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SUMMABILITY FACTORS INVOLVING ORLICZ METRICS

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For Cesàro summability, there is a theorem (see [6], p. 105) which gives necessary and sufficient conditions on f_k such that $\sum f_k x_k$ is C_β-summable whenever $\sum x_k$ is C_a-summable ($0 \leqslant \beta \leqslant a$). The sequence f_k is called a ***summability factor***. It is therefore natural to ask whether a similar theorem holds for strong summability. This paper gives an affirmative answer. We shall formulate the result in terms of Orlicz metrics.

Following Waszak [5] and Antoni [1], we define a modular space of strongly summable sequences as follows. Let $A = (a_{mn})$ be a regular matrix such that $a_{mn} \geqslant 0$ and no column consists of zeros only. Let $\Phi = \{\varphi_n\}$ be a sequence of Orlicz functions, i.e. each φ_n is a non-decreasing continuous function defined on $[0, \infty)$, vanishes only at 0, and $\varphi_n(u) \to \infty$ as $u \to \infty$. For $x = \{t_n\}$ we write

$$\sigma_m(\Phi; x) = \sum_{n=1}^{\infty} a_{mn}\varphi_n(|t_n|).$$

Denote by $T_0(\Phi)$ the space of all sequences x such that $\sigma_m(\Phi; \lambda x) \to 0$ as $m \to \infty$ for all $\lambda > 0$. We may define an F-norm in $T_0(\Phi)$ by

$$\|x\|_\Phi = \inf\{\varepsilon > 0: \varrho(\Phi; x/\varepsilon) \leqslant \varepsilon\}, \tag{1}$$

where

$$\varrho(\Phi; x) = \sup\{\sigma_m(\Phi; x): m \geqslant 1\}.$$

It is interesting to note that for each m we may define a norm $\|x\|_{\Phi,m}$ as in (1) with $\varrho(\Phi; \cdot)$ replaced by $\sigma_m(\Phi; \cdot)$. Then, following the same argument as in [3], we have

$$\|x\|_\Phi = \sup\{\|x\|_{\Phi,m}: m \geqslant 1\}. \tag{2}$$

An Orlicz function φ is said to be *r-convex* ($0 < r \leqslant 1$) if

$$\varphi(au + \beta v) \leqslant a^r\varphi(u) + \beta^r\varphi(v)$$

for all $\alpha, \beta \geqslant 0$ and $\alpha^r + \beta^r = 1$. If all functions φ_n are r-convex, then an r-homogeneous norm can be given by

$$\|x\|_\Phi = \inf\{\varepsilon > 0:\ \varrho(\Phi; x/\varepsilon^{1/r}) \leqslant 1\}. \tag{3}$$

Similarly, (2) holds with $\|x\|_\Phi$ and $\|x\|_{\Phi,m}$ being r-homogeneous norms.

A sequence $x = \{t_n\}$ is said to be *Φ-strongly summable* to a number $t = t(x)$ if $\{t_n - t\} \in T_0(\Phi)$, and t is called the *Φ-strong limit* of x. We denote by $T(\Phi)$ the set of all Φ-strongly summable sequences. Again, following Antoni [1], we say that a sequence $\Phi = \{\varphi_n\}$ of Orlicz functions is *lower-regular* at the point zero if there exist an Orlicz function ψ and a fixed neighbourhood U of the point zero such that $\psi(u) \leqslant \inf\{\varphi_n(u): n \geqslant 1\}$ for all $u \in U$. It was shown (see [1], Theorem 7) that if $\Phi = \{\varphi_n\}$ is lower-regular and equicontinuous at the point zero, then $T(\Phi)$ is an FK-space ([6], p. 29) with norm given by (1) or (3). Furthermore, the Φ-strong limit $t = t(x)$ of $x \in T(\Phi)$ is uniquely determined ([1], Theorem 8).

