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UNIVERSAL ALGEBRAS WITH ALL OPERATIONS OF BOUNDED RANGE

BY

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1. Introduction. In this paper we study universal algebras $\langle A; U_k\rangle$, where U_k is the set of all finitary operations f such that f depends on at most one variable or the range of f has cardinality at most k (k cardinal $\leqslant |A|$). Finite algebras of this type were studied in [4]. Let $I \neq \emptyset$ be a set. Any subset ϱ of A^I is called an *I-relation* on A and A_ϱ is the set of all finitary operations f with $fg_1 \ldots g_n \in \varrho$ whenever all $g_i \in \varrho$. A relation ϱ is *stable relative* $\langle A; F\rangle$ if $F \subseteq A_\varrho$. First all I-relations stable relative $\langle A; U_k\rangle$ are determined in terms of the lattice of all equivalence relations on I. Further all relations ϱ such that $A_\varrho = U_k$ are characterized for all finite $k \leqslant |A|$ and for $k = |A|$. Here the least cardinality of I with an I-relation ϱ satisfying $A_\varrho = U_k$ is $k+1$ for $1 < k < \min(|A|, \aleph_0)$.

In the last section the equational class K generated by $\langle A; F\rangle$ such that U_k is the set of all polynomials over $\langle A; F\rangle$ is determined. This result extends Węglorz's representation theorem for Post-like algebras [26], which in its turn generalizes Foster's representation theorem for primal algebras [8].

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2. Preliminaries. To make this paper self-contained we will first briefly introduce the terminology and notation.

Let A be a set with $\alpha = |A| > 1$. Let $O^{(n)}$ be the set of all n-ary operations on A $(n = 1, 2, \ldots)$ and let

$$O = \bigcup_{n=1}^{\infty} O^{(n)}.$$

The coordinates (or components) of $a \in A^n$ will be throughout denoted by $(a_1, \ldots, a_n)$. The image of a in the mapping f will be designated by fa or $fa_1 \ldots a_n$ and the image of $B = B_1 \times \ldots \times B_n \subseteq A^n$ by fB or $fB_1 \ldots B_n$. The operations $e_i^n \in O^{(n)}$ $(1 \leqslant i \leqslant n)$, defined by $e_i^n a = a_i$ for every $a \in A^n$, are called *projections* [9] (others names: trivial, identity, or selective operations). The set of all projections will be denoted by J.

A universal algebra is a pair $\langle A; F\rangle$ where $F \subseteq O$. The set $F \cap O^{(n)}$ will be denoted by $F^{(n)}$.

We can construct the set $[F]$ of all *compositions* (or superpositions or compound operations) over F as follows. Let $F_0 = F$. Suppose F_i has already been constructed. Let F_{i+1} be the set of all operations obtained

(i) by replacing a variable in an operation from F_i by an operation from F_i,

(ii) by identifying some variables in operations from F_i, and

(iii) by permuting variables in operations from F_i.

Finally let

$$[F] = \bigcup_{i=0}^{\infty} F_i.$$

The sets $C \subseteq O$ satisfying $[C] = C$ are called *closed classes*. Because $[F \cup J]$ is the set of all polynomials ([9], § 12) (or algebraic operations) over $\langle A; F\rangle$, we will call any closed class C containing J a *polynomial class* (preferring it to the name *clone* used in [6]). The closed classes can be described as subalgebras of a certain algebra on O [14] and, therefore, form an algebraic lattice with respect to inclusion.

Note that this lattice is countable if $a = 2$ (see [16]), has $2^{\aleph_0}$ elements if $2 < a < \aleph_0$ (see [12] and [5]), and $2^{(2^a)}$ elements if $a \geqslant \aleph_0$.

For simplicity we have excluded zero operations. If these are needed, we can to each closed class $C \subseteq O$ assign the set $C^* = C \cup \{a \in A \mid a_1 \in C\}$, where a_1 denotes the mapping $A \to \{a\}$. For this the Mal'cev [14] preiterative algebra $\langle 0; \zeta, \tau, \Delta, *\rangle$ can be extended to $O \cup A$ so that $\langle O \cup A; *\rangle$ is still a monoid and the subalgebras are the closed classes, the sets C^*, and the sets of zero operations. Most of our results can be modified to include closed classes of the type C^*.

