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CHARACTERISTIC SETS OF A SYSTEM OF EQUIVALENCE RELATIONS

BY

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The meet $Q = \bigcap R_i$ of equivalence relations $R_1, \ldots, R_n$ in a set X is again an equivalence relation and the partition of X into co-classes of Q is the upper-lower bound of the partitions of X into co-classes of R_i, $i = 1, \ldots, n$.

In connection with some problems in information storage and retrieval Professor Z. Pawlak asked the following question:

Given a system of equivalence relations $R_1, \ldots, R_n$, how to check whether all $R_i (i = 1, \ldots, n)$ are needed to build $Q = \bigcap R_i$, i.e., is $Q = \bigcap_{i \neq j} R_i$ satisfied for some $j = 1, \ldots, n$ or not?

The problem is of a general nature and can be asked of an arbitrary system $A_1, \ldots, A_n$ of subsets of a set Y. It is easy to see that it is invariant under (Boolean) isomorphisms of a system of sets, a notion investigated by Marczewski [1].

Let $F(A_1, \ldots, A_n)$ be the field of sets generated by $A_1, \ldots, A_n$ in Y, i.e., the least class containing all A_i's and closed under meet and complementation (with respect to Y). A system $A_1, \ldots, A_n$ of subsets of Y is isomorphic to the system $B_1, \ldots, B_n$ of subsets of Z if and only if there exists an isomorphism f (in usual algebraic sense) of $F(A_1, \ldots, A_n)$ onto $F(B_1, \ldots, B_n)$ which maps $\{A_1, \ldots, A_n\}$ onto $\{B_1, \ldots, B_n\}$ (i.e., $f(A_1), \ldots, f(A_n)$ is a permutation of $B_1, \ldots, B_n$).

With the help of characteristic functions it is easy to determine whether two systems are isomorphic. Let $h_i(y) = 1$ if $y \in A_i$ and let $h_i(y) = 0$ if $y \notin A_i$. These are the characteristic functions of single sets A_i. The characteristic function of the system $A_1, \ldots, A_n$ is defined as $h(y) = (h_1(y), \ldots, h_n(y))$. It is a function which maps Y into the product $\{0, 1\}^n$. Let us denote the elements of this product consisting only of 0's and only of 1's by $\mathit{0}$ and $\mathit{1}$, respectively.

[1] E. Szpilrajn-Marczewski, *On the isomorphism and equivalence classes and sequence s of sets*, Fundamenta Mathematicae 32 (1939), p. 133-148.

The image of Y by h, i.e.,

$$h(A_1, \ldots, A_n) = h(y) = \{(h_1(y), \ldots, h_n(y)) \mid y \in Y\},$$

is called the *characteristic set* of the system $A_1, \ldots, A_n$.

It is obvious that

Two systems $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ are isomorphic if and only if

$$h(A_1, \ldots, A_n) = h(B_1, \ldots, B_n).$$

An answer to the question whether

$$\bigcap A_i = \bigcap_{i \neq j} A_i \quad \text{for some } j = 1, \ldots, n$$

depends only on the isomorphic type of the system $A_1, \ldots, A_n$; thus it will be the same for two systems such that $h(A_1, \ldots, A_n) = h(B_1, \ldots, B_n)$. Moreover, it is easy to check that the answer is the same for the system $A_1, \ldots, A_n$ of subsets of Y and for the system $C_1 = A_1 \setminus \bigcap A_i, \ldots, C_n = A_n \setminus \bigcap A_i$ of subsets of $W = \bigcup A_i \setminus \bigcap A_i$. For these systems we have

$$h(A_1, \ldots, A_n) \setminus \{\mathit{0}, \mathit{1}\} = h(C_1, \ldots, C_n).$$

Indeed, an easy algorithm which uses only $h(A_1, \ldots, A_n) \setminus \{\mathit{0}, \mathit{1}\}$ can be developed, which solves the problem whether $\bigcap A_i = \bigcap_{i \neq j} A_i$ or not.

