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A LOCAL METRIC CHARACTERIZATION OF BANACH SPACES

BY

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1. Introduction. The purpose of this paper is to give a generalization of the Andalafte and Blumenthal characterization of Banach spaces among the class of complete, convex, externally convex, metric spaces which have the two-triple property. This generalization answers a question of Martin and Valentine [4] (P 962). We also provide a negative answer to one of their questions (P 961). See [3] for definitions and a detailed study of these concepts. A metric space satisfies the *Young Postulate* provided, for each three of its points p, q, r, if q' and r' are the respective midpoints of p and q and p and r, then $q'r' = qr/2$ (juxtaposition denotes distance). Andalafte and Blumenthal [1] showed that a complete, convex, externally convex, metric space which has the two-triple property is a real Banach space if and only if it satisfies the Young Postulate. We will say a metric space satisfies the *Local Young Postulate* provided, for each point t of the space, there is a spherical neighborhood S_t with center t such that S_t satisfies the Young Postulate. The main result of this paper is that a complete, convex, externally convex, metric space is a real, rotund Banach space if and only if it satisfies the Local Young Postulate.

A local version of the following theorem, which is found in [1], p. 29, is useful in the sequel.

THEOREM AB. *Let Y denote a complete, convex, externally convex, metric space which has the two-triple property and satisfies the Young Postulate. If q' and r' are points of the lines $L(p, q)$ and $L(p, r)$, respectively, with $pq' = \lambda \cdot pq$ and $pr' = \lambda \cdot pr$ $(0 \leqslant \lambda \leqslant 1)$, then $q'r' = \lambda \cdot qr$.*

The local version of Theorem AB simply requires that the points p, q, r of the theorem lie in a sphere which satisfies the Young Postulate.

Throughout this paper, M will denote a complete, convex, externally convex, metric space which satisfies the Local Young Postulate.

2. Immediate consequences of the Local Young Postulate. Before proceeding with the characterization of real, rotund Banach spaces, we first show that each pair of distinct points lies on a unique metric line and we prove that "small" spherical neighborhoods are convex.

THEOREM 2.1. *Each two distinct points of M lie on a unique metric line.*

Proof. If the contrary is assumed, then distinct points p, q, r, r' and segments $S(q, r)$ and $S(q, r')$ can be found such that q is a midpoint of p and r, q is a midpoint of p and r', and the segments $S(q, r)$ and $S(q, r')$ have only the point q in common. By the Local Young Postulate, there is a sphere S_q with center q and radius ε, say, such that S_q satisfies the Young Postulate. Let t, t' and s be points on the segments $S(q, r)$, $S(q, r')$ and a segment $S(p, q)$, respectively, such that $qs = qt = qt' = \varepsilon/2$. Then q, s, t, t' are elements of S_q, q is a midpoint of s and t, and q is a midpoint of s and t'. By the Young Postulate for S_q, we have $qq = tt'/2$, contrary to the fact that $t \neq t'$.

THEOREM 2.2. *If $S_{p,\varepsilon}$ is a sphere which satisfies the Young Postulate and if x and y are distinct points in $S_{p,\varepsilon}$, then $S_{p,\varepsilon}$ contains the segment joining x and y.*

Proof. Let x and y be distinct points in $S_{p,\varepsilon}$ and let z be between x and y. Without loss of generality assume $xz \geqslant yz$. Let x' be the point such that xpx' holds and $xp/xx' = xz/xy = \lambda < 1$. Note that $\lambda \geqslant 1/2$ and $px' \leqslant px < \varepsilon$. Now

$$yx' \leqslant px' + py = [(1-\lambda)/\lambda]px + py < [(1-\lambda)/\lambda]\varepsilon + \varepsilon = (1/\lambda)\varepsilon.$$

Since $S_{p,\varepsilon}$ satisfies the Young Postulate and, consequently, the local version of Theorem AB, we have

$$pz = \lambda \cdot x'y < \lambda(1/\lambda)\varepsilon = \varepsilon.$$

Thus z lies in $S_{p,\varepsilon}$ and $S_{p,\varepsilon}$ contains the segment joining x and y.

3. Parallelograms and trapezoids. It is now possible to define parallelograms and trapezoids, at least locally, and to show that they have properties similar to those of parallelograms and trapezoids in the Euclidean plane.

Definition 3.1. Points q, r, q', r' will be called *vertices of a trapezoid* provided there is some point p such that p is between q and q', p is between r and r', and $qp/qq' = rp/rr'$.