Polynomial classes can be described using relations on A. Let I be a non-empty set. An *I-relation* or *$|I|$-ary relation* ϱ on A is a subset of the set A^I of all mappings $I \to A$. If $|I| = k < \aleph_0$, we will identify A^I and A^k and the I-relations are simply the k-ary relations. Throughout ϱ denotes an I-relation on A. We say [18] that $f \in O^{(n)}$ (*weakly*) *preserves* ϱ if $fg_1 \ldots g_n \in \varrho$ whenever all $g_i \in \varrho$. Here and in the sequel $h = fg_1 \ldots g_n$ is the mapping $I \to A$ defined by $hi = f(g_1 i)\ldots(g_n i)$ for every $i \in I$. Let A_ϱ be the set of all $r \in O$ preserving ϱ. It is easy to see that A_ϱ is a polynomial class [19]. The converse is also true; namely [17], given any polynomial class P, there is a ϱ such that $|I| \leqslant a + \aleph_0$ and $P = A_\varrho$. The *relational degree* of P is the least cardinality of I for which there is a ϱ with $A_\varrho = P$.

The set A_ϱ can also be described as the set of all homomorphisms (or compatible mappings) of the I-relations ϱ^n into ϱ ($n = 1, 2, \ldots$).

We say that ϱ is *stable relative* $\langle A; F\rangle$ or $[F]$ if $F \subseteq A_\varrho$ ([7] if $|I| < \aleph_0$; in [27] a binary stable relation is called an invariant relation). It is easy to see that for $\varrho \neq \emptyset$ the following conditions are equivalent:

(i) ϱ stable relative F,

(ii) ϱ subalgebra of $\langle A^I; F\rangle$, and

(iii) $\langle \varrho; F\rangle \in \mathrm{SP}(A)$ [9].

In this sense, for $\varrho \neq \emptyset$, A_ϱ is the largest set $Q \subseteq O$ such that ϱ is a subalgebra of $\langle A^I; Q\rangle$.

Let $f \in O^{(m)}$. We say that f *depends on its i-th variable* if there are $a \in A^n$ and $b \in A$ such that

$$fa \neq fa_1 \ldots a_{i-1} b\, a_{i+1} \ldots a_n.$$

We say that $(b_0, b_1, b_2) \in A^3$ is an *essential triple for* f if b_0, b_1, and b_2 are distinct and there are $a^j \in A^n$ $(j = 0, 1, 2)$ and $1 \leqslant i \leqslant n$ such that $b_j = fa^j$, $a_i^0 = a_i^2 \neq a_i^1$, and $a_l^0 = a_l^1 \neq a_l^2$ for $l = 1, \ldots, n$; $l \neq i$ [15]. We will need the following result due to Iablonskiĭ ([10], Basic Lemma) and Salomaa [24] (see also [15]).

LEMMA 1. *Let* $f \in O^{(n)}$ *depend on at least two variables and let* $2 < k < \aleph_0$. *If* $|fA^n| \geqslant k$, *then there exist* $a^j \in A^n$ $(j = 0, \ldots, k-1)$ *such that all* fa^j *are pairwise distinct and* (fa^0, fa^1, fa^2) *is an essential triple for* f.

3. Stable relations relative U_k. Let U_1 be the set of all operations depending on at most one variable. Let $0 < k \leqslant a$ be a cardinal and let $U_k = U_1 \cup \{f \in O^{(n)} \mid |fA^n| \leqslant k,\ n = 1, 2, \ldots\}$. It is easy to see that each U_k is a polynomial class and that $U_a = O$. Let U consist of U_1 and all $f \in O^{(n)}$ $(n = 2, 3, \ldots)$ defined for every $a \in A^n$ by

$$fa = \lambda(\varphi_1 a_{j_1} \oplus \ldots \oplus \varphi_k a_{j_k}),$$

where $\lambda\colon \{0, 1\} \to A$, $\varphi_j\colon A \to \{0, 1\}$, $1 \leqslant i_1 < \ldots < i_k \leqslant n$, and $\oplus$ is the sum mod 2 on the set $\{0, 1\}$. Burle [4] has shown that for $a < \aleph_0$ the classes $U_1, L, U_2, \ldots, U_a$ form a unique maximal chain from U_1 to U_a in the lattice of closed classes (maximal in the sense that it cannot be properly extended to a chain from U_1 to U_a). For $a \geqslant \aleph_0$ the situation seems to be more complicated. Using the ideas from [15], § 3, and Lemma 1 it is easy to obtain

LEMMA 2. *The classes* $U_2, U_3, \ldots, U_k$ *form a unique maximal chain from* U_2 *to* U_k *for every* $2 < k < \min(a, \aleph_0)$.

