum_{l=[n\rho]+1}^{\infty}\cdots\Big)^{1/2} = (A_1 + A_2)^{1/2},\end{aligned}$$

where $[n\rho]$ is defined as in [7, Section I.5], that is, if $\rho = 0$ then $[n\rho] = 0$ for all $n \ge 0$, whereas for $0 < \rho < 1$ we require that

$$\mu(G_{[n\rho]+1}) \le (\mu(G_n))^\rho < \mu(G_{[n\rho]}).$$

Note that $n \ge [n\rho]$ for $n \ge 0$. Also, if $l \le [n\rho]$ and $\xi \in \Gamma_l$ then $\xi \in \Gamma_n$ so that for every $\gamma \in \Gamma$ the elements $\gamma - \xi$ and γ belong to the same coset of Γ_n. Consequently,

$$(\sigma_2\widehat{a})(\gamma-\xi) - (\sigma_2\widehat{a})(\gamma) = (\sigma_2(\gamma-\xi) - \sigma_2(\gamma))\widehat{a}(\gamma).$$

Thus, for each of the terms in the sum in A_1 and any $\kappa > b$ we have

$$\begin{aligned}&\sup\{\|\tau_\xi(\sigma_2\widehat{a}) - \sigma_2\widehat{a}\|_2^2 : \xi \in \Gamma_l\}\\ &\quad= \sup\Big\{\int_{\Gamma\setminus\Gamma_n}|\sigma_2(\gamma-\xi) - \sigma_2(\gamma)|^2|\widehat{a}(\gamma)|^2\,d\lambda(\gamma) : \xi \in \Gamma_l\Big\}\\ &\quad\le C_\kappa^2(m_l)^{2\kappa}(m_n)^{-2(\mu+\rho\kappa)}\|\widehat{a}\|_2^2 \le C_\kappa^2(m_l)^{2\kappa}(m_n)^{2\rho(1/p-1/2-\kappa)}.\end{aligned}$$

Therefore,

$$A_1 \le C_\kappa^2(m_n)^{2\rho(1/p-1/2-\kappa)}\sum_{l=-\infty}^{[n\rho]}(m_l)^{-2(b-\kappa)} \le C_\kappa^2(m_n)^{2\rho(1/p-1/2-b)}.$$

To estimate A_2 we observe that, since $|\sigma_2(\gamma)| \le (\langle\gamma\rangle)^{-\mu}$,

$$A_2 \le 4 \sum_{l=[n\rho]+1}^{\infty} (m_l)^{-2b} \|\sigma_2 \widehat{a}\|_2^2$$

$$\le C \sum_{l=[n\rho]+1}^{\infty} (m_l)^{-2b} (m_n)^{2\rho(1/p-1/2)} \le C(m_n)^{2\rho(1/p-1/2-b)}.$$

Thus we see that if $b > 0$ then

$$\| |\cdot|^b (\sigma\widehat{a})^\vee \|_2 \le C(m_n)^{\rho(1/p-1/2-b)}.$$

(ii) If $n < 0$, then

$$\|(\sigma\widehat{a})^\vee\|_2 \le \|\sigma\|_\infty \|a\|_2 \le C(m_n)^{1/p-1/2}.$$

Also, an argument like that used in (i) shows that for $b > 0$,

$$\| |\cdot|^b (\sigma\widehat{a})^\vee \|_2 \le C(m_n)^{1/p-1/2-b}.$$

Thus in both cases we see that if we choose $b = 2(1/p - 1/2)$ then

$$\|(\sigma\widehat{a})^\vee\|_2^{1/2} \| |\cdot|^b (\sigma\widehat{a})^\vee \|_2^{1/2} \le C.$$

Furthermore, it follows easily from the inequalities just proved that $(\sigma\widehat{a})^\vee \in L^1(G)$. Therefore, we have

$$\int_G (\sigma\widehat{a})^\vee(x)\, d\mu(x) = (\sigma\widehat{a})(\gamma_0) = 0.$$

