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ON THE COMPLETENESS OF $\mathscr{L}_p$-SPACES OVER A CHARGE

BY

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Introduction. In [6] Green gave necessary and sufficient conditions for $\mathscr{L}_p(X, \mathscr{F}, \mu)$, $1 \leqslant p < \infty$, to be a complete normed linear space for a positive bounded charge space $(X, \mathscr{F}, \mu)$. But in [1] K. P. S. Bhaskara Rao and V. Aversa have shown that the necessary part of Green's result is not correct. However, in [5] Greco partially solved the problem of completeness of $\mathscr{L}_p(X, \mathscr{F}, \mu)$. In this paper we give a complete solution of the problem using a different method and we do not restrict ourselves to the bounded charge spaces nor do we impose any restriction on p except that it is non-negative. We also show that $\mathscr{L}_\infty(X, \mathscr{F}, \mu)$ is complete for every charge space $(X, \mathscr{F}, \mu)$. See also [7] for some related results.

We follow mainly K. P. S. Bhaskara Rao and M. Bhaskara Rao [2] for the notation and results which we use in our proofs. See also [1] and [4].

1. Definitions and notation. An extended real-valued finitely additive function μ on a field $\mathscr{F}$ of subsets of a set X is called a *charge*, and $(X, \mathscr{F}, \mu)$ is called a *charge space*. A charge space is called *positive* (*bounded*) if the charge is positive (bounded). In the following we assume that μ is a positive charge.

The set function $\mu^*: \mathscr{P}(X) \to [0, \infty]$ ($\mathscr{P}(X)$ denotes the power set of the set X) is defined by

$$\mu^*(A) = \inf\{\mu(B): B \supset A, B \in \mathscr{F}\}.$$

For $f, g: X \to \boldsymbol{R}$ we write

$$f = g \text{ a.e. } [\mu] \quad \text{if } \mu^*\{x: |f(x)-g(x)| > \varepsilon\} = 0$$

for all $\varepsilon > 0$. A function $f: X \to \boldsymbol{R}$ is said to be a *null function* if $f = 0$ a.e. $[\mu]$. A set $A \subset X$ is said to be a *null set* if I_A is a null function. We shall say that a sequence $\{f_n\}$ of real-valued functions on X *converges hazily* to a real-valued function f on X if

$$\lim_{n\to\infty} \mu^*\{x: |f_n(x)-f(x)| > \varepsilon\} = 0 \quad \text{for all } \varepsilon > 0.$$

A function f is said to be T_1*-measurable* if there exists a sequence $\{s_n\}$ of simple functions which converges to f hazily. A simple function

$$s = \sum_{i=1}^{n} c_i I_{B_i},$$

where c_i's are real numbers and $\{B_1, B_2, \ldots, B_n\} \subset \mathscr{F}$ is a partition of X, is called *integrable* if $\mu(B_i) < \infty$ whenever $c_i \neq 0$, and the integral of s, denoted by $\int s d\mu$, is defined to be the real number $\sum_{i=1}^{n} c_i \mu(B_i)$. A real-valued function f on X is said to be *integrable* if there is a sequence $\{s_n\}$ of integrable simple functions, converging to f hazily, and

$$\int |s_n - s_m| \, d\mu \to 0 \quad \text{as } n, m \to \infty.$$

In [2] the functions of this type are called *D-integrable.*

Denote by $L_0(X, \mathscr{F}, \mu)$, or by $L_0(\mu)$ for short, the linear space of all T_1-measurable functions and put

$$N = \{f \in L_0(\mu) \colon f = 0 \ \text{ a.e. } [\mu]\} \quad \text{and} \quad \mathscr{L}_0(\mu) = L_0(\mu)/N.$$

For $f \in L_0(\mu)$ and $\varepsilon > 0$ define

$$\psi(f, c) = c + \mu^*\{x \colon |f(x)| > c\}.$$

Define

$$\|f\|_0 = \begin{cases} \inf\limits_{c>0} \dfrac{\psi(f, c)}{1+\psi(f, c)} & \text{if } \psi(f, c) < \infty \text{ for some } c > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Now it is easy to see that convergence in this F-seminorm coincides with hazy convergence ([2], 4.3.5).

