COLLOQUIUM MATHEMATICUM

VOL. XXIII 1971 FASC. 2

ON THE SPECTRAL RADIUS IN $L_1(G)$

BY

E. PORADA (WROCŁAW)

The aim of this note is to present some properties of the spectral radius of the elements in the l_1-group algebra of an infinite non-abelian discrete group. After introducing some preliminary notions we formulate a list of properties of the group algebra of an abelian group, which later will be discused in the case of non-abelian groups. Most of them are devoted to spectral radius and the regular norm of hermitian elements of the group algebra and the behavior of these while passing from the group to separating family of homomorphic images of it.

1. Let G be a locally compact group with the unit 1. Let $\|x\|$, $\nu(x)$, $\lambda(x)$ denote the norm in the algebra $L_1(G)$, the spectral radius, and the regular norm of x, respectively, i.e.

$$\|x\| = \int_G |x(g)|\,\mu(dg),$$

where μ is the left invariant Haar measure in G,

$$\nu(x) = \lim_{n\to\infty} \sqrt[n]{\|x^n\|}, \qquad \lambda(x) = \|T_x\|,$$

where x^n is the n-fold convolution and T the regular representation of $L_1(G)$ in the algebra of the linear operators in the space $L_2(G)$. For hermitian elements x in $L_1(G)$ we clearly have

$$\lambda(x) \leqslant \nu(x) \leqslant \|x\|,$$

since

$$\|T_x\|^{2n} = \|T_x^{2n}\| = \|T_{x^{2n}}\| \leqslant \|x^{2n}\| \leqslant \|x\|^{2n}.$$

It is known that if G is a finite group, then ν and λ are norms in the finite-dimensional algebra of hermitian elements in $l_1(G)$. Consequently, for certain constants c_1, c_2,

$$c_1\lambda(x) \leqslant \nu(x) \leqslant c_2\lambda(x) \qquad \text{for } x = x^*,\ x \in l_1(G),$$

and, in particular,

$$c_1\lambda(x^n) \leqslant c_1\lambda(x)^n \leqslant \nu(x^n) = \nu(x)^n \leqslant c_2\lambda(x)^n \quad \text{for } n = 1, 2, \ldots,$$

whence $\nu(x) = \lambda(x)$ for $x = x^*$, $x \in l_1(G)$.

For a normal and closed subgroup $H \subset G$ let m_H and $m_{G/H}$ be the left invariant Haar measure in the group H and G/H, respectively, such that for $x \in C_0(G)$, i.e. for any continuous function x on the group G vanishing outside a compact set, there is

$$\int_G x(g)\mu(dg) = \int_{G/H}\Big(\int_H x(gh)m_H(dh)\Big)\, m_{G/H}(d\bar{g}),$$

where $\bar{g}$ denotes the element gH of G/H. The map

$$x \to \varphi_H(x),$$

where $\varphi_H(x) \in C_0(G/H)$ and

$$\varphi_H(x)\,(\bar{g}) = \int_H x(gh)m_H(dh),$$

is the natural homomorphism of $C_0(G)$ onto $C_0(G/H)$. In the case where G is discrete, φ_H is the linear extension of the natural homomorphism

$$C_0(G) \supset G \to G/H \subset C_0(G/H).$$

The continuous extension of φ_H to $L_p(G)$ we denote still by φ_H. It is clear that $\|\varphi_H(x)\| \leqslant \|x\|$.

1.1. LEMMA. *Let G be a locally compact group and $\mathcal{M}$ a centered (i.e. closed under finite intersection) family of normal subgroups such that $\bigcap_{H \in \mathcal{M}} H = \{1\}$. Then, for each $x \in L_p(G)$,*

$$\lim_{H \to \{1\}} \|\varphi_H(x)\|_p = \|x\|_p.$$

Proof. It is sufficient to prove that for the f's from a dense (in the L_p-norm) subset of $L_p(G)$ we have

$$(*) \qquad \int_{G/H}\Big|\int_H f(gh)m_H(dh)\Big|^p m_{G/H}(d\bar{g}) = \int_G |f(g)|^p \mu(dg)$$

for some $H \in \mathcal{M}$. Let f be a simple function, i.e. such that $\operatorname{supp}(f)$ is a finite union $K_1 \cup \ldots \cup K_s$ of disjoint compact sets and f is constant on each K_i, $i = 1, \ldots, s$. (The set of simple functions is dense in $L_p(G)$, $0 < p < \infty$.) Of course, $K_iK_j^{-1}$ is compact for $i, j = 1, \ldots, s$ and $1 \notin K_iK_j^{-1}$ if $i \neq j$. By assumption there exists an $H \in \mathcal{M}$ such that $H \cap K_iK_j^{-1} = \emptyset$ whenever $i \neq j$, $i, j = 1, \ldots, s$, whence, for a g in G, $gH \cap K_i \neq \emptyset$ for at most one i. Consequently,

$$\int_H |f(gh)|^p m_H(dh) = \int_H |(f(gh))^p| m_H(dh) = \Big|\int_H f(gh)m_H(dh)\Big|^p$$

and $(*)$ follows.