Hence (see [3, Definition III.7.13]) $(\sigma\widehat{a})^\vee$ is a $(p, 2, b)$ molecule centered at $0 \in G$ for $b = 2(1/p - 1/2)$. We shall from here on denote $(\sigma\widehat{a})^\vee(x)$ by $M(x)$ and we shall now show that $M \in H^p$ by proving that M can be decomposed in a suitable sum of $(p, 2)$ atoms. Like before, we shall give a detailed proof only for the case $n \ge 0$; the case $n < 0$ is similar but easier. First we define sets B_k and D_k by $B_{-1} = \emptyset$ and, for $k \ge 0$, $B_k = G_{[n\rho]-k}$ and $D_k = B_k \setminus B_{k-1}$. Next we define functions M_k, $k \ge 0$, by

$$M_k(x) = (\mu(D_k))^{-1} \Big(\int_{D_k} M(t)\, d\mu(t) \Big) \chi_{D_k}(x).$$

Then

$$\|M_0\|_2^2 \le \|M\|_2 \le C(m_n)^{\rho(1/p-1/2)}$$

and, for $k \ge 1$,

$$\|M_k\|_2 \le (m_{[n\rho]-k})^b (m_{[n\rho]-k})^{-b} \Big(\int_{D_k} |M(x)|^2\, d\mu(x) \Big)^{1/2}$$

$$\le (\mu(B_k))^{-b} \Big(\int_{D_k} |x|^{2b} |M(x)|^2\, d\mu(x) \Big)^{1/2}$$

$$\leq C(\mu(B_k))^{-b}(m_n)^{\rho(1/p-1/2-b)}.$$

Now we decompose $M(x)$ as follows:

$$\begin{aligned}
M(x) &= \sum_{k=0}^{\infty}(M(x)-M_k(x))\chi_{D_k}(x) + \sum_{k=0}^{\infty} M_k(x)\chi_{D_k}(x) \\
&= \sum_{k=0}^{\infty}(M(x)-M_k(x))\chi_{D_k}(x) \\
&\quad + \sum_{k=1}^{\infty}\Big(\int_{G\setminus B_{k-1}} M(t)\,d\mu(t)\Big)\Big(\frac{\chi_{D_k}(x)}{\mu(D_k)} - \frac{\chi_{D_{k-1}}(x)}{\mu(D_{k-1})}\Big) \\
&= \sum_{k=0}^{\infty} a_k(x) + \sum_{k=1}^{\infty} b_k(x).
\end{aligned}$$

We have

$$\|a_0\|_2 \leq 2C(m_n)^{\rho(1/p-1/2)} \leq C(\mu(B_0))^{-b}(m_n)^{\rho(1/p-1/2-b)}$$

and, for $k \geq 1$,

$$\|a_k\|_2 \leq 2\Big(\int_{D_k} |M(x)|^2\,d\mu(x)\Big)^{1/2} \leq C(\mu(B_k))^{-b}(m_n)^{\rho(1/p-1/2-b)}.$$

Thus, if $a_k^*(x) = (\|a_k\|_2^{-1})a_k(x)(\mu(B_k))^{-(1/p-1/2)}$ then each a_k^* is a $(p,2)$ atom and $\sum_{k=0}^{\infty} a_k(x) = \sum_{k=0}^{\infty} \lambda_k a_k^*(x)$, where

$$\begin{aligned}
\sum_{k=0}^{\infty} |\lambda_k|^p &\leq C\sum_{k=0}^{\infty}((m_n)^{\rho(1/p-1/2-b)}(\mu(B_k))^{1/p-1/2-b})^p \\
&\leq C\sum_{k=0}^{\infty}\Big(\frac{m_{[n\rho]}}{m_{[n\rho]-k}}\Big)^{(1/p-1/2-b)p} \leq C\sum_{k=0}^{\infty} 2^{kp(1/p-1/2-b)} < C,
\end{aligned}$$

provided $b > 1/p - 1/2$. Next, to find a suitable estimate for the $\|b_k\|_2$ we first observe that for every $k \geq 1$,

$$\begin{aligned}
\int_{G\setminus B_{k-1}} |M(x)|\,d\mu(x) &\leq \Big(\int_G |x|^{2b}|M(x)|^2\,d\mu(x)\Big)^{1/2} \\
&\quad \times \Big(\int_{G\setminus B_{k-1}} |x|^{-2b}\,d\mu(x)\Big)^{1/2} \\
&\leq C(m_n)^{\rho(1/p-1/2-b)}(\mu(B_k))^{(-2b+1)/2},
\end{aligned}$$