For $0 < p < \infty$ put

$$L_p(\mu) = \{f \in L_0(\mu) \colon |f|^p \text{ is integrable}\}$$

and

$$\mathscr{L}_p(\mu) = L_p(\mu)/N.$$

The space $\mathscr{L}_p(\mu)$ is equipped with an F-norm $\|\cdot\|_p$ which is defined as follows:

$$\|f\|_p = \begin{cases} \int |f|^p \, d\mu & \text{for } 0 < p < 1, \\ (\int |f|^p \, d\mu)^{1/p} & \text{for } 1 \leqslant p < \infty. \end{cases}$$

A function $f \colon X \to \boldsymbol{R}$ is called *essentially bounded* if there exists a positive real number k such that

$$\mu^*\{x \colon |f(x)| > k\} = 0.$$

Denote by $L_\infty(\mu)$ the linear space of all essentially bounded T_1-measurable functions on X and put

$$\mathscr{L}_\infty(\mu) = L_\infty(\mu)/N.$$

The space $\mathscr{L}_\infty(\mu)$ is equipped with the norm

$$\|f\|_\infty = \inf\{k > 0: \mu^*\{x: |f(x)| > k\} = 0\}.$$

A sequence $\{A_n\}$, where $A_n \subset X$, is said to be *μ-Cauchy* if

$$\mu^*(A_n \Delta A_m) \to 0 \quad \text{as } n, m \to \infty.$$

2. Completeness of $\mathscr{L}_p$, $0 \leqslant p < \infty$. Before proving our main theorem we need the following three lemmas.

LEMMA 2.1. *If $A_n \subset X$ and the sequence $\{I_{A_n}\}$ converges hazily to f, then $f = I_A$ a.e. $[\mu]$ for some $A \subset X$ and $\mu^*(A_n \Delta A) \to 0$.*

Proof. We shall show that there is a set $A \subset X$ such that

$$\mu^*\{x: |f(x) - I_A(x)| > 1/k\} = 0 \quad \text{for all } k \geqslant 1.$$

Consider

$$B_k = \{x: f(x) \in (-\infty, -1/k) \cup (1/k, 1-1/k) \cup (1+1/k, \infty)\},$$

where $k > 3$.

Since $B_k \subset \{x: |f(x) - I_{A_n}(x)| > 1/k\}$ for all n, we have $\mu^*(B_k) = 0$. Let

$$A = \{x: \tfrac{1}{2} < f(x) < 1 + \tfrac{1}{2}\}.$$

Then

$$\{x: |f(x) - I_A(x)| > 1/k\} \subset B_k \quad \text{for all } k \geqslant 3.$$

Therefore

$$\mu^*\{x: |f(x) - I_A(x)| > 1/k\} = 0 \quad \text{for all } k \geqslant 1.$$

Thus $f = I_A$ a.e. $[\mu]$. Hence $\{I_{A_n}\}$ converges hazily to I_A or, equivalently, $\mu^*(A_n \Delta A) \to 0$.

LEMMA 2.2. *Suppose for every μ-Cauchy sequence $\{A_n\} \subset \mathscr{F}$ with $\mu(A_n) < \infty$ there exists $A \subset X$ with $\mu^*(A_n \Delta A) \to 0$. Then for every sequence $\{B_n\} \subset \mathscr{F}$ with $\sum_{n=1}^{\infty} \mu(B_n) < \infty$ there exists $B \subset X$ such that*

(i) *$\mu^*(B_n \backslash B) = 0$ for all n;*

(ii) *$\mu^*(B) \leqslant \sum_{n=1}^{\infty} \mu(B_n)$.*

Proof. Let

$$\{B_n\} \subset \mathscr{F} \quad \text{with } \sum_{n=1}^{\infty} \mu(B_n) < \infty.$$

Put

$$A_k = \bigcup_{n=1}^{k} B_n.$$

Then $\mu(A_k) < \infty$ and $\{A_k\}$ is a μ-Cauchy sequence in $\mathcal{F}$ since

$$\mu(A_k \Delta A_{k+1}) \leqslant \mu(B_{k+1})$$

and

$$\mu(A_k \Delta A_{k+l}) \leqslant \sum_{n=k}^{k+l-1} \mu(A_n \Delta A_{n+1}).$$

Take $B \subset X$ with $\mu^*(A_k \Delta B) \to 0$. Now,

$$\mu^*(A_k \backslash B) \leqslant \mu^*(A_k \Delta B).$$

But $\mu^*(A_k \backslash B)$ is an increasing sequence of positive real numbers. It follows that $\mu^*(A_k \backslash B) = 0$ for all k. This yields (i).

