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REMARKS ON DIAGONAL AND GENERALIZED DIAGONAL ALGEBRAS

BY

J. PŁONKA (WROCŁAW)

Introduction. In [2] we have described so-called diagonal algebras, i. e. algebras with one fundamental operation $d(x_1, \ldots, x_n)$ satisfying the axioms 1° and 2° (see § 1 below).

In the present paper some other characterizations of those algebras are given, and the properties of algebras obtained by rejection of the axiom 2° are discussed. The last algebras will be called generalized diagonal algebras.

1. Definitions. An algebra $\mathfrak{D} = \big(X; d(x_1, \ldots, x_n)\big)$ will be called an *n-dimensional diagonal algebra* if its fundamental operation d satisfies two axioms:

1° $d\big(d(x_1^1, \ldots, x_n^1), \ldots, d(x_1^n, \ldots, x_n^n)\big) = d(x_1^1, \ldots, x_n^n)$,

2° $d(x, x, \ldots, x) = x$.

A representation theorem for these algebras as well as some other properties have been given in [2].

An algebra $\mathfrak{D} = \big(X; d(x_1, \ldots, x_n)\big)$ will be called a *generalized n-dimensional diagonal algebra* if its operation d satisfies axiom 1°.

2. Properties of generalized diagonal algebras. Now we shall prove some simple properties of the algebras just defined.

(i) *If* $y = d(x_1, \ldots, x_n)$, *then* $d(y, \ldots, y) = y$.

Indeed, we have $d(y, \ldots, y) = d\big(d(x_1, \ldots, x_n), \ldots, d(x_1, \ldots, x_n)\big) = d(x_1, \ldots, x_n) = y$.

(ii) *The condition* 1° *is equivalent to the following set of conditions*:

3° $d\big(x_1, \ldots, x_{j-1}, d(x_j^1, \ldots, x_j^n), x_{j+1}, \ldots, x_n\big) = d(x_1, \ldots, x_{j-1}, x_j^j, x_{j+1}, \ldots, x_n)$, $j = 1, 2, \ldots, n$.

Indeed, using 1° and (i) we get

$$d(x_1, \ldots, x_{j-}, d(x_j^1, \ldots, x_j^n), x_{j+1}, \ldots, x_n)$$
$$= d\big(d(x_1, \ldots, x_1), \ldots, d(x_{j-1}, \ldots, x_{j-1}), d\big(d(x_j^1, \ldots, x_j^n), \ldots,$$
$$d(x_j^1, \ldots, x_j^n)\big), d(x_{j+1}, \ldots, x_{j+1}), \ldots, d(x_n, \ldots, x_n)\big)$$
$$= d\big(d(x_1, \ldots, x_1), \ldots, d(x_{j-1}, \ldots, x_{j-1}), d(x_j^1, \ldots, x_j^n), d(x_{j+1}, \ldots, x_{j+1}\big), \ldots,$$
$$d(x_n, \ldots, x_n)\big) = d(x_1, \ldots, x_{j-1}, x_j^j, x_{j+1}, \ldots, x_n).$$

The reverse implication is obvious.

(iii) *Every algebraic operation is trivial or is of the form*

$$f(x_1, \ldots, x_m) = d(x_{i_1}, \ldots, x_{i_n}) \quad (1 \leqslant i_k \leqslant m,\ k = 1, \ldots, n).$$

This follows at once from 3°.

(iv) *Either the operation d is constant, or in the algebra there are no algebraic constants at all.*

Indeed, if d is not constant, and for suitable $i_1, \ldots, i_n$ there is $d(x_{i_1}, \ldots, x_{i_n}) = c$, an algebraic constant, then $d(x, x, \ldots, x) = c$ and, by 1°, $d(x_1, \ldots, x_n) = d\big(d(x_1, \ldots, x_1), \ldots, d(x_n, \ldots, x_n)\big) = d(c, c, \ldots, c)$ will be a constant, a contradiction.

Now we prove

THEOREM 1. *Every generalized n-dimensional diagonal algebra is of the form $\mathfrak{A} = (X; d)$, where $X = A \cup D$, $A \cap D = \emptyset$ and $(D; d)$ is a subalgebra of $\mathfrak{A}$ is an n-dimensional diagonal algebra. Moreover, there exists a retraction $f\colon A \cup D \to D$ such that*

$$d(a_1, \ldots, a_n) = d\big(f(a_1), \ldots, f(a_n)\big).$$

Proof. Let $\mathfrak{A}$ be an arbitrary generalized n-dimensional diagonal algebra, and let D be the set of all elements $x \in X$ such that $d(x, \ldots, x) = x$. Evidently, (D, d) is a subalgebra of $\mathfrak{A}$ in which the axioms of diagonal algebras are satisfied. If $A = X \setminus D$ and for $a \in A$ we define $f(a) = d(a, \ldots, a)$, then by 3° we get

$$d(x_1, \ldots, x_{j-1}, a, x_{j+1}, \ldots, x_n) =$$
$$d\big(x_1, \ldots, x_{j-1}, d(a, \ldots, a), x_{j+1}, \ldots, x_n\big) = d\big(x_1, \ldots, x_{j-1}, f(a), x_{j+1}, \ldots, x_n\big)$$

and so all assertions of the theorem are proved.