1.2. LEMMA. *If $\varphi\colon B_1 \to B_2$ is a continuous homomorphism of a Banach algebra B_1 into a Banach algebra B_2, then*

$$\nu(\varphi(x)) \leqslant \nu(x).$$

Proof. By assumption there exist a constant c such that

$$\|\varphi(x)\|_{B_2} \leqslant c\|x\|_{B_1}$$

for $x \in B_1$. Hence

$$\nu(\varphi(x)) = \lim_{n\to\infty} \sqrt[n]{\|\varphi(x)^n\|_{B_2}} \leqslant \lim_{n\to\infty} \sqrt[n]{c} \lim_{n\to\infty} \sqrt[n]{\|x^n\|_{B_1}} = \nu(x).$$

Consequently, if H is a normal subgroup of G, then for $x \in L_1(G)$

$$\nu(\varphi_H(x)) \leqslant \nu(x).$$

2. In this section G will denote an abelian locally compact group.

2.1. THEOREM. *The algebra $L_1(G)$ is commutative and the spectral radius ν satisfies following conditions characteristic for a norm:*

(A) $\nu(x+y) \leqslant \nu(x)+\nu(x)$ *for* $x, y \in L_1(G)$,

(B) $\nu(x) = 0$ *if and only if* $x = 0$,

(C) *spectral radius (now spectral norm) is continuous with respect to the usual norm in* $L_1(G)$,

(D) *if $\mathscr{M}$ is a centered family of normal subgroups of G such that $\bigcap_{H\in\mathscr{M}} H = \{1\}$, then*

$$\nu(x) = \lim_{H\to\{1\}} \nu(\varphi_H(x))$$

and

(D′) $\lambda(x) = \lim_{H\to\{1\}} \lambda(\varphi_H(x))$.

Properties (A) and (B) follow from the following fact (cf., e.g., [6], p. 264): if $x \in L_1(G)$ for a locally compact abelian group G, then

$$\nu(x) = \|\hat{x}\|_\infty = \max\{|\hat{x}(\chi)|: \chi \in \hat{G}\},$$

where $\hat{x}(\chi) = \int_G x(g)\overline{\chi(g)}\mu(dg)$. Properties (A), (B), and the inequality $\nu(x) \leqslant \|x\|$ for $x \in L_1(G)$ imply immediately (C). Properties (D) and (D′) are obtained via the following lemma:

2.2. LEMMA. *If $\mathscr{M}$ is a centered family of subgruops of G such that $\bigcap_{H\in\mathscr{M}} H = \{1\}$, then $\Gamma = \bigcup_{H\in\mathscr{M}} H^\perp$, where $H^\perp$ (the annihilator of H) is the group of all characters of G which map H onto 1, is a dense subgroup of $\hat{G}$.*

Proof. Since $\mathscr{M}$ is centered, Γ is a subgroup. By Pontrjagin's duality theorem, if $\overline{\Gamma} \neq \hat{G}$, then there exists a non-trivial character $u \in (\hat{G})\hat{} = G$ such that $u(\overline{\Gamma}) = \{1\}$. So $u(H^\perp) = \{1\}$ for $H \in \mathscr{M}$, whence $u \in \bigcap_{H\in\mathscr{M}} (H^\perp)^\perp = \bigcap H = \{1\}$, which is a contradiction.

Because of $(G/H)\hat{} = H^{\perp}$ (cf. [6], p. 35), for $a\ \chi$ in $(G/H)\hat{}$ we write

$$\big(\varphi_H(x)\big)\hat{}\,(\chi) = \int\limits_{G/H}\Big(\int\limits_{H} x(gh)\overline{\chi(\bar{g})}\,m_H(dh)\Big)m_{G/H}(d\bar{g})$$
$$= \int\limits_{G/H}\Big(\int\limits_{H} x(gh)\overline{\chi(g)}\,m_H(dh)\Big)m_{G/H}(d\bar{g})$$
$$= \int\limits_{G} x(g)\overline{\chi(g)}\,\mu(dg) = \hat{x}(\chi).$$

Hence

$$\lim_{H\to\{1\}} \nu\big(\varphi_H(x)\big) = \lim_{H\to\{1\}}\ \max_{\chi\in(G/H)\hat{}} \big|\big(\varphi_H(x)\big)\hat{}\,(\chi)\big|$$
$$= \lim_{H\to\{1\}}\ \max_{\chi\in H^{\perp}} |\hat{x}(\chi)|$$
$$= \max\Big\{|\hat{x}(\chi)|:\ \chi\in\bigcup_{H\in\mathcal{M}} H^{\perp}\Big\}$$
$$= \max\{|\hat{x}(\chi)|:\ \chi\in\hat{G}\} = \nu(x),$$

which completes the proof of (D). By Plancherel's theorem $\lambda = \nu$ and we obtain (D′).