It is easy to characterize relations stable relative U_1. For this we will denote the set of all (binary) equivalence relations on I by C_I. It is well-

-known (e. g. [6] and [9]) that C_I ordered by inclusion is a complete lattice.

A set $V \subseteq C_I$ is an *upper order ideal* if $x \in V$ implies $y \in V$ for every $x \in C_I$, $y \supseteq x$. The least upper order ideal containing $W \subseteq C_I$ will be denoted by (W). A *filter* (dual ideal) is an upper order ideal that is a sublattice of C_I. The least filter containing W will be denoted by $[W)$.

For any $0 < k \leqslant a$ we set $W^k = \{\gamma \in W \mid \gamma$ has at most k equivalence classes$\}$ and $\{W\}^k = [W)^k \setminus W^k$.

For any $f\colon M \to N$ we designate by $\varkappa_f$ the *kernel* of f, i. e. the element of C_M defined by $(x, y) \in \varkappa_f \Leftrightarrow fx = fy$. Finally, for $\gamma \in C_I$ we set $\Delta_\gamma = \{f \in A^I \mid x \equiv y(\gamma) \Rightarrow fx = fy\}$.

Note that Δ_γ is an I-relation on A and that $\gamma' \subseteq \gamma'' \Rightarrow \Delta_{\gamma'} \supseteq \Delta_{\gamma''}$. The relation $\bigcup\limits_{\gamma \in G} \Delta_\gamma$ with $G \subseteq C_I$ will be denoted by ΔG.

Now we can easily characterize relations stable relative U_1 and U_k.

PROPOSITION 1. *The relation ϱ is stable relative U_1 if and only if $\varrho = \Delta G$ with $G \subseteq C_I$.*

Proof. The sufficiency is immediate in view of $fg \in \Delta_\gamma$ for any $f \in O^{(1)}$, $\gamma \in G$ and $g \in \Delta_\gamma$.

For the necessity it suffices to prove $r \in \varrho \Rightarrow \Delta_{\varkappa_r} \subseteq \varrho$ because we can take $G = \{\varkappa_r \mid r \in \varrho\}$. Let $r \in \varrho$ and let $p \in \Delta_{\varkappa_r}$.

We define $f \in O^{(1)}$ as follows: 1. for $i \in rI$ we choose any $j_i \in r^{-1}i$ and set $fi = pj_i$ and 2. for all $i \in A \setminus rI$ let fi be any element of A.

The operation thus defined satisfies $p = fr$. Indeed, for any $y \in I$ we have $(y, j_{ry}) \in \varkappa_r$ and in view of $p \in \Delta_{\varkappa_r}$ we get $py = pi_{ry} = fry$. Finally $f \in U_1 \subseteq A_\varrho$ and $r \in \varrho$ give the required $p = fr \in \varrho$.

THEOREM 1. *The relation ϱ is stable relative U_k if and only if $\varrho = \Delta G$, where $G \subseteq C_I$ satisfies $\{G\}^k = \emptyset$.*

Proof. Necessity. If ϱ is stable relative U_k, it is stable relative U_1, i. e., by Proposition 1, $\varrho = \Delta G$. Let $\gamma_i \in G$ $(i = 1, \ldots, n)$, $\gamma_0 \in C_I^k$ and

$$\gamma_0 \supseteq \sigma = \bigcap_{i=1}^{n} \gamma_i.$$

It suffices to prove $\Delta_{\gamma_0} \subseteq \varrho$. In view of $k \leqslant a$ there exist $g_j \in \Delta_{\gamma_j}$ such that $\varkappa_{g_j} = \gamma_j$ $(j = 0, 1, \ldots, n)$. Let a mapping $h\colon I \to A_n$ be defined by $hi = (g_1 i, \ldots, g_n i)$ for every $i \in I$.

Define $f \in O^{(n)}$ as follows. For $a \in hI$ choose any $i_a \in h^{-1}a$ and set $fa = g_0 i_a$; for any $a \in A^n \setminus hI$ let $fa = g_0 i$, where i is an arbitrary fixed element of I.