Definition 3.2. Points q, r, q', r' will called *vertices of a parallelogram* provided there is some point p such that p is between q and q', p is between r and r', and $qp/qq' = rp/rr' = 1/2$.

It should be noted that, according to our definitions, a parallelogram is a special trapezoid. The following theorem lends some credence to our choice of terminology.

THEOREM 3.1. *Let $S_{a,\varepsilon}$ be a sphere which satisfies the Young Postulate. If q, r, q', r' lie in $S_{a,\varepsilon/3}$ and are vertices of a parallelogram and if p is the midpoint of q and q' and of r and r', then $qr = q'r'$, $qr' = rq'$, p is the midpoint of the midpoint of q and r and the midpoint of q' and r', and p is the midpoint of the midpoint of q and r' and the midpoint of r and q'.*

Proof. Let s, s', t, t' be the respective midpoints of q and r, q' and r', r and q', and q and r'. The Young Postulate applied to the points q, r, q' yields $pt = qr/2$. Similar consideration of r, q', r' gives $pt = q'r'/2$ and, consequently, $qr = q'r'$. In exactly the same manner we obtain $rq' = qr'$. Now let t^* be the point such that p is the midpoint of t and t^*. Since

$$at^* \leqslant ap+pt^* = ap+pt \leqslant ap+pa+at < \varepsilon,$$

r, t, r', t^* lie in $S_{a,\varepsilon}$, r, t, r', t^* are vertices of a parallelogram, and hence $rt = r't^*$. Moreover, t, q', t^*, q are vertices of a parallelogram and lie in $S_{a,\varepsilon}$. Thus $qt^* = q't$. Now

$$qt^*+r't^* = q't+tr = q'r = qr';$$

and since the segment joining q and r' is unique, by Theorem 2.1 we have $t^* = t'$. A similar argument shows p is the midpoint of s and s', which completes the proof.

THEOREM 3.2. *Suppose $S_{a,\varepsilon}$ satisfies the Young Postulate. Let x, y, z, w be vertices of a trapezoid in $S_{a,\varepsilon/3}$ and let r be the point between x and z and y and w such that $xr/xz = yr/yw = \lambda$. Then $xy/wz = \lambda/(1-\lambda)$. If p is between x and w and if t is between y and z such that $xp/xw = yt/yz = \lambda$, then r is the midpoint of p and t. Moreover, if l and m are points on the segments joining x and r and y and r, respectively, such that $rl/rx = rm/ry$, then m is between l and the point n on the segment joining y and z for which $zl/zx = zn/zy$.*

Proof. Without loss of generality, assume $\lambda < 1/2$. Let u and v be points such that r is the midpoint of x and v, and r is the midpoint of y and u. By Theorem 2.2 (since $\lambda < 1/2$), u and v lie in $S_{a,\varepsilon/3}$. Thus x, y, u, v are vertices of a parallelogram, and if q and s are the respective midpoints of x and u and y and v, then r is the midpoint of q and s, and $qs = xy = uv$. We have $ru/rw = rv/rz = \lambda/(1-\lambda)$. Let o be the point between y and z such that $yo/yz = yu/yw = 2\lambda$; then by the local version of Theorem AB in $S_{a,\varepsilon/3}$ we have $ou = 2\lambda\cdot wz$. Furthermore,

$$xy = uv = [\lambda/(1-\lambda)]wz$$

and

$$ov = (1-2\lambda)xy = (1-2\lambda)[\lambda/(1-\lambda)]wz.$$

Thus

$$uv+vo = [\lambda/(1-\lambda)]wz+(1-2\lambda)[\lambda/(1-\lambda)]wz = 2\lambda\cdot wz$$

and v is between u and o. Since

$$yr/yu = ys/yv = yt/yo = 1/2,$$

$rs = uv/2$, $st = vo/2$, $rt = uo/2$, and s is between r and t. We now infer that q, r, s, t are linear, and since lines are unique, r is between q and t. An argument similar to the above shows that p, q, r, s are linear and r is between p and s. It is now easily seen that $pr = rt$.

Let l and m be points on the segments joining x and r and y and r, respectively, such that $rl/rx = rm/ry$, and let n be the point on the segment joining y and z such that $zl/zx = zn/zy$. That m is between l and n follows from applying the local version of Theorem AB to the triple of points r, x, y and y, r, t and z, x, y.