Now it is easy to extend some results proved in [2] to generalized diagonal algebras.

We say that an element a of a generalized diagonal algebra is *collinear* with an element b in i-th direction ($i = 1, 2, \ldots, n$), or shortly, $a \equiv_i b$, if $d(a, \ldots, a) = d(a, \ldots, a, b, a, \ldots, a)$, where the element b takes i-th place on the right-hand side of the formula.

It is easy to prove that

(v) *Each of the relations* $\equiv_i$ *is a congruence,*

(vi) $d(a_1, \ldots, a_n) = d(b_1, \ldots, b_n)$ *if and only if*

$$a_i \equiv_i b_i \quad (i = 1, 2, \ldots, n).$$

(vii) *Every two minimal sets of generators have the same cardinality, and every two maximal independent sets have the same cardinality provided the algebra in question is proper,*

(viii) *The algebra* $(A \cup D; d)$ *has a basis if and only if the algebra* $(D; d)$ *has the form* $\mathfrak{P}_{B,\ldots,B}$, $|A| = |B|$ (¹), *and the retraction* f *(see Theorem* 1) *is a one-to-one mapping of* A *onto an independent subset of* $(D; d)$.

(ix) *The cartesian product of the algebras* $(A \cup D; d)$ *and* $(A' \cup D'; d)$ *has the form* $(A \times A' \cup A \times D' \cup D \times A' \cup D \times D'; d)$.

(x) *The free product of the algebras* $\mathfrak{A}_1$ *and* $\mathfrak{A}_2$ *has the form* $\big((A \cup A') \cup (D \cup D'); d\big)$, *where* $(A \cup D; d)$ *is isomorphic with* $\mathfrak{A}_1$, $(A' \cup D'; d)$ *is isomorphic with* $\mathfrak{A}_2$, *and* $(D \cup D'; d)$ *is the free product of* $(D; d)$ *and* $(D'; d)$.

(xi) *Generalized diagonal algebras satisfy the condition of exchange of independent sets* (see [3]).

3. Some characterizations of diagonal algebras. We say that an operation $f(x_1, \ldots, x_n)$ is *associative* if it satisfies the formula

$$f\big(f(x_1, \ldots, x_n), y_2, \ldots, y_n\big) = f\big(x_1, f(x_2, y_2, x_3, \ldots, x_n), y_3, \ldots, y_n\big)$$
$$= \ldots = f\big(x_1, y_2, \ldots, y_{n-1}, f(x_2, \ldots, x_n, y_n)\big).$$

If f is an operation of two variables, then the above defined associativity is the usual one.

We say that an n-tuple of elements $(a_1, \ldots, a_n)$ of an algebra $\mathfrak{A} = \big(X; g(x_1, \ldots, x_n)\big)$ *is regularly conjugate* (²), if the following formula is satisfied: $g\big(a_i, \ldots, a_i, g(a_1, \ldots, a_n), a_i, \ldots, a_i\big) = a_i$, where $g(a_1, \ldots, a_n)$ is on the i-th place of the left-hand side of the equation ($i = 1, 2, \ldots, n$)

Assuming associativity, it suffices to write in this definition

$$g\big(g(a_1, \ldots, a_n), a_1, \ldots, a_1\big) = a_1. \tag{1}$$

Let $\mathfrak{A} = \big(X; g(x_1, \ldots, x_n)\big)$ be an algebra with one fundamental operation g.

(¹) For the definition of the algebra $\mathfrak{P}_{B,\ldots,B}$ see [2]; $|X|$ denotes the cardinal of X; A and D are defined as in the proof of Theorem 1.

(²) This notion was employed by Liapin in the book [4], p. 108, for $n = 2$; we can also find there a proof of theorem 2 given below for the operations of two variables, by the aid of the usual associativity (the case of a semigroup).

THEOREM 2. *The following conditions are equivalent:*

(c_1) $\mathfrak{A}$ *is a diagonal algebra, i. e. it satisfies the formulas* 1° *and* 2°.