Turning again to the problem of equivalence relations we see that, since $R_1, \ldots, R_n$ are subsets of X^2, the problem proposed at the beginning can be solved with the help of the general algorithm. This algorithm, however, does not make any use of the fact that R_i's are equivalence relations in X^2, thus sets of a special type. It could be improved if the $h(R_1, \ldots, R_n) \setminus \{\mathit{0}, \mathit{1}\}$, at which the algorithm starts, cannot be an arbitrary subset of $\{0, 1\}^n$ to which $\mathit{0}$ and $\mathit{1}$ do not belong. Unfortunately, it is not the case.

For any subset T of $\{0, 1\}^n$ to which $\mathit{0}$ and $\mathit{1}$ do not belong there exist an X and a system $R_1, \ldots, R_n$ of equivalence relations in X such that

$$h(R_1, \ldots, R_n) \setminus \{\mathit{0}, \mathit{1}\} = T.$$

For a given $T = \{t_1, \ldots, t_k\}$ we construct the relations R_i in the following way. Let $X = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ be a set of $2k$ elements. By definition the relations R_i do not hold between two different a's or two different b's. They do not hold either between a_l and b_m if $l \neq m$. However, both $a_l R_i b_l$ and $b_l R_i a_l$ hold if and only if $t_{li} = 1$ and, for the obvious reason, $a_l R_i a_l$ and $b_l R_i b_l$ always hold. It is easy to see that

$$h(a_l, a_l) = h(b_l, b_l) = \mathit{1},$$

$$h(a_l, a_m) = h(b_l, b_m) = h(a_l, b_m) = h(b_m, a_l) = \mathit{0},$$

whenever $l \neq m$, and $h(a_l, b_l) = t_l$ for all $l = 1, \ldots, k$. Therefore, for so defined relations we have $h(R_1, \ldots, R_n) = T$.

The construction performed above shows also that for any subset T of $\{0, 1\}^n$ containing both $\mathit{0}$ and $\mathit{1}$ there exists a system of equivalence relations with T being its characteristic set. Of course, to be a characteristic set, T must contain $\mathit{1}$, but what about $\mathit{0}$?

An easy proof (by checking all possibilities) shows that if P and S are two equivalence relations in X, then for all x, y, w, z in X the following is true ($\bar{P}$ and $\bar{S}$ denote negations of P and S, respectively):

If xPy, wSz, $x\bar{S}y$, $w\bar{P}z$, *then*
either $x\bar{P}w$ *and* $x\bar{S}w$,
or $x\bar{P}z$ *and* $x\bar{S}z$,
or $y\bar{P}z$ *and* $y\bar{S}z$,
or yPw *and* ySw.

It follows that if $P \cup S = X^2$, then either $P = X^2$ or $S = X^2$. It follows also that if $T \subset \{0, 1\}^n$ is a set which can serve as a characteristic set for a system of relations, then the following is true:

(*) *For every* $t_1, t_2 \in T$ *there exist* $t_3, t_4, t_5, t_6 \in T$ *such that, for any* i *and* j $(i, j = 1, \ldots, n)$, *if* $t_{1i} = t_{2j} = 1$ *and* $t_{1j} = t_{2i} = 0$, *then either* $t_{3i} = t_{3j} = 0$ *or* $t_{4i} = t_{4j} = 0$, *or* $t_{5i} = t_{5j} = 0$, *or* $t_{6i} = t_{6j} = 0$.

If $\mathit{0} \in T$, then (*) is satisfied by $t_3 = t_4 = t_5 = t_6 = \mathit{0}$. But if $n = 2$ and T consists of three elements $\mathit{1}$, $(0, 1)$, $(1, 0)$, then (*) fails to be true, which shows that this set is not a characteristic set of any system with two equivalence relations.

The following problem has not been solved yet:

PROBLEM. Is (*) a sufficient condition for a set $T \subset \{0, 1\}^n$ containing $\mathit{1}$ to be a characteristic set of some system of n equivalence relations? **(P 1140)**

The solution of this problem could plausibly help in developing a specific algorithm eliminating superfluous equivalence relations from a system; in the case we know from other sources that each two elements of the considered set are linked at least by one relation.

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