4. The characterization. We are now ready to show that M is a real, rotund, Banach space. Since we have already shown M has a unique line through each pair of its distinct points, we need only to show M is a Banach space over the reals (see [2], Theorem 3.2, p. 369, for alternate criteria that a Banach space be rotund). We accomplish this by showing M satisfies the Young Postulate. The Andalafte and Blumenthal result [1] then completes the proof.

THEOREM 4.1. *If p, x, y are points of M with x', y' the midpoints of p and x and p and y, respectively, then $x'y' = xy/2$.*

Proof. Since M satisfies the Local Young Postulate, there is a sphere S with center p which satisfies the Young Postulate. Let s, t be points in S and on the segments joining p and x and p and y, respectively, such that $ps/px = pt/py$. Let Λ be the set of all $\lambda > 0$ such that, for each μ, $0 \leqslant \mu \leqslant \lambda$ whenever s' and t' are points on the lines joining p and s and p and t, respectively, such that

(i) if $ps'/ps = pt'/pt = \mu$, then $s't'/st = \mu$;

(ii) for each point m between s and t there is a point m' between s' and t' such that p, m, m' satisfy the same betweenness relation as p, s, s', and $pm'/pm = \mu$ and $m's'/ms = \mu$.

If Λ is not bounded from above, then since $px/ps = py/pt = a$ implies $xy/st = a$ and $px'ps = (px/ps)/2 = (py/pt)/2 = py'/pt$ implies $x'y'/st = a/2$, we have $x'y'/xy = 1/2$. By the way of contradiction suppose Λ is bounded from above. By Theorems 2.2, 3.1 and the local version of Theorem AB, $\Lambda \neq \emptyset$. Thus $\sup \Lambda$ exists, say equals l. It follows from the continuity of the metric that l is an element of Λ. Let u and v be points on the rays of p and s and p and t, respectively, such that $pu/ps = pv/pt = l$. For each point w on the segment $S(u, v)$ joining u and v, there is a sphere $S_{w,\epsilon}$ which satisfies the Young Postulate. Since $S(u, v)$ is compact, there exists a finite collection of spheres, say $S_{w_1,(1/3)\epsilon_1}$, $S_{w_2,(1/3)\epsilon_2}, \ldots$ $\ldots, S_{w_m,(1/3)\epsilon_m}$, of the set of spheres $S_{w,\epsilon/3}$ that covers $S(u, v)$. Let τ be the Lebesgue number for the open cover $\{S_{w_j,(1/3)\epsilon_j} | j = 1, 2, \ldots, m\}$. Let $u = s_0, s_1, s_2, \ldots, s_n = v$ be points on $S(u, v)$ such that $s_{i-1}s_is_{i+1}$ $(i = 1, 2, \ldots, n-1)$ holds and $s_{i-1}s_i = s_is_{i+1} = uv/n < \tau/2$. Note that s_{i-1}, s_i, s_{i+1} lie in a sphere which satisfies the Young Postulate. For each i and j $(0 \leqslant i \leqslant n, 1 \leqslant j \leqslant m)$ such that $s_i \in S_{w_j,(1/3)\epsilon_j}$ choose $r_{ij} = (1/3)\epsilon_j - w_js_i$. Then $S_{s_i,r_{ij}}$ is contained in $S_{w_j,(1/3)\epsilon_j}$ and if t_i is chosen on the ray from p to s_i such that $s_it_i = r = \min r_{ij}$, we conclude that $t_i \in S_{w_j,(1/3)\epsilon_j}$. No generality is lost if we assume $pu \geqslant ps_i$ $(i = 1, 2, \ldots, n)$. Let s_0', s_0'' be points such that $ps_0'u$ and pus_0'' hold and $us_0', us_0'' \leqslant r$. Let $c = ps_0'/pu$ and $k = pu/ps_0''$. It follows that $us_0' = (1-c)pu$ and $us_0'' = [(1-k)/k]pu$. Let s_i' and s_i'' be points such that $ps_i's_i$ and ps_is_i'' hold, and we have $ps_i'/ps_i = c$ and $ps_i/ps_i'' = k$ $(i = 1, 2, \ldots, n)$. Then $s_{i-1}', s_i', s_{i+1}', s_{i-1}, s_i, s_{i+1}, s_{i-1}'', s_i'', s_{i+1}''$ lie in a sphere $S_{w_j,(1/3)\epsilon_j}$. Since $ps/pu = pt/pv = l$ and s_0' is between p and u and s_n' is between p and v, we have $ps/ps_0' = pt/ps_n' = a$, $st = a\cdot s_0's_n'$ and, in fact, $s_0's_n' = (l/a)uv$. It is easily seen that