For (ii), notice that

$$\mu^*(B) \leqslant \mu^*(A_k \cup (A_k \Delta B)) \leqslant \mu(A_k) + \mu^*(A_k \Delta B)$$
$$\leqslant \sum_{n=1}^{k} \mu(B_n) + \mu^*(A_k \Delta B).$$

Since $\mu^*(A_k \Delta B) \to 0$, we get (ii).

LEMMA 2.3. *Let $0 < p < \infty$ and let $\{f_n\}$ be a Cauchy sequence in $\mathcal{L}_p(\mu)$ which converges hazily to f. Then*

$$f \in \mathcal{L}_p(\mu) \quad \text{and} \quad \|f_n - f\|_p \to 0.$$

Proof. We shall show that if $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}_p(\mu)$, $0 < p < \infty$, then it satisfies the following two conditions:

(i) The charges λ_n on $\mathcal{F}$ defined as

$$\lambda_n(F) = \int_F |f_n|^p \, d\mu, \qquad F \in \mathcal{F},$$

are uniformly absolutely continuous with respect to μ, i.e., given $\varepsilon > 0$ there exists $\delta > 0$ such that $\lambda_n(E) < \varepsilon$ for all n whenever $\mu(E) < \delta$.

(ii) For each $\varepsilon > 0$ there exists $E_\varepsilon \in \mathcal{F}$ such that

$$\mu(E_\varepsilon) < \infty \quad \text{and} \quad \lambda_n(E_\varepsilon^c) < \varepsilon \text{ for all } n.$$

The assertion follows from this by Theorem 4.6.10 in [2]. (In fact, that theorem is formulated in [2] for $1 \leqslant p < \infty$, but the proof can be easily adapted to the case where $0 < p < 1$.)

We first prove (i) and (ii) for $0 < p < 1$. Fix $\varepsilon > 0$. Since $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}_p(\mu)$, there exists $N > 1$ such that

$$\int |f_n - f_m|^p \, d\mu < \varepsilon/2 \quad \text{for all } n, m \geqslant N.$$

Now,

$$\int_E |f_n|^p \, d\mu \leqslant \int_E |f_n - f_N|^p \, d\mu + \int_E |f_N|^p \, d\mu \quad \text{for all } n \geqslant N.$$

Since $f_1, \ldots, f_N \in \mathscr{L}_p(\mu)$, there exists $\delta > 0$ such that

$$\lambda_1(E), \ldots, \lambda_N(E) < \varepsilon/2 \quad \text{whenever } \mu(E) < \delta$$

(see [2], Theorem 4.4.13 (xi)). It follows that $\lambda_n(E) < \varepsilon$ whenever $\mu(E) < \delta$ and n is arbitrary. This proves (i) for $0 < p < 1$. With the same notation, there exists $E_\varepsilon \in \mathscr{F}$ such that $\mu(E_\varepsilon) < \infty$ and $\lambda_n(E_\varepsilon^c) < \varepsilon/2$ for $n = 1, 2, \ldots, N$ (see [2], Lemma 4.4.15). This yields (ii) for $0 < p < 1$. For $0 < p < \infty$ the same argument goes through except that we use the inequality

$$\Big(\int_E |f_n|^p \, d\mu\Big)^{1/p} \leqslant \Big(\int_E |f_n - f_N|^p \, d\mu\Big)^{1/p} + \Big(\int_E |f_N|^p \, d\mu\Big)^{1/p}.$$

Now, we are ready to prove our main theorem. Notice that, in the case $1 \leqslant p < \infty$ and $\mu(x) < \infty$, the following theorem is essentially due to Greco ([5], Corollario 2.5) by a different method (see Remark 2.5 below).