Let $\Phi = \{\varphi_n\}$ and $\Lambda = \{\lambda_n\}$ be two sequences of Orlicz functions. For each n, we define a complementary function of φ_n with respect to λ_n as follows:

$$\psi_n(v) = \sup\{\lambda_n(uv) - \varphi_n(u): u > 0\}.$$

For example, for $\varphi_n(u) = u^p/p$ and $\lambda_n(u) = u^r/r$, where $p > r > 0$, we have $\psi_n(v) = v^q/q$ with $1/p + 1/q = 1/r$. We say that $\Psi = \{\psi_n\}$ is the *sequence of Λ-complementary functions* of Φ. In order to obtain a meaningful Hölder inequality, we shall assume in what follows that both φ_n and λ_n are r-convex for each n, and that ψ_n is also an r-convex Orlicz function.

The next lemma gives a sufficient condition for the complementary function to be also an Orlicz function. A function λ is said to satisfy the *(Δ_2, δ_2)-condition* if $\lambda(2u) \leqslant M\lambda(u)$ for some constant $M > 0$ and for all $u \geqslant 0$.

LEMMA. *Let φ and λ be r-convex Orlicz functions and let λ satisfy the (Δ_2, δ_2)-condition. If the properties*

$$\varphi(u)/\lambda(u) \to 0 \text{ as } u \to 0, \qquad \varphi(u)/\lambda(u) \to \infty \text{ as } u \to \infty$$

are satisfied, then the complementary function ψ of φ with respect to λ is an r-convex Orlicz function.

The proof is elementary.

Suppose that $\alpha = \|x\|_{\Phi,m}$ and $\beta = \|y\|_{\Psi,m}$ as given by (3). Then for $x = \{t_n\}$ and $y = \{s_n\}$ we have

$$\lambda_n\big(|t_n s_n/(2\alpha\beta)^{1/r}|\big) \leqslant \frac{1}{2}\varphi_n(|t_n/\alpha^{1/r}|) + \frac{1}{2}\psi_n(|s_n/\beta^{1/r}|).$$

Multiplying by a_{mn} throughout and summing over $n = 1, 2, \ldots,$ we obtain

$$\sigma_m\big(\Lambda;\, xy/(2\alpha\beta)^{1/r}\big) \leqslant 1 \qquad \text{and} \qquad \|xy\|_{\Lambda,m} \leqslant 2\|x\|_{\Phi,m}\|y\|_{\Psi,m},$$

where $xy = \{t_n s_n\}$. Taking the supremum over all m, we get

$$\|xy\|_{\Lambda} \leqslant 2\|x\|_{\Phi}\|y\|_{\Psi}.$$

This is the Hölder inequality.

We may define a Λ-associate norm of $\|\cdot\|_{\Phi}$ as follows:

$$\|y\|_{\Phi}^{(\Lambda)} = \sup\{\|xy\|_{\Lambda}\colon \|x\|_{\Phi} \leqslant 1\}.$$

Obviously,

$$\|xy\|_{\Lambda} \leqslant \|x\|_{\Phi}\|y\|_{\Phi}^{(\Lambda)}.$$

This is the strengthened Hölder inequality. Similarly, we can define $\|y\|_{\Phi,m}^{(\Lambda)}$ and obtain

$$\|xy\|_{\Lambda,m} \leqslant \|x\|_{\Phi,m}\|y\|_{\Phi,m}^{(\Lambda)}.$$

A modular $\varrho(\Phi;\, \cdot)$ is said to be *r-homogeneous* if, for $a \geqslant 0$,

$$\varrho(\Phi;\, ax) = a^r \varrho(\Phi;\, x).$$

Now we state our main theorem. Again, we write $xy = \{t_n s_n\}$ for $x = \{t_n\}$ and $y = \{s_n\}$.