Using $\gamma_0 \supseteq \sigma = \varkappa_h$ we see that f is well defined.

Proceeding as in the proof of Lemma 3 we prove that $g_0 = fg_1 \ldots g_n$. Since $f \in U_k \subseteq A_\varrho$ and $g_1, \ldots, g_n \in \varrho$, we get $g_0 \in \varrho$ as required.

Sufficiency. We can assume that G is an upper order ideal. Let $f \in U_k^{(n)}$ and $r_1, \ldots, r_n \in \varrho$. From $r_i \in \varrho$ it follows that $\varkappa_{r_i} \in G$. Designating $h = fr_1 \ldots r_n$, we get

$$\varkappa_h \supseteq \bigcap_{i=1}^{n} \varkappa_{r_i}.$$

In view of $f \in U^k$ we have $\varkappa_h \in C_I^k$ and, therefore, $\varkappa_h \in [G)^k$. From $[G)^k \subseteq G^k$ we get $\varkappa_h \in G$ and, finally, $h \in \Delta_{\varkappa_h} \subseteq \varrho$. Thus $f \in A_\varrho$, as we wished to show.

Corollary 1. *The relation ϱ satisfies $A_\varrho = O$ if and only if $\varrho = \Delta F$, where F is a filter on C_I.*

Proof. Let $\varrho = \Delta G$, where G is an upper order ideal in C_I. For any $\gamma \in C_I$ we have

$$\Delta_\gamma = \bigcup \{\Delta_\eta \mid \eta \in C_I^a, \eta \supseteq \gamma\}.$$

This shows that $\varrho = \Delta G = \Delta G^a$.

Let $\sigma = \Delta[G)$. By the same argument we get $\sigma = \Delta[G) = \Delta[G)^a$. Thus

$$A_\varrho = O \Rightarrow G^a = [G)^a \Rightarrow \varrho = \sigma \Rightarrow \varrho = \Delta[G),$$

where $[G)$ is a filter.

Conversely, if G is a filter, then obviously $\{G\}^a = \emptyset$ and $A_\varrho = U_a = O$ by Theorem 1.

If I is finite, then any non-empty filter has a least element. Thus we have the following result (proved in [18] for $|I| \leqslant a < \aleph_0$; slightly incorrect result for $a < \aleph_0$ and $|I| < \aleph_0$ is in [2]):

Corollary 2. *Let I be finite. Then $A_\varrho = O$ if and only if $\varrho = \emptyset$ or $\varrho = \Delta_\gamma$ with $\gamma \in C_I$.*

4. The equality $A_\varrho = U_k$. Let P be a polynomial class. It is natural to study the following problem:

Characterize all relations ϱ with $A_\varrho = P$.

Such characterization was given in Corollary 1 for $P = O = U_a$.

Now we are going to characterize the ϱ's with $A_\varrho = U_k$ for finite k's.

For infinite k's the situation seems to be far more complicated. Using Post's results [16] it is possible to characterize $A_\varrho = P$ for any polynomial class P on a two-element set (e. g. $A_\varrho = J$ was characterized in [21]) while nothing is known for $a > 2$. It seems that this is a rather difficult problem.

Theorem 2. *Let $1 < k < \min(a, \aleph_0)$. Then $A_\varrho = U_k$ if and only if $\varrho = \Delta G$, where $G \subseteq C_I$ satisfies $\{G\}^k = \emptyset$ and $\{G\}^{k+1} \neq \emptyset$.*

Proof. By Theorem 1 we have $U_k \subseteq A_\varrho \Leftrightarrow \{G\}^k = \emptyset$ and $U_{k+1} \nsubseteq A_\varrho \Leftrightarrow \{G\}^{k+1} \neq \emptyset$. In view of Lemma 2 we have $U_k \subset C \subset U_{k+1}$ for no closed class C and the theorem follows.

COROLLARY 3. *For* $1 < k < \min(a, \aleph_0)$ *the relational degree of* U_k *is* $k+1$.