THEOREM 2.4. *Let $0 \leqslant p < \infty$. Then $\mathscr{L}_p(\mu)$ is complete if and only if for every μ-Cauchy sequence $\{A_n\} \subset \mathscr{F}$ with $\mu(A_n) < \infty$ there exists $A \subset X$ with $\mu^*(A_n \Delta A) \to 0$.*

Proof. Necessity. Let $\{A_n\} \subset \mathscr{F}$ be a μ-Cauchy sequence with $\mu(A_n) < \infty$. Then $\{I_{A_n}\}$ is a Cauchy sequence in $\mathscr{L}_p(\mu)$. Hence, by our assumption of completeness, there exists $f \in \mathscr{L}_p(\mu)$ such that $\|I_{A_n} - f\|_p \to 0$. By Theorem 4.6.10 of [2], $\{I_{A_n}\}$ converges hazily to f. This yields, in view of Lemma 2.1, the desired conclusion.

Sufficiency.

Case $p = 0$. Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{L}_0(\mu)$. By passing through a subsequence, we may assume that

$$\mu^*\{x \in X \colon |f_n(x) - f_{n+1}(x)| > 2^{-n}\} < 3^{-n}.$$

Define

$$A_n = \{x \in X \colon |f_n(x) - f_{n+1}(x)| > 2^{-n}\}.$$

Since f_n's are T_1-measurable, without loss of generality we may assume that f_n's are simple functions to ensure that $A_n \in \mathscr{F}$. Then

$$\sum_{n=k}^{\infty} \mu(A_n) < \sum_{n=k}^{\infty} 3^{-n} = 2^{-1} \cdot 3^{-(k-1)} \quad \text{for all } k.$$

Thus, by our assumption and Lemma 2.2, for each sequence $\{A_n\}_{n=k}^{\infty}$, $k = 1, 2, \ldots$, we can get a set $B_k \subset X$ such that

(i) $\mu^*(A_n \backslash B_k)_\infty = 0$ for all $n \geqslant k$;

(ii) $\mu^*(B_k) \leqslant \sum_{n=k}^{\infty} (A_n) < 2^{-1} \cdot 3^{-(k-1)}$.

Without loss of generality, we may assume that $B_k \subset B_{k+1}$ for all k (because, otherwise, take B_1, $B_1 \cap B_2, \ldots$). Let

$$B = \bigcap_k B_k.$$

Then, clearly, $\mu^*(B) = 0$.

We shall now define our function f to which $\{f_n\}$ converges hazily.

If $x \in B$, define $f(x)$ arbitrarily. If $x \notin B$, let $k(x)$ be the smallest k such that $x \notin B_k$. Note that if

$$x \notin \bigcup_{i=n}^{m-1} A_i \quad \text{and} \quad m > n,$$

then

(iii) $|f_n(x) - f_m(x)| \leqslant \sum_{i=n}^{m-1} 2^{-i} < 2^{-(n-1)}$.

Now, consider the following two cases:

Case 1. $x \notin \bigcup_{n \geqslant k(x)} A_n$. Then, by (iii), $\{f_n(x)\}$ is a Cauchy sequence of real numbers. Define

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Case 2. $x \in \bigcup_{n \geqslant k(x)} A_n$. Let $n(x)$ be the smallest $n \geqslant k(x)$ such that $x \in A_n$. Define $f(x) = f_{n(x)}(x)$.

To prove that $\{f_n\}$ converges hazily to f define

$$C_k = B \cup \Big(\bigcup_{n=1}^{k} A_n \backslash B_1\Big) \cup \Big(\bigcup_{n=2}^{k} A_n \backslash B_2\Big) \cup \ldots \cup (A_k \backslash B) \cup B_k.$$

In view of (i) and (ii) we have

$$\mu^*(C_k) \leqslant \mu^*(B_k) < 2^{-1} \cdot 3^{-(k-1)}.$$

We claim that $x \notin C_k$ implies $|f(x) - f_k(x)| < 2^{-(k-1)}$.

Indeed, we have $k(x) \leqslant k$. If x is as in Case 1, we get

$$x \notin \bigcup_{n \geqslant k} A_n,$$

and so, in view of (iii),

$$|f(x)-f_k(x)| = \lim_{n\to\infty} |f_n(x)-f_k(x)| \leqslant 2^{-(k-1)}.$$

Now, let x be as in Case 2. Since $x \notin C_k$, we have $k(x) \leqslant k \leqslant n(x)$. Hence, by (iii)

$$|f(x)-f_k(x)| \leqslant 2^{-(k-1)},$$

and so the claim is proved.

Since $\mu^*(C_n) \to 0$ as $n \to \infty$, we have $f_n \to f$ hazily. This proves the sufficiency for $p = 0$.