(c_2) *The operation* g *of the algebra* $\mathfrak{A}$ *is associative and satisfies the implication*

$$g(y, x, \ldots, x) = g(x, y, x, \ldots, x) = \ldots = g(x, \ldots, x, y) \to x = y.$$

(c_3) *The operation* g *of the algebra* $\mathfrak{A}$ *is associative and each* n*-tuple of the algebra* $\mathfrak{A}$ *is conjugate with respect to this operation.*

Proof. (c_1) $\to$ (c_2). By (ii) the operation $d(x_1, \ldots, x_n)$ of a diagonal algebra is obviously associative. If the left-hand side of the implication given in (c_2) is fulfilled, then, by virtue of (vi) the elements x and y are collinear in every direction. In view of (v) and (vi) we can write

$$x = d(x, \ldots, x) = d(y, \ldots, y) = y.$$

(c_2) $\to$ (c_3). For the sake of brevity we shall write $(x_1 \ldots x_n)$ instead of $g(x_1, \ldots, x_n)$.

Because of associativity we can write

$$\big((x\ldots x)x\ldots x\big) = \big(x(x\ldots x)x\ldots x\big) = \ldots = \big(x\ldots x(x\ldots x)\big),$$

hence $(x\ldots x) = x$. Thus the operation is idempotent.

Now we have

$$\Big(\big((x_1\ldots x_n)x_1\ldots x_1\big)x_1\ldots x_1\Big) = \big((x_1\ldots x_n)(x_1\ldots x_1)x_1\ldots x_1\big)$$
$$= \big((x_1\ldots x_n)x_1\ldots x_1\big) = \big(\underbrace{x_1\ldots x_1}_{i-1}(x_2\ldots x_i x_1 x_{i+1}\ldots x_n)x_1\ldots x_1\big)$$
$$= \big((x_1\ldots x_1)x_1\ldots x_1(x_2\ldots x_i x_1 x_{i+1}\ldots x_n)x_1\ldots x_1\big)$$
$$= \big(\underbrace{x_1\ldots x_1}_{i-1}\big(\underbrace{x_1\ldots x_1}_{i-1}(x_2\ldots x_i x_1 x_{i+1}\ldots x_n)x_1\ldots x_1\big)x_1\ldots x_1\big)$$
$$= \big(\underbrace{x_1\ldots x_1}_{i-1}\big((x_1 x_2\ldots x_n)x_1\ldots x_1\big)x_1\ldots x_1\big).$$

Thus, applying (c_2), we obtain $\big((x_1\ldots x_n)x_1\ldots x_1\big) = x_1$, whence ($c_2$)$\to$($c_3$) in view of associativity of the operation.

($c_3$$\to$$c_1$). We shall show first that the operation g in the condition (c_3) is idempotent. We shall prove it for the operations of three variables — the proof for an arbitrary n does not differ essentially.

For the sake of brevity we shall write y instead of (xxx). Thus we have:

$$\big((yyy)(yyy)(yyy)\big) = \Big(\big((yyy)yy\big)(yyy)y\Big)$$
$$= \Big(\Big(\big((yyy)yy\big)yy\Big)yy\Big) = \Big(\Big(\big(y(yyy)y\big)yy\Big)yy\Big)$$
$$= \Big(\Big(y\big((yyy)yy\big)y\Big)yy\Big) = y.$$

The last step is obtained from (1) by putting y instead of x_1. Proceeding similarly we obtain the formula $\big((yyy)(yyy)(yyy)\big) = x$. Hence, $(xxx) = x$.

Now we shall prove that the formulas 3° are fulfilled for $j = 1$, and therefore by virtue of associativity that they are fulfilled for every j. By (c_3) we have

$$\big((x_1 \ldots x_n) y_2 \ldots y_n\big) = \Big((x_1 \ldots x_n)\big(y_2 (x_1 y_2 x_1 \ldots x_1) y_2 \ldots y_2\big) y_3 \ldots y_n\Big)$$
$$= \Big(\big((x_1 \ldots x_n) y_2 \ldots y_2\big)(x_1 y_2 x_1 \ldots x_1) y_3 \ldots y_n\Big)$$
$$= \Big(\big(x_1 y_2 \ldots y_2 (x_2 \ldots x_n y_2)\big)(x_1 y_2 x_1 \ldots x_1) y_3 \ldots y_n\Big)$$
$$= \Big(\Big(\big(x_1 y_2 \ldots y_2 (x_2 \ldots x_n y_2)\big) x_1 \ldots x_1\Big) y_2 \ldots y_n\Big) = (x_1 y_2 \ldots y_n).$$

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