provided $b > 1/2$. Also, $\|(\mu(D_k))^{-1}\chi_{D_k}\|_2 \leq C(\mu(B_k))^{-1/2}$. Thus, if $b > 1/2$, then

$$\|b_k\|_2 \leq C(m_n)^{\rho(1/p-1/2-b)}(\mu(B_k))^{-b}.$$

If we define b_k^* by

$$b_k^*(x) = (\|b_k\|_2^{-1})b_k(x)(\mu(B_k))^{-(1/p-1/2)},$$

then b_k^* is a $(p,2)$ atom and $\sum_{k=1}^{\infty} b_k(x) = \sum_{k=1}^{\infty} \gamma_k b_k^*(x)$, where

$$\sum_{k=1}^{\infty} |\gamma_k|^p = \sum_{k=1}^{\infty} (\|b_k\|_2(\mu(B_k))^{1/p-1/2})^p$$
$$\leq C\sum_{k=1}^{\infty} ((m_n)^{\rho(1/p-1/2-b)}(\mu(B_k))^{1/p-1/2-b})^p < \infty,$$

whenever $b > 1/p - 1/2$. Thus M is a sum of multiples of $(p,2)$ atoms and $\|M\|_{H^p} \leq C$ with C independent of a. Consequently $\sigma \in \mathcal{M}(H^p)$.

3. Multipliers for power-weighted Hardy spaces. In [5] the present authors proved a multiplier theorem for power-weighted Hardy space H_α^p. We restate the theorem here, together with two of its corollaries (see [5, Theorem 4.7 and Corollaries (4.8) and (4.14)]).

THEOREM OQ. *Let $0 < p \leq 1$. If $\varphi \in L^\infty(\Gamma)$ satisfies*

$$\sup_k (m_k)^{1-p} \sum_{j=k}^{\infty} \|(\varphi^j)^\vee\|_{K(1/p-1/r,r,p)}^p < \infty$$

for some r with $1 \leq r < \infty$ then $\varphi \in \mathcal{M}(H_\alpha^p)$ for $-1 + p/r < \alpha \leq 0$. Here $\varphi^j = \varphi\chi_{\Gamma_{j+1}\setminus\Gamma_j}$ for $j \in \mathbf{Z}$.

COROLLARY OQ 1. *Let $0 < p < 1$. If $\varphi \in L^\infty(\Gamma)$ satisfies*

$$\sup_k (m_k)^{1/p-1}\|(\varphi^k)^\vee\|_{K(1/p-1/r,r,p)} < \infty$$

for some r with $1 \leq r < \infty$ then $\varphi \in \mathcal{M}(H_\alpha^p)$ for $-1 + p/r < \alpha \leq 0$.

COROLLARY OQ 2. *Let $0 < p \leq 1$. If $\varphi \in L^\infty(\Gamma)$ satisfies*

$$\sup_k (m_k)^{1/p-1+\varepsilon}\|(\varphi^k)^\vee\|_{K(1/p-1/r+\varepsilon,r,\infty)} < \infty$$

for some $\varepsilon > 0$ and r with $1 \leq r < \infty$, then $\varphi \in \mathcal{M}(H_\alpha^p)$ for $-1 + p/r < \alpha \leq 0$.

As an application of these results we prove the following.

THEOREM 2. (i) *Let $0 < p \leq 1$ and $0 \leq \rho < 1$. If $\mu = (1-\rho)(1/p - 1/r)$ for some r with $2 < r < \infty$ and if $\sigma \in S_\rho^{-\mu}$ then $\sigma \in \mathcal{M}(H_\alpha^p)$ for $-1+p/r < \alpha \leq 0$.*

(ii) *Let $0 < p < 1$ and $0 < \rho < 1$. If $\mu = (1-\rho)(1/p - 1/2)$ and if $\sigma \in S_\rho^{-\mu}$ then $\sigma \in \mathcal{M}(H_\alpha^p)$ for $-1 + p/2 < \alpha \leq 0$.*