Case $0 < p < \infty$. As easily seen, a Cauchy sequence in $\mathscr{L}_p(\mu)$ is also a Cauchy sequence in $\mathscr{L}_0(\mu)$. Hence the assertion follows from the case $p = 0$ and Lemma 2.3.

Remark 2.5. We shall compare the condition of Theorem 2.4 with Greco's condition (**) (see [5], p. 244). Define

$$\bar{\mathscr{F}} = \{E \subset X\colon (\forall \varepsilon > 0)(\exists A, B \in \mathscr{F})\ B \subset E \subset A \text{ and } \mu(A \backslash B) < \varepsilon\}$$

and denote by $\bar{\mu}$ the unique extension of μ to a positive charge on the field $\bar{\mathscr{F}}$ (the Peano–Jordan completion of μ). Then Greco's condition reads as follows:

(**) For every increasing sequence $\{E_n\} \subset \bar{\mathscr{F}}$ there exists $E \in \bar{\mathscr{F}}$ with $\bar{\mu}(E_n \Delta E) \to 0$.

By Theorem 1.1 of [3], under the assumption that $\mu(x) < \infty$, (**) is equivalent to

(**)′ $\bar{\mu}$ is Cauchy-complete.

We shall check that (**)′ is equivalent to the following condition:

(**)″ For every μ-Cauchy sequence $\{A_n\} \subset \mathscr{F}$ there exists $A \subset X$ with $\mu^*(A_n \Delta A) \to 0$.

Indeed, since $\bar{\mu} = \mu^*|\bar{\mathscr{F}}$, (**)′ implies (**)″. To prove the converse, let first $\{A_n\} \subset \mathscr{F}$ be a μ-Cauchy sequence and take A as in (**)″. Fix $\varepsilon > 0$ and choose n_0 and $B \in \mathscr{F}$ with $A \Delta A_{n_0} \subset B$ and $\mu(B) < \varepsilon$. Then

$$A_{n_0} \backslash B \subset A \subset A_{n_0} \cup B \quad \text{and} \quad (A_{n_0} \cup B) \backslash (A_{n_0} \backslash B) = B.$$

Hence $A \in \bar{\mathscr{F}}$. The general case follows from this since, given $E_n \in \bar{\mathscr{F}}$, we can choose $A_n \in \mathscr{F}$ with $A_n \subset E_n$ and $\bar{\mu}(E_n \backslash A_n) \to 0$.

We shall illustrate our results by three examples.

Example 1 (cf. [1], Example 2). Let $X = [0, 1]$, $\mathscr{F}$ be the field generated by all open sets of X and μ be any positive finite countably additive measure on $\mathscr{F}$. In the notation of Remark 2.5, $\bar{\mathscr{F}}$ is the σ-field of μ-measurable sets in X and $\bar{\mu}$ is a positive regular measure on $\bar{\mathscr{F}}$. Hence (**) is satisfied since $\mu(X) < \infty$. It follows from Remark 2.5 that the condition of Theorem 2.4 is satisfied. Therefore $\mathscr{L}_p(\mu)$, $0 \leqslant p < \infty$, is complete.

EXAMPLE 2. Let $X = [0, 1]$, let

$$\mathscr{F} = \{\bigcup_{i=1}^{n} [a_i, b_i): n \in N, a_i < b_i, a_i, b_i \in (0, 1)\},$$

and μ be the Lebesgue measure restricted to $\mathscr{F}$. Then $(X, \mathscr{F}, \mu)$ does not satisfy the condition of Theorem 2.4, and so $\mathscr{L}_p(X, \mathscr{F}, \mu)$ $(0 \leqslant p < \infty)$ is not complete.

Indeed, let $\{r_1, r_2, \ldots\}$ be an enumeration of the rationals in $[0, 1)$ and put

$$A_n = [r_n - 2^{-(n+2)}, r_n + 2^{-(n+2)}) \cap X, \quad n = 1, 2, \ldots$$

Then

$$A_n \in \mathscr{F} \quad \text{and} \quad \sum_{n=1}^{\infty} \mu(A_n) \leqslant 1/2.$$

Suppose $(X, \mathscr{F}, \mu)$ satisfies the condition of Theorem 2.4. Then, by Lemma 2.2, there exists a subset $A \subseteq X$ such that

(i) $\mu^*(A_n \backslash A) = 0$ for all n;

(ii) $\mu^*(A) \leqslant 1/2$.