THEOREM. *Let $\Phi = \{\varphi_n\}$, $\Lambda = \{\lambda_n\}$ and $\Psi = \{\psi_n\}$ be three sequences of r-convex Orlicz functions, all lower-regular and equicontinuous at the point zero. Let Ψ be the sequence of Λ-complementary functions of Φ, and let $\varrho(\Lambda;\, \cdot)$ be r-homogeneous. In order that $xy \in T(\Lambda)$ whenever $x \in T(\Phi)$, it is necessary and sufficient that*

(i) $\|y\|_{\Phi}^{(\Lambda)} < \infty$,

(ii) $y \in T(\Lambda)$.

Remark. If $y \in T(\Psi)$, then (i) holds. However, it is possible to obtain (i) without $y \in T(\Psi)$. Here y is called a *summability factor from $T(\Phi)$ to $T(\Lambda)$*.

Proof. To prove the sufficiency of the theorem, we consider

$$|t_n s_n - ts| \leqslant |t|\;|s_n - s| + |s_n|\;|t_n - t|,$$

where $t = t(x)$ and $s = s(y)$ are the Φ-strong limit of $x = \{t_n\}$ and the Λ-strong limit of $y = \{s_n\}$, respectively. In view of (i), (ii), and the inequality

$$\|xy - ts\|_{\Lambda,m} \leqslant |t|^r\|y - s\|_{\Lambda,m} + \|y\|_{\Phi,m}^{(\Lambda)}\|x - t\|_{\Phi,m}$$

the result follows. Here we have used the fact that if $\sigma_m(\Phi;\, \lambda x) \to 0$ as $m \to \infty$ for all $\lambda > 0$, then $\|x\|_{\Phi,m} \to 0$ as $m \to \infty$.

Conversely, since $e = \{1, 1, \ldots\} \in T(\Phi)$, condition (ii) is necessary. If we regard xy as the image of a diagonal matrix with entries $s_1, s_2, \ldots$ applying to $x \in T(\Phi)$, then using a theorem of Zeller (see [6], p. 29, Theorem III) we have, for some constant $M > 0$ and for all $x \in T(\Phi)$,

$$\|xy\|_A \leqslant M\|x\|_\Phi.$$

Obviously, the A-associate norm of $\|\cdot\|_\Phi$ is finite and condition (i) is necessary. The proof is complete.

COROLLARY 1. *Let Φ, A and Ψ be given as in the Theorem. Then in order that $xy \in T_0(A)$ whenever $x \in T_0(\Phi)$, it is necessary and sufficient that $\|y\|_\Phi^{(A)} < \infty$.*

In particular, if $\varphi_n(u) = u^p/p$, $a_{mn} = 1/m$ for $1 \leqslant n \leqslant m$, and $a_{mn} = 0$ for $n > m$, then $T(\Phi)$ is the set of all sequences $x = \{t_n\}$ such that, for some $t = t(x)$,

$$\frac{1}{m}\sum_{n=1}^{m} |t_n - t|^p \to 0 \quad \text{as } m \to \infty.$$

Denote the space by $T(p)$. It is easy to see that $x = \{t_n\} \in T(p)$ if and only if

$$2^{-m+1}\sum_{n=2^{m-1}}^{2^m-1} |t_n - t|^p \to 0 \quad \text{as } m \to \infty.$$

Now, let $T(\Phi) = T(p)$ and $T(A) = T(r)$, where $p > r > 0$. Then $T(\Psi) = T(q)$, where $1/p + 1/q = 1/r$, and we have

COROLLARY 2. *$xy \in T(r)$ whenever $x \in T(p)$ if and only if $y = \{s_n\}$ is in $T(r)$ and*

$$\sup\Big\{2^{-m+1}\sum_{n=2^{m-1}}^{2^m-1} |s_n|^q : m \geqslant 1\Big\} < \infty.$$

We remark that, as in Corollary 1, the condition $y \in T(r)$ in Corollary 2 can be dropped if we consider only sequences strongly summable to zero. In this special case, $\|\cdot\|_\Phi^{(A)}$ (Orlicz norm) and $\|\cdot\|_\Psi$ (Luxemburg norm) are equivalent. However, the problem remains open in general. **(P 1197)**

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