$$s_{i-1}'s_i' = (l/a)s_{i-1}s_i \quad (i = 1, 2, \ldots, n).$$

Since $ps_i'/ps_i = c$ and $ps_i/ps_i'' = k$, we obtain

$$s_i's_i/s_i's_i'' = [k(1-c)]/[k(1-c)+1-k].$$

Since $s_{i-1}', s_i', s_{i+1}', s_{i-1}, s_i, s_{i+1}, s_{i-1}'', s_i'', s_{i+1}''$ lie in a sphere $S_{w_j,(1/3)\epsilon_j}$, where $S_{w_j,(1/3)\epsilon_j}$ satisfies the Young Postulate and

$$s_i's_i/s_i's_i'' = [k(1-c)]/[k(1-c)+1-k] \quad (i = 0, 1, 2, \ldots, n),$$

it follows from Theorem 3.2 that

$$s_{i-1}'s_i/s_{i-1}'s_{i+1}'' = s_{i+1}'s_i/s_{i+1}'s_{i-1}'' = [k(1-c)]/[k(1-c)+1-k];$$

that is, $s_{i-1}', s_{i+1}', s_{i+1}'', s_{i-1}''$ are vertices of a trapezoid and

$$s_{i-1}'s_{i+1}' = \frac{[k(1-c)]/[k(1-c)+1-k]}{1-[k(1-c)]/[k(1-c)+1-k]}\, s_{i-1}''s_{i+1}'' \qquad (i = 1, 2, \ldots, n-1).$$

Since $s_0', s_1', \ldots, s_n'$ are linear, we see that $s_0'', s_1'', \ldots, s_n''$ are linear and

$$s_0''s_n''/st = (s_0''s_n''/uv)(uv/st) = l/k > l.$$

By the construction above, (i) and (ii) of the definition of Λ are seen to be satisfied. Thus l/k is an element of Λ; that is, $l \neq \sup \Lambda$. This contradiction proves the theorem.

Applying the result of Andalafte and Blumenthal [1], we have

THEOREM 4.2. *A complete, convex, externally convex, metric space is a real, rotund, Banach space if and only if it satisfies the Local Young Postulate.*

5. Wilson angles and their supplements. Wilson [5] defined angles in arbitrary metric spaces as follows. If a, b, c are points of a metric space, then it follows from the triangle inequality that

$$-1 \leqslant (ab^2+ac^2-bc^2)/2ab\cdot bc \leqslant 1.$$

Thus bac is called an *angle with vertex* a, and its value is given by

$$bac = \arccos[(ab^2+ac^2-bc^2)/2ab\cdot ac].$$

Wilson defined the angle (ϱ, σ) of two metric rays ϱ and σ (congruent images of half-lines) with common initial point a by $(\varrho, \sigma) = \lim bac$ as b and c tend to a on the rays ϱ and σ, respectively, provided this limit exists. Martin and Valentine [4] conjectured (P 961) that the following properties imply that a complete, convex, externally convex, metric space M is an inner-product space:

(i) each two intersecting rays determine an angle as defined above,

(ii) the space M also has supplementary angles.

This is false, for the hyperbolic plane is seen to have these properties as the following argument shows.

Let p, q, r, and s be points in the hyperbolic plane such that p is between q and s. By the hyperbolic law of cosines,

$$\cos < qpr = (\cosh px \cosh py - \cosh xy)/\sinh px \sinh py$$

for any points x and y on the rays $R(p, q)$ and $R(p, r)$, respectively. A similar statement holds for $< rps$ and

$$rps + < rpq = \pi.$$

By considering the Taylor expansions of the respective hyperbolic functions we have

$$\frac{\cosh px \cosh py - \cosh xy}{\sinh px \sinh py}$$

$$= \left[\left(1+\frac{px^2}{2!}+\frac{px^4}{4!}+\dots\right)\left(1+\frac{py^2}{2!}+\frac{py^4}{4!}+\dots\right)-\right.$$

$$\left.-\left(1+\frac{xy^2}{2!}+\frac{xy^4}{4!}+\dots\right)\right]\left[\left(px+\frac{px^3}{3!}+\dots\right)\left(py+\frac{py^3}{3!}+\dots\right)\right]^{-1}$$

$$= \lim_{x,y\to p}\left(\frac{px^2}{2!}+\frac{py^2}{2!}-\frac{xy^2}{2!}\right)(px\cdot py)^{-1},$$

which is Wilson's definition of the angle between two rays.

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