Proof. By Theorem 2 we must find the least cardinality of I for which there is $G \subseteq C_I$ with $\{G\}^k = \emptyset$. Since $|I| \leqslant k$ implies $\{G\}^{k+1} = \{G\}^k = \emptyset$, we have $|I| \geqslant k+1$. Thus let $I = \{0, 1, \ldots, k\}$ and let $\tau_n \in C_I^k$ $(n = 0, 1)$ be defined by

$$(i, j) \in \tau_n \Leftrightarrow \{i, j\} = \{0, 1, 2\} \setminus \{n\} \quad \text{for } i, j \in I,\ i \neq j.$$

Let $G = C_I^k \cup \{\tau_0, \tau_1\}$. Since $\tau_0 \cap \tau_1$ is the identity equivalence ι, we have $\iota \in \{G\}^{k+1}$ and $\{G\}^{k+1} \neq \emptyset$. Thus $\varrho = \Delta G$ satisfies $A_\varrho = U_k$.

COROLLARY 4. *Let a polynomial class* P *have relational degree* r. *If* $1 < k < \aleph_0$, *then the relational degree of* $P \cap U_k$ *is at most* $r + k + 1$.

This follows from Corollary 3 if we use the concatenation of relations [20].

5. The equational class generated by $\langle A; U_k \rangle$**.** In this section we will characterize all algebras in the least equational class K (or primitive class or variety) containing $\boldsymbol{A} = \langle A; F \rangle$ with $[F] = U_k$. To prove our representation theorem we will start from Birkhoff's characterization $K = \mathrm{HSP}\boldsymbol{A}$ (see e. g. [9], § 23).

In view of Theorem 1 it suffices to study the homomorphic images of $\langle \varrho; F \rangle$, where $\varrho \neq \emptyset$ and ϱ is stable relative A or, equivalently, the congruences on $\langle \varrho; F \rangle$ with $\varrho = \Delta G$, where $\emptyset \neq G \subseteq C_I$ satisfies $\{G\}^k = \emptyset$. Let θ be a fixed congruence on $\langle \varrho; F \rangle$. Given $g_i \in A^I$ $(i = 1, 2)$, we set $E(g_1, g_2) = \{x \in I \mid g_1 x = g_2 x\}$.

First we derive a necessary condition for θ.

LEMMA 3. *Let* $1 \leqslant n < \aleph_0$ *and let* $c_{pj} \in \varrho$ *and* $c_{1j} \equiv c_{2j}(\theta)$ $(p = 1, 2;$ $j = 1, \ldots, n)$. *If* $m_1 \in \varrho$, $m_2 \in \varrho$, *and* $|m_1 I \cap m_2 I| \leqslant k$, *then*

$$\bigcap_{l=1}^{n} E(c_{1l}, c_{2l}) \subseteq E(m_1, m_2) \Rightarrow m_1 \equiv m_2(\theta). \tag{1}$$

Proof. Let $\varphi\colon I \to A^{n+2}$ be defined for every $i \in I$ by

$$\varphi i = (c_{21} i, \ldots, c_{2n} i, m_1 i, m_2 i).$$

We claim that there exists $f \in O^{(2n+2)}$ such that

$$f c_{l1} \ldots c_{ln} \varphi = m_l \quad (l = 1, 2). \tag{2}$$

A necessary and sufficient condition for this is

$$(c_{p1} i, \ldots, c_{pn} i, \varphi i) = (c_{q1} j, \ldots, c_{qn} j, \varphi j) \Rightarrow m_p i = m_q j \tag{3}$$

for every $i, j \in I$ and $p, q \in \{1, 2\}$. We will verify this condition.

First of all, $\varphi i = \varphi j$ implies

$$(i, j) \in \varkappa_{c_{21}} \cap \ldots \cap \varkappa_{c_{2n}} \cap \varkappa_{m_1} \cap \varkappa_{m_2}.$$

Now, if $p = q$, then $(i, j) \in \varkappa_{m_p}$ gives already the required $m_p i = m_p j$. Thus consider the case $\{p, q\} = \{1, 2\}$. From $(i, j) \in \varkappa_{c_{2l}}$ $(1 \leqslant l \leqslant n)$ we get $c_{2l} i = c_{2l} j$. According to the premiss of (3), $c_{1l} i = c_{2l} j$ and, therefore, $c_{1l} i = c_{2l} i$, $i \in \bigcap_{l=1}^{n} E(c_{1l}, c_{2l})$ and by assumption $i \in E(m_1, m_2)$. This and $(i, j) \in \varkappa_{m_2}$ finally give the required $m_1 i = m_2 i = m_2 j$.

This proves that there exists f satisfying (2).