Property (i) implies that $A_n \cap A$ is non-empty for all $n \geqslant 1$. Hence A is dense in X. Therefore, $\mu^*(A) = 1$, which contradicts (ii).

EXAMPLE 3 (cf. [3], Example 2.5, and [2], p. 125). Let $X = N$, the set of positive integers, and let $\mathscr{F} = \mathscr{P}(N)$. Define μ on $\mathscr{F}$ as follows:

$$\mu(A) = \begin{cases} \sum_{n \in A} 2^{-n} & \text{if } A \text{ is finite}, \\ 2 - \sum_{n \in A^c} 2^{-n} & \text{if } A^c \text{ is finite}. \end{cases}$$

Extend μ to $\mathscr{F}$ as a positive real-valued charge (see [2], 3.3.4). Then $(X, \mathscr{F}, \mu)$ does not satisfy the condition of Theorem 2.4. Indeed, put $A_n = \{1, 2, \ldots, n\}$ and suppose $\mu(A_n \backslash A) \to 0$ for some $A \subset X$. Then $A = X$, whence $\mu(A \backslash A_n) > 1$.

3. Completeness of $\mathscr{L}_\infty$. The next result generalizes Nota 1 in [5] and, partially, Proposition 4.7.9 in [2].

THEOREM 3.1. *The space $\mathscr{L}_\infty(X, \mathscr{F}, \mu)$ is complete for every charge space $(X, \mathscr{F}, \mu)$.*

Proof. Let $\{f_n\}$ be a Cauchy sequence of functions in $\mathscr{L}_\infty(\mu)$. We shall define a function $f \in \mathscr{L}_\infty(\mu)$ such that $f_n \to f$ in $\mathscr{L}_\infty(\mu)$. By passing to a subsequence, we may assume that

$$\|f_n - f_{n+1}\|_\infty < 2^{-n}.$$

Let

$$A_n = \{x: |f_n(x) - f_{n+1}(x)| > 2^{-n}\}, \quad n \geqslant 1.$$

Then $\mu^*(A_n) = 0$ for all n. Moreover, as in the proof of Theorem 2.4, if

$$x \notin \bigcup_{i=n}^{m-1} A_i \quad \text{and} \quad m > n,$$

then

$$|f_n(x) - f_m(x)| < 2^{-(n-1)}.$$

Now, we define the desired function f.

Case 1. $x \notin \bigcup_{n=1}^{\infty} A_n$. Then $\{f_n(x)\}$ is a Cauchy sequence of real numbers. Define

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Case 2. $x \in \bigcup_{n=1}^{\infty} A_n$. Let $n(x)$ be the smallest n such that $x \in A_n$. Define $f(x) = f_{n(x)}(x)$.

To show that $f_n \to f$ in $\mathscr{L}_\infty$, define

$$H_n = \bigcup_{k=1}^{n} A_k.$$

Then $\mu^*(H_n) = 0$. We claim that $x \notin H_n$ implies

$$|f_n(x) - f(x)| \leqslant 2^{-(n-1)}.$$

This is clear if x is as in Case 1. Let x be as in Case 2. Since $x \notin H_n$, we have $n(x) > n$, and so $|f_n(x) - f(x)| \leqslant 2^{-(n-1)}$. Thus, the claim is proved.

Now, since $\mu^*(H_n) = 0$, it follows immediately that $f_n \to f$ hazily and f is essentially bounded. Therefore, in view of Proposition 4.6.13 in [2], $f \in \mathscr{L}_\infty(\mu)$. Moreover, $\|f_n - f\|_\infty \to 0$. Thus f is as desired.

Remark 3.2. We have obtained our results only for positive charge spaces. But this restriction on μ can be removed if we work with the total variation of μ, denoted now by $|\mu|$. This change will not affect our argument anyway because the definition of the $\mathscr{L}_p$-spaces, $0 \leqslant p \leqslant \infty$, over general charge spaces only involves $|\mu|$. The total variation $|\mu|$ is defined on F as follows:

$$|\mu|(A) = \sup\Big\{\sum_{i=1}^{n} |\mu(B_i)| \colon \{B_1, B_2, \ldots, B_n\} \subset \mathscr{F} \text{ is a partition of } A\Big\}.$$

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