Proof. Let $\sigma = \sigma\chi_{\Gamma_0} + \sigma\chi_{\Gamma\setminus\Gamma_0} = \sigma_1 + \sigma_2$. We first prove that $\sigma_1 \in \mathcal{M}(H^p_\alpha)$ by applying Corollary OQ 2. It follows from Theorem O that for any $k < 0$ and $2 \le r < \infty$,

$$\|(\sigma_1^k)^\vee\|_{K(1/p-1/r+1,r,\infty)} \le \sup_l (m_l)^{-(1/p-1/r+1)} \sup\{\|\tau_\xi\sigma_1^k - \sigma_1^k\|_{r'} : \xi \in \Gamma_l\}.$$

For $l \le k < 0$ and $\xi \in \Gamma_l$ we have $\gamma - \xi \in \Gamma_{k+1}\setminus\Gamma_k$ if and only if $\gamma \in \Gamma_{k+1}\setminus\Gamma_k$. Thus, taking $\kappa = 1/p - 1/r + 1$ we see that for $\xi \in \Gamma_l$,

$$\|\tau_\xi\sigma_1^k - \sigma_1^k\|_{r'} = \Big(\int_{\Gamma_{k+1}\setminus\Gamma_k} |\sigma_1^k(\gamma-\xi) - \sigma_1^k(\gamma)|^{r'}\,d\lambda(\gamma)\Big)^{1/r'}$$
$$\le C_\kappa|\xi|^\kappa(\langle\gamma\rangle)^{-(\mu+\rho\kappa)}(m_k)^{1/r'} \le C_{p,r}(m_l)^{1/p-1/r+1}(m_k)^{1/r'},$$

because $\langle\gamma\rangle = 1$ for $\gamma \in \Gamma_{k+1}\setminus\Gamma_k \subset \Gamma_0$. Therefore, for $l \le k < 0$, we have

$$(m_l)^{-(1/p-1/r+1)}\sup\{\|\tau_\xi\sigma_1^k - \sigma_1^k\|_{r'} : \xi \in \Gamma_l\} \le C_{p,r}(m_k)^{1/r'} \le C_{p,r}(m_k)^{-1/p},$$

because $m_k < 1$ and $-1/p < 0 < 1/r'$. On the other hand, if $l > k$ and $\xi \in \Gamma_l$ then

$$\|\tau_\xi\sigma_1^k - \sigma_1^k\|_{r'} \le 2\|\sigma_1^k\|_{r'} \le C(m_k)^{1/r'},$$

so that we again have

$$(m_l)^{-(1/p-1/r+1)}\sup\{\|\tau_\xi\sigma_1^k - \sigma_1^k\|_{r'} : \xi \in \Gamma_l\} \le C(m_k)^{-1/p}.$$

Thus we see that

$$\sup_k (m_k)^{1/p}\|(\sigma_1^k)^\vee\|_{K(1/p-1/r+1,r,\infty)} \le C,$$

and it follows from Corollary OQ 2 (with $\varepsilon = 1$) that $\sigma_1 \in \mathcal{M}(H^p_\alpha)$ under each of the assumptions of (i) and (ii).

In order to prove that $\sigma_2 \in \mathcal{M}(H^p_\alpha)$ we need to distinguish between parts (i) and (ii) of the theorem.

(i) Given α with $-1 + p/r < \alpha \le 0$, choose t so that $2 < t < r$ and $-1 + p/r < -1 + p/t < \alpha$. Also, let $\delta = (1-\rho)(1/t - 1/r)$; then $\mu > \delta > 0$. For each $k \ge 0$ we have

$$\|(\sigma_2^k)^\vee\|^p_{K(1/p-1/t,t,p)} \le \sum_{l=-\infty}^{\infty} (m_l)^{-(1/p-1/t)p}\sup\{\|\tau_\xi\sigma_2^k - \sigma_2^k\|^p_{t'} : \xi \in \Gamma_l\}$$
$$= \sum_{l=-\infty}^{0}\cdots + \sum_{l=1}^{k}\cdots + \sum_{l=k+1}^{\infty}\cdots = A_1 + A_2 + A_3.$$