Note that there is such an operation with values only in $m_1 I \cup m_2 I$. Since $|m_1 I \cup m_2 I| \leqslant k$, this shows that such f exists in U_k. Finally, applying $c_{pl} \in \varrho$, $m_p \in \varrho$, $c_{1l} \equiv c_{2l}(\theta)$, $f \in U_k$, formula (2), and the substitution property, we obtain $m_1 \equiv m_2(\theta)$.

Let $P(I)$ be the set of all subsets of I partially ordered by inclusion. Lemma 3 leads to the following definition:

Definition 1. An equivalence relation θ on the set ϱ is a *k-equivalence* if

1° $m_1 \equiv m_2(\theta) \Rightarrow fm_1 \equiv fm_2$ for any $f: A \to A$,

2° the set $L = \{E(m_1, m_2) \mid m_1 \equiv m_2(\theta)\}$ is a non-empty filter on $P(I)$ satisfying

$$E(m_1, m_2) \in L,\ |m_1 I \cup m_2 I| \leqslant k \Rightarrow m_1 \equiv m_2(\theta)$$

for every $m_1, m_2 \in \varrho$.

LEMMA 4. *Let θ be a k-equivalence on ϱ. Then θ is a congruence on $\langle \varrho; F \rangle$.*

Proof. Let L be the filter corresponding to θ, $f \in U_k^{(n)}$ and $m_{pl} \in \varrho$, $m_{1l} \equiv m_{2l}(\theta)$ $(p = 1, 2;\ l = 1, \ldots, n)$. Further, let $h_p = fm_{p1} \ldots m_{pn}$. Because L is a filter and

$$\bigcap_{l=1}^{n} E(m_{1l}, m_{2l}) \subseteq E(h_1, h_2)$$

we obtain $E(h_1, h_2) \in L$.

Moreover, $|h_1 I \cup h_2 I| \leqslant |fA^n| \leqslant k$, hence, by definition, $h_1 \equiv h_2(\theta)$ as we wished to show.

Let $\mathrm{Id}(A)$ denote the set of all identities satisfied in A ([9], § 26). Then we have the following representation theorem:

THEOREM 3. *Let $\boldsymbol{A} = \langle A; F \rangle$ with $[F] = U_k$ $(1 < k \leqslant |A|)$, $\boldsymbol{B}$ a non-trivial algebra of the same type, and K the equational class generated by $\boldsymbol{A}$. Then the following conditions are equivalent:*

(i) $\boldsymbol{B} \in K$.

(ii) $\mathrm{Id}(\boldsymbol{A}) = \mathrm{Id}(\boldsymbol{B})$.

(iii) *There are non-empty sets I and $G \subseteq C_I$ with $\{G\}^k = \emptyset$ and a k-equivalence θ on ΔG such that $\boldsymbol{B} \simeq \langle \Delta G; F \rangle / \theta$.*

Proof. (i) and (ii) are equivalent by definition, (i) $\Rightarrow$ (iii) was proved in Lemma 3, and (iii) $\Rightarrow$ (i) in Lemma 4.

Remarks. Let $k = a$. Then any a-equivalence θ is simply defined by a non-empty filter L on $P(I)$ and by $E(m_1, m_2) \in L \Leftrightarrow m_1 \equiv m_2$ for every $m_1, m_2 \in \varrho$. Thus $\langle \varrho; U_a \rangle / \theta$ is an extension of reduced direct powers ([9], § 22, Definition 6) (reduced direct powers constitute the special case $\varrho = \Delta C_I$). Hence as a corollary we get Węglorz's representation theorem for *Post-like algebras* [26] (i. e. algebras in the equational class generated by $\langle A; F \rangle$ with $[F] = O$). This in its turn extends a representation theorem for the equational class generated by a *primal algebra* (i. e. $\langle A; F \rangle$ with $[F] = O$ and $a < \aleph_0$) due to Wade [25], Rosenbloom [23] and Foster [8] (see also [9], § 27, T5). In this special case the reduction by θ can be removed while for $a \geqslant \aleph_0$ this cannot be done [26].

Note that, for $k < a$, in K are algebras whose lattice of congruences is not distributive ([13] or [9], Example 5.70), thus Jónsson's theorem ([13] or [9], § 39, T6) is not applicable. For $k = a$ it is applicable and $U_a = \mathrm{IP}_S\mathrm{HSP}_P(A)$, where P_S and P_P are the operators of the formation of subdirect and prime products, respectively.

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