3. Now we turn to non-abelian groups. We are going to show that (A), (B), (D) *and* (D′) *are in general not satisfied for a non-abelian discrete group.*

First we prove the following proposition.

3.1. PROPOSITION. *Let F be a free group freely generated by two generators a and b. Let A and B denote the cyclic subgroups generated by a and b, respectively. If $x, y \in l_1(F)$ and* $\operatorname{supp}(x)\subset A$, $\operatorname{supp}(y)\subset B$, *then*

$$v(x+y) \geqslant \sqrt{\|x\|\,\|y\|}.$$

The proof of 3.1 is a simplification of the proof of a theorem of [2], where a more refined result is proved for a soluble group.

Proof. Let us note first that if

$$a^{k_1}b^{l_1}\dots a^{k_n}b^{l_n} = u_1w_1\dots u_nw_n,$$

where k_i, l_i are non-zero integers and $u_i, w_i\in A\cup B$, then

$$a^{k_i} = u_i,\quad b^{l_i} = w_i\quad (i = 1,\dots,n).$$

Hence (the summation is all over the set of sequences $\{k_i, l_i\} = \{k_1, l_1, \dots, k_n, l_n\}$, $\{k_i\} = \{k_1,\dots,k_n\}$, $\{l_i\} = \{l_1,\dots,l_n\}$, where k_i, l_i are non-zero integers),

$$\|(x+y)^{2n}\| = \sum_{w\in F}|(x+y)^{2n}(w)|$$
$$\geqslant \sum_{\{k_i,l_i\}}|(x+y)^{2n}(a^{k_1}b^{l_1}\dots a^{k_n}b^{l_n})|$$

$$= \sum_{\{k_i, l_i\}} \Big| \sum_{a^{k_1} b^{l_1} \ldots a^{k_n} b^{l_n} = u_1 w_1 \ldots u_n w_n} (x+y)(u_1)(x+y)(w_1) \ldots (x+y)(w_n) \Big|$$

$$= \sum_{\{k_i\}, \{l_i\}} |x(a^{k_1}) y(b^{l_1}) \ldots x(a^{k_n}) y(b^{l_n})|$$

$$= \sum_{\{k_i\}, \{l_i\}} |x(a^{k_1})| \ldots |x(a^{k_n})| \, |y(b^{l_1})| \ldots |y(b^{l_n})|$$

$$= \Big(\sum_{\{k_i\}} |x(a^{k_1})| \ldots |x(a^{k_n})|\Big) \Big(\sum_{\{l_i\}} |y(b^{l_1})| \ldots |y(b^{l_n})|\Big)$$

$$= \Big(\sum_{k=-\infty}^{\infty} |x(a^k)|\Big)^n \Big(\sum_{l=-\infty}^{\infty} |y(b^l)|\Big)^n = \|x\|^n \|y\|^n.$$

Thus

$$\lim_{n\to\infty} \sqrt[2n]{\|(x+y)^{2n}\|} \geqslant \lim_{n\to\infty} \sqrt[2n]{\|x\|^n \|y\|^n} = \sqrt{\|x\| \, \|y\|}$$

and the proof is complete.

For $\varepsilon > 0$ there exists a trigonometrical polynomial

$$f(t) = \sum_{-M}^{M} d_n e^{int}$$

such that

$$\max_{t \in [0, 2\pi)} |f(t)| \leqslant \frac{\varepsilon}{2} \sum_{-M}^{M} |d_n|.$$

(For existence see [7], p. 388.) Let

$$g(t) = f(t) \exp\big(i(2M+1)t\big)$$

and

$$T(t) = g(t) + \overline{g(t)} = \sum_{n=1}^{2M+1} d_{n-M-1} e^{int} + \overline{d_{n-M-1}} e^{-int}$$
$$= \sum_{n=1}^{N} c_n e^{int} + \bar{c}_n e^{-int},$$

where $c_n = d_{n-M-1}$, $n = 1, \ldots, N$, $N = 2M+1$. Let us define $x, y \in l_1(F)$ by setting

$$x(w), y(w) = \begin{cases} c_n & \text{if } w = a^n, b^n \\ \bar{c}_n & \text{if } w = a^{-n}, b^{-n} \qquad n = 1, \ldots, N, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, since $\operatorname{supp}(x) \subset A$, $\operatorname{supp}(y) \subset B$, and A, B are abelian groups,

$$\nu(x) = \nu(y) = \max_t |g(t) + \overline{g(t)}| \leqslant \varepsilon \sum_{n=1}^{N} |c_n| = \frac{\varepsilon}{2}\|x\| = \frac{\varepsilon}{2}\|y\|.$$