For $l \leq k$ and $\xi \in \Gamma_l$ we have (taking $\kappa = 1/p - 1/t + 1$)

$$\begin{aligned}\|\tau_\xi \sigma_2^k - \sigma_2^k\|_{t'} &= \Big(\int_\Gamma |\sigma_2^k(\gamma - \xi) - \sigma_2^k(\gamma)|^{t'} \, d\lambda(\gamma) \Big)^{1/t'} \\ &\leq C_\kappa |\xi|^\kappa \langle\gamma\rangle^{-(\mu+\rho\kappa)} (m_k)^{1/t'} \\ &\leq C_{p,t}(m_l)^{1/p-1/t+1}(m_k)^{-\mu-\rho(1/p-1/t+1)+1/t'} \\ &= C_{p,t}(m_l)^{1/p-1/t+1}(m_k)^{-(1/p-1)-\rho-\delta}.\end{aligned}$$

Thus

$$A_1 \leq C_{p,t}^p (m_k)^{p-1-\rho p-\delta p} \sum_{l=-\infty}^{0} (m_l)^p \leq C_{p,t}^p (m_k)^{p-1-\delta p/2},$$

because $\sum_{l=-\infty}^0 (m_l)^p \leq \sum_{l=-\infty}^0 (1/2)^{lp} < C$ and $(m_k)^{-\rho p - \delta p/2} < 1$ since $m_k \geq 1$ and $-\rho p - \delta p/2 < 0$.

Similarly, for any $\varepsilon > 0$ we have

$$\begin{aligned} A_2 &\leq \sum_{l=1}^{k} C_{\varepsilon,p,t}^p (m_l)^{-(1/p-1/t)p}(m_l)^{(1/p-1/t+\varepsilon)p}(m_k)^{(-1/p+1-\rho\varepsilon-\delta)p} \\ &\leq C_{\varepsilon,p,t}^p \sum_{l=1}^{k} (m_k)^{\varepsilon p}(m_k)^{p-1-\rho\varepsilon p-\delta p} \\ &\leq C_{\varepsilon,p,t}^p (m_k)^{p-1-\delta p/2}(m_k)^{(1-\rho)\varepsilon p-\delta p/4} k (m_k)^{-\delta p/4}.\end{aligned}$$

Choose $\varepsilon > 0$ so that $(1-\rho)\varepsilon p - \delta p/4 < 0$. Then $(m_k)^{(1-\rho)\varepsilon p - \delta p/4} \leq 1$ for all $k \geq 0$. Next, choose $N \in \mathbf{N}$ so that if $k \geq N$ then $k(m_k)^{-\delta p/4} \leq 1$. Let

$$C = C_{\varepsilon,p,t}^p (1 + \max\{k(m_k)^{-\delta p/4} : 1 \leq k \leq N\}).$$

Then we have for all $k \geq 1$,

$$A_2 \leq C(m_k)^{p-1-\delta p/2}.$$

Finally, for A_3 we have

$$\begin{aligned} A_3 &\leq \sum_{l=k+1}^{\infty} (m_l)^{-(1/p-1/t)p} 2^p \|\sigma_2^l\|_{t'}^p \\ &\leq C \sum_{l=k+1}^{\infty} (m_l)^{-(1/p-1/t)p}(m_l)^{-\mu p}(m_l)^{p/t'} \\ &\leq C(m_k)^{p-1-\mu p} \leq C(m_k)^{p-1-\delta p/2},\end{aligned}$$

because $\mu p > \delta p/2$ and $m_k \geq 1$. Thus we may conclude that for every $k \geq 0$

$$\|(\sigma_2^k)^\vee\|_{K(1/p-1/t,t,p)}^p \leq C(m_k)^{p-1-\delta p/2}.$$

Therefore,

$$\sup_k (m_k)^{1-p} \sum_{j=k}^{\infty} \|(\sigma_2^j)^\vee\|^p_{K(1/p-1/t,t,p)} \le C \sup_{k\ge 0} (m_k)^{1-p} \sum_{j=k}^{\infty} (m_j)^{p-1-\delta p/2}$$
$$\le C \sup_{k\ge 0} (m_k)^{-\delta p/2} \le C_2$$

and, according to Theorem OQ, this implies that $\sigma_2 \in \mathcal{M}(H^p_\alpha)$. Thus we see that $\sigma \in \mathcal{M}(H^p_\alpha)$ for $-1 + p/r < \alpha \le 0$.

(ii) If $0 < p < 1$, $0 < \rho < 1$ and $-1 + p/2 < \alpha \le 0$, we have for every $k \ge 0$,

$$\|(\sigma_2^k)^\vee\|^p_{K(1/p-1/2,2,p)} \le \sum_{l=-\infty}^{\infty} (m_l)^{-(1/p-1/2)p} \sup\{\|\tau_\xi \sigma_2^k - \sigma_2^k\|_2^p :\ \xi \in \Gamma_l\}$$
$$= \sum_{l=-\infty}^{0} \cdots + \sum_{l=1}^{k} \cdots + \sum_{l=k+1}^{\infty} \cdots = B_1 + B_2 + B_3 .$$

If $l \le k$ and $\xi \in \Gamma_l$ we have, like in the proof of (i), for any $\kappa > 0$,

$$\|\tau_\xi \sigma_2^k - \sigma_2^k\|_2 \le C_\kappa |\xi|^\kappa (\langle\gamma\rangle)^{-\mu-\rho\kappa} (m_k)^{1/2}$$
$$\le C_p (m_l)^{1/p+1/2} (m_k)^{-\mu-\rho(1/p+1/2)+1/2}$$
$$\le C_p (m_l)^{1/p+1/2} (m_k)^{-(1/p-1)-\rho} .$$

Thus,

$$B_1 \le C_p^p (m_k)^{p-1-\rho p} \sum_{l=-\infty}^{0} (m_l)^p \le C_p (m_k)^{p-1}$$

because $(m_k)^{-\rho p} \le 1$ for $k \ge 0$ and $\sum_{l=-\infty}^{0} (m_l)^p \le C$ for $0 < p < 1$.

Now, choosing $\kappa = 1/p + 1/2 + \varepsilon$ for some $\varepsilon > 0$ to be determined later on, we see that

$$B_2 \le \sum_{l=1}^{k} C^p_{\varepsilon,p} (m_l)^{-(1/p-1/2)p} (m_l)^{(1/p+1/2+\varepsilon)p} (m_k)^{-\mu-\rho(1/p+1/2+\varepsilon)+1/2}$$
$$\le C^p_{\varepsilon,p} (m_k)^{p-1-\rho p-\rho\varepsilon p} \sum_{l=1}^{k} (m_l)^{\varepsilon p}$$
$$\le C^p_{\varepsilon,p} (m_k)^{p-1} (m_k)^{(1-\rho)\varepsilon p-\rho p/2} k (m_k)^{-\rho p/2} .$$

Choose $\varepsilon > 0$ so that $(1-\rho)\varepsilon p - \rho p/2 < 0$. Then $(m_k)^{(1-\rho)\varepsilon p-\rho p/2} \le 1$ for all $k \ge 1$. Next, choose $N \in \mathbb{N}$ so that if $k \ge N$ then $k(m_k)^{-\rho p/2} \le 1$. Let

$$C = C^p_{\varepsilon,p}(1 + \max\{k(m_k)^{-\rho p/2} :\ 1 \le k \le N\}) .$$

Then we have for all $k \geq 1$,

$$B_2 \leq C(m_k)^{p-1}.$$

Finally, for B_3 we have

$$\begin{aligned} B_3 &\leq \sum_{l=k+1}^{\infty} (m_l)^{-(1/p-1/2)p} 2^p \|\sigma_2^l\|_2^p \\ &\leq C \sum_{l=k+1}^{\infty} (m_l)^{-(1/p-1/2)p} (m_l)^{p/2} \leq C \sum_{l=k+1}^{\infty} (m_l)^{p-1} \leq C(m_k)^{p-1}, \end{aligned}$$

because $0 < p < 1$. Consequently,

$$\sup_k (m_k)^{1/p-1} \|(\sigma_2^k)^\vee\|_{K(1/p-1/2,2,p)} < C$$

and Corollary OQ 1 implies that $\sigma_2 \in \mathcal{M}(H_\alpha^p)$, and we see that $\sigma \in \mathcal{M}(H_\alpha^p)$ for $-1 + p/2 < \alpha \leq 0$. This completes the proof of Theorem 2.

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