3.2. COLLORARY. *For every $\varepsilon > 0$ there exist hermitian elements $x, y \in l_1(F)$ such that*

$$\varepsilon\nu(x+y) \geqslant \nu(x) + \nu(y).$$

Proof. Let x, y be as above. Then $x = x^*$, $y = y^*$, of course, and, by 3.1,

$$\varepsilon\nu(x+y) \geqslant \varepsilon\sqrt{\|x\|\,\|y\|} = \varepsilon\|x\| \geqslant \nu(x)+\nu(y).$$

We have proved that (A) does not hold in $l_1(F)$.

We do not know whether the algebra $l_1(F)$ satisfies the condition (B) (**P 745**) but we can show that this condition does not hold in the algebra $l_1(G \oplus H)$, where G is any finite non-abelian group and H is any discrete group. This follows from a well known fact (cf., e.g., [1]) that if G is a finite non-abelian group, then

$$l_1(G) = L(V_1) \oplus \ldots \oplus L(V_k),$$

where $L(V_i)$ is the algebra of linear operators in a finite-dimensional linear space V_i and $\dim(V_i) > 1$ for some i. If $\dim(V) > 1$, then there exists an operator T in $L(\mathrm{R})$ such that $\mathrm{T}^n = 0$. Thus there exist non-trivial nilpotent elements in $l_1(G)$. If $x \in l_1(G)$, $x^2 = 0$ and $y \in l_1(H)$, then the function $x \otimes y$ defined by the formula

$$x \otimes y(g \oplus h) = x(g)y(h), \qquad g \in G,\ h \in H,$$

is an element of $l_1(G \oplus H)$ and $(x \otimes y)^2 = 0$.

We prove now that $l_1(F)$ does not have property (D).

The family $\mathscr{M}$ of all normal subgroups H of the group F such that $|F/H| < \infty$ is a centered family and $\bigcap_{H \in \mathscr{M}} H = \{1\}$ ([4], p. 232). Let us remark that if $x, y \in l_1(F)$ are hermitian elements such that $\nu(x+y) > \nu(x) + \nu(y)$ and if $H \in \mathscr{M}$, then

$$\nu\big(\varphi_H(x+y)\big) = \nu\big(\varphi_H(x) + \varphi_H(y)\big) \leqslant \nu\big(\varphi_H(x)\big) + \nu\big(\varphi_H(y)\big).$$

Thus we have

$$\nu\big(\varphi_H(x+y)\big) \leqslant \nu(x) + \nu(y) < \nu(x+y)$$

for any $H \in \mathscr{M}$, whence we infer that $\nu\big(\varphi_H(x+y)\big)$ does not tend to $\nu(x+y)$, which shows that (D) does not hold.

It is proved in a paper of H. Kesten ([3], Theorem 3) that if F is a free group with n generators, then there exists an hermitian $x \in l_1(F)$ such that

(i) $x(w) \geqslant 0$ for $w \in F$,

(ii) $\|x\| = 1$,

(iii) $\lambda(x) = \sqrt{(2n-1)/n^2}$.

Let us fix $n \geqslant 2$ and $x \in l_1(F)$ satisfying (i), (ii) and (iii). Let $\mathcal{M}$ denote the family of all subgroups $H \subset F$ such that F/H is a finite group. If $H \in \mathcal{M}$, then $\lambda\big(\varphi_H(x)\big) = \nu\big(\varphi_H(x)\big)$. Thus

$$\lambda\big(\varphi_H(x)\big) = 1 > \sqrt{\frac{2n-1}{n^2}} = \lambda(x)$$

and we infer that $l_1(F)$ does not satisfy (D$'$).

REFERENCES

[1] M. Hall, *The theory of groups*, New York 1959.

[2] A. Hulanicki, *On the spectral radius in group algebras*, Studia Mathematica 34 (1970), p. 209-214.

[3] H. Kesten, *Symmetric random walks on groups*, Transactions of the American Mathematical Society 92 (1959).

[4] А. Г. Курош, *Теория групп*, Москва 1967.

[5] C. E. Rickart, *General theory of Banach algebras*, 1960.

[6] W. Rudin, *Fourier analysis on groups*, London — New York 1962.

[7] A. Zygmund, *Trigonometrical series*, vol. 1, Cambridge 1959.

Reçu par la Rédaction le 27